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Feedback theory for neutral equations in infinite dimensional state spaces [i]

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Abstract. This paper is concerned with neutral partial differential equations with atomic difference operators in infinite dimensional state Hilbert spaces. Here we introduce an approach based essentially on representation of closed loop systems of well-posed and regular linear systems in Salamon-Weiss sense. This technique allows us to introduce a new formulation of generalized solutions of such equations. Furthermore we study spectral theory and positivity of the semigroup solution.

Keywords: translation semigroup, neutral equations, regular linear systems, feedback

MSC 2000 classification: primary 47D06, 34K40, secondary 93C05, 93B52

Dedicated to Rainer Nagel on the occasion of his 65^{th} *birthday*

Introduction and notation

The well-posedness of neutral equations with *atomic difference operators* in finite dimensional state spaces is, by now, a well-developed area in the theory of differential equations (see e.g., [3], [16], [15, Chap. 9], [19], [20], [23], [31]). At the same time the theory of such equations in infinite dimensional state space and p–integrable phase space seems to be not investigated. We note that in the C–setting the authors of [17] have presented a purely semigroup approach

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based on Hille-Yosida theorem to prove well-posedness of the neutral equations in Banach spaces.

In the current paper, we are interested in studying well-posedness of neutral equations with atomic difference operators in infinite dimensional state spaces and in the L_2–setting. Further, we work with a general class of difference and delay operators (not necessarily represented as Riemann-Stieltjes integrals) that are issued as observation operators from appropriate regular linear systems (in Salamon-Weiss sense [30]). Our approach is mainly based on feedback theory of regular linear systems.

In order to sketch precisely our results, let us introduce some notation that will be used throughout this work. The Hilbert space X, the real number $r \in (0, +\infty)$ are supposed to be fix. We denote by $L_2 := L^2([-r, 0], X)$ the Lebesgue space of square-integrable X–valued functions on $[-r, 0]$, $W^{1,2} := W^{1,2}([-r, 0], X)$ the Sobolev space, and $\mathcal{E} := C([-r, 0], X)$, the Banach space of X–valued continuous functions on $[-r, 0]$. If $x : [-r, \infty) \to X$, then the history of x is the function $x_t(\cdot) : [-r, 0] \to X$ defined by $x_t(s) = x(t+s)$ for $t \geq 0$. The Fréchet space $L_{2,loc} := L^2_{loc}(\mathbb{R}_+, X)$ is the space of all locally square-integrable functions.

Let us consider the neutral equation (we write shortly $(\mathrm{NE})_f$)

$$\begin{aligned} \frac{d}{dt}\,[x(t) - Kx_t] &= A[x(t) - Kx_t] + Px_t + f(t), \\ x_0 = \varphi \in L_2, &\ \lim_{t \to 0}(x(t) - Kx_t) = \eta \in X. \end{aligned}$$

Here A is the generator of a C_0-semigroup T on X, $K, P : W^{1,2} \to X$ are bounded linear operators, and $f \in L_{2,loc}$.

Instead of equation $(\mathrm{NE})_f$ one can first introduce the linear system (we write shortly $(\mathrm{LS})_f$)

$$\begin{aligned} \dot{z}(t) &= Az(t) + Px_t + f(t), \quad x_0 = \varphi, \\ y(t) &= z(t) + Kx_t, \quad \lim_{t \to 0} z(t) = \eta\,. \end{aligned}$$

So by invoking the feedback law $x(t) = y(t)$, we retrieve our initial neutral equation $(\mathrm{NE})_f$. Due to this fact one has only to solve the closed loop system associated with $(\mathrm{LS})_f$. Since the feedback theory is only introduced for standard (i.e. without delay) linear systems, so the first step in our approach is devoted to reformulate $(\mathrm{LS})_0$ (i.e. when $f = 0$) as a free delay system on a suitable infinite dimensional state space. To that purpose, as in [13], we shall assume that K and P are issued as observation operators from regular linear systems with input space X, state space L_2, output space X and the left shift semigroup on L_2 (we denote these systems by Σ_K and Σ_P). Under these assumptions it is shown in [13] that $(\mathrm{LS})_0$ can be written as a regular linear system (denoted by $\mathbf{\Sigma}$) of the form

$$\begin{cases} \dot{w}(t) = \mathcal{A}w(t) + \mathcal{B}x(t), & w(0) = \binom{\eta}{\varphi}, \quad t \geq 0, \\ y(t) = \mathcal{C}w(t), & t \geq 0, \end{cases} \tag{1}$$

with input space X, state space $\mathcal{X} := X \times L_2$ and output space X, where $\mathcal{A}(\eta,\varphi)^T = (A\eta + P\varphi, \varphi')^T$ for $(\eta,\varphi) \in D(\mathcal{A}) := D(A) \times D(Q)$, the control operator $\mathcal{B} = [\mathbb{P}\ B]^\top$ and the observation operator $\mathcal{C} = [I\ K]$. Here Q, B and $\mathbb{P}$ are respectively the generator of the left shift semigroup on L_2, the control and feed-through operators of Σ_P. Moreover, the state and output functions of $\mathbf{\Sigma}$ satisfy

$$w(t) = (z(t), x_t)^\top, \qquad t \geq 0, \tag{2}$$

and

$$y(t) = z(t) + \tilde{K}x_t, \qquad a.e.\ t \geq 0, \tag{3}$$

where

$$z(t) = T(t)\eta + \int_0^t T(t-\tau)[\mathbb{P}x(\tau) + \tilde{P}x_\tau]\, d\tau, \quad t \geq 0, \tag{4}$$

and $\tilde{K}$, $\tilde{P}$ are the Yosida extensions of K and P with respect to the left shift semigroup (see (15)). Note that the formulas (3) and (4) are well defined since x_t is the state trajectory of the both regular systems Σ_K and Σ_P (see Theorem 7). If in addition the identity operator I_X in X is an admissible feedback operator for Σ_K, then we prove that it is also an admissible feedback for $\mathbf{\Sigma}$. Now if we consider the feedback law $x(t) = y(t) + u(t)$, $t \geq 0$, where u is a new input, then (see [30]) we obtain another regular system $\mathbf{\Sigma}^I$, called closed loop system, with input space X, state space $\mathcal{X}$, output space X, state trajectory w and input function u. Observe that if we take $u = 0$ then the solution of $\mathbf{\Sigma}^I$ corresponds to that of $(\mathrm{NE})_0$. In particular, using (2)–(3) we will see that the solution of $(\mathrm{NE})_0$ is a pair of functions (z, x) with $z : [0,\infty) \to X$ and $x : [-r,\infty) \to X$, where z and x satisfy (4) and the equation $x(t) = z(t) + \tilde{K}x_t$ for a.e. $t \geq 0$. So we call (z, u) the *generalized solution* of the homogeneous neutral equation $(\mathrm{NE})_0$. Now if we denote by $\mathfrak{T}$ the C_0-semigroup corresponding to the system $\mathbf{\Sigma}^I$ then

$$\mathfrak{T}(t)\binom{\eta}{\varphi} = (z(t), x_t)^\top, \qquad \binom{\eta}{\varphi} \in \mathcal{X}, \quad t \geq 0, \tag{5}$$

where (z, x) is the generalized solution of $(\mathrm{NE})_0$. On the other hand, we will prove that the generator of $\mathfrak{T}$ (which is the part of $\mathcal{A}_{-1} + \tilde{\mathcal{C}}\mathcal{B}$ in $\mathcal{X}$, where $\tilde{\mathcal{C}}$ is the Yosida extension of $\mathcal{C}$ with respect to $\mathcal{A}$) coincides with the following operator

$$\mathfrak{A} = \begin{pmatrix} A & P \\ 0 & Q_m \end{pmatrix}, \qquad D(\mathfrak{A}) = \left\{ \binom{\eta}{\varphi} \in D(A) \times W^{1,2} : \varphi(0) - K\varphi = \eta \right\}, \tag{6}$$

where the maximal operator Q_m is defined by $Q_m\varphi = \varphi'$ for $\varphi \in D(Q_m) := W^{1,2}$. Thus there is a natural connection between the generalized solution of the neutral equation $(\mathrm{NE})_0$ and the Cauchy problem given by the generator $(\mathfrak{A}, D(\mathfrak{A}))$ of $\mathfrak{T}$. This connection has been already given in [3] in the case $X = \mathbb{R}^n$, $A = 0$, and K, P are given by Riemann-Stieltjes integrals.

Let us now deal with the non-homogeneous equation $(\mathrm{NE})_f$. To this purpose it suffices to think of f as a second control function of $\mathbf{\Sigma}$. Then in (1), we replace $\mathcal{B}$ and $\mathcal{C}$ by $\mathfrak{B} = [\mathcal{B}\ I]$ and $\mathfrak{C} = [\mathcal{C}\ 0]^\top$, so that we obtain another regular linear system with the same input space, state space and output space $\mathcal{X}$, semigroup generated by $\mathcal{A}$, control operator $\mathfrak{B}$, observation operator $\mathfrak{C}$, and has $I_{\mathcal{X}}$ as an admissible feedback. Using the same approach explained above we deduce that the generalized solution of $(\mathrm{NE})_f$ is related to the solution of the non-homogeneous Cauchy problem

$$\begin{cases} \dot{w}(t) = \mathfrak{A}w(t) + (f(t), 0), & t \geq 0, \\ w(0) = \binom{\eta}{\varphi} \in \mathcal{X}. \end{cases} \tag{nCP}$$

We shall see that the generalized solution (z, x) of $(\mathrm{NE})_f$ satisfies

$$z(t) = T(t)\eta + \int_0^t T(t-\tau)[\mathbb{P}x(\tau) + \tilde{P}x_\tau + f(\tau)]\, d\tau, \qquad t \geq 0. \tag{7}$$

We note that the authors of [4] have introduced several notions of well-posedness for $(NE)_f$ (see Section 4). In this paper we have improved (even in the infinite dimensional state spaces) all these definitions by introducing the new variation of constants formula (7).

The second aim of this paper (see Section 5) is to compute the spectrum $\sigma(\mathfrak{A})$ of $\mathfrak{A}$. This is not an easy task to do, since the domain of $\mathfrak{A}$ is not diagonal. To overcome this difficulty, we will use the theory of one-side coupled operator matrices introduced by Engel [7]. To that purpose we consider the difference equation

$$x(t) = Kx_t, \quad x_0 = \varphi \in L_2, \tag{8}$$

where K is the same operator with the same assumptions as before. We prove that (8) has a unique (mild) solution $x : [-r, \infty) \to X$ given by $x_t = S_K(t)\varphi$, where

$$[S_K(t)\varphi](s) = \begin{cases} \varphi(s+t), & t+s \leq 0, \\ \tilde{K}[S_K(s+t)\varphi], & t+s \geq 0 \end{cases}$$

is a C_0-semigroup. Its generator is given by

$$Q_K\psi := Q_m\psi \quad \text{for} \quad \psi \in D(Q_K) := \{\psi \in W^{1,2} :\ \psi(0) = K\psi\}.$$

Moreover, $\lambda \in \rho(Q_K)$ if and only if $1 \in \rho(Ke_\lambda)$, where the operator $e_\lambda : X \to L_2$ is defined by $(e_\lambda \eta)(s) = e^{\lambda s}\eta$ for $\eta \in X$ and $s \in [-r, 0]$. By setting $\mathcal{D}_\lambda := e_\lambda(\mathbf{1} - Ke_\lambda)^{-1}$ for $\lambda \in \rho(Q_K)$, we show that $\lambda \in \rho(\mathfrak{A})$ if and only if $\lambda \in \rho(A + P\mathcal{D}_\lambda)$.

We end the last section of this paper by studying the positivity of the solution of $(\mathrm{NE})_0$.

Finally, we note that the results of this paper are still hold for Banach spaces (see Remark 30). Here we have worked with Hilbert spaces since they are important for authors dealing with control theory and in practice they correspond to energy spaces.

1 A background on regular systems

In this section we recall the concept of infinite dimensional well-posed and regular linear systems (see [5, 29, 30] for more details).

In the sequel, Z, U and Y are Hilbert spaces and $V := (V(t))_{t\geq 0}$ is a C_0-semigroup on Z with generator $(G, D(G))$ and type (or growth bound) $\omega_0(V)$. Let Z_{-1} be the *extrapolation space* of Z for G, i.e. the completion of Z with respect to the norm $\|R(\lambda_0, G)z\|$ for some fixed $\lambda_0 \in \rho(G)$. We recall that G can be extended to a bounded operator $G_{-1} : Z \to Z_{-1}$ which generates a $\mathcal{C}_0$–semigroup $V_{-1} := (V_{-1}(t))_{t\geq 0}$ on Z_{-1} extending V (see e.g. [8, Chap.V, Theorem 5.5]).

The pair $(V, \Phi) := (V, (\Phi(t))_{t\geq 0})$ is called a *control system* on Z, U if $\Phi(t) : L^2([0, t], U) \to Z,\ t \geq 0$, are bounded linear operators satisfying

$$\Phi(t + s)u = \Phi(t)(u(\cdot + s)|[0, t]) + V(t)\Phi(s)(u|[0, s]) \tag{9}$$

for $u \in L^2([0, s+t], U)$ and $t, s \geq 0$. By the representation theorem due to Weiss [28, Theorem 3.9], there exists a unique operator $B \in \mathcal{L}(U, Z_{-1})$, called *control operator* for the semigroup V, such that

$$\Phi(t)u = \int_0^t V_{-1}(t - \sigma)Bu(\sigma)\, d\sigma \tag{10}$$

for any $t \geq 0$ and $u \in L^2([0, t], U)$, where the integral exists in Z_{-1}. Each control system (V, Φ) with control operator B is completely determined by an abstract differential equation of the form

$$\dot{x}(t) = Gx(t) + Bu(t), \quad x(0) = \eta, \quad t \geq 0, \tag{11}$$

which has a unique strong solution (called state trajectory) and is given by

$$x(t) = V(t)\eta + \Phi(t)u, \qquad t \geq 0.$$

We now look at the following observation system

$$\begin{aligned} \dot{x}(t) &= Gx(t), \quad x(0) = \eta, \\ y(t) &= Cx(t), \quad t \geq 0, \end{aligned} \tag{12}$$

where $C : D(G) \to Y$ is a bounded linear operator with respect to the graph norm of G. It has been shown by Weiss [27] that the well-posedness of (12) requires a certain "admissibility" of C for the semigroup V. More precisely, C is called an *admissible observation operator* for G (or for V) if

$$\int_0^{t_0} \|CV(\tau)\eta\|^2 \, d\tau \leq \gamma^2 \|\eta\|^2 \tag{13}$$

holds for some (hence for all) $t_0 \geq 0$ and all $\eta \in D(G)$ with a constant $\gamma := \gamma(t_0) > 0$. Due to (13), the operator defined by

$$(\Psi(t)z)(\tau) = CV(\tau)z \quad \text{for} \quad z \in D(G) \quad \text{and} \quad 0 \leq \tau \leq t, \tag{14}$$

can be extended to a bounded linear operator from X to $L^2_{loc}(\mathbb{R}_+, Y)$, which will be denoted also by $\Psi(t)$. In this case we say that $(V, \Psi) := (V, (\Psi(t))_{t\geq 0})$ is an *observation system* on Z, Y, and the observation equation $y(\cdot)$ of (12) satisfies $y(\tau) = (\Psi(t)\eta)(\tau)$ for a.e. $0 \leq \tau \leq t$ and all $\eta \in Z$. So, in order to have a representation like (14) on the hull space Z, Weiss [27] has introduced an extension of C, called the *Yosida extension* which is defined by

$$\begin{aligned} \tilde{C}z &:= \lim_{\lambda \to +\infty} C\lambda R(\lambda, G)z \\ D(\tilde{C}) &:= \{z \in Z : \text{ the above limit exists in } Y\}. \end{aligned} \tag{15}$$

As shown in [27, Theorem 4.5], the admissibility of C for V implies that $V(t)z \in D(\tilde{C})$ for all $z \in Z$ and a.e. $t \geq 0$, the map $\Psi_\infty : Z \to L^2_{loc}(\mathbb{R}_+, Y), \Psi_\infty z := \tilde{C}V(\cdot)z$, is linear and bounded, called the *extended output map*, and the output function $y(\cdot)$ of (12) is given by $y(t) = (\Psi_\infty \eta)(t)$ for all $\eta \in Z$ and a.e. $t \geq 0$.

In this section we assume that (V, Φ) and (V, Ψ) are control and observation systems with control and observation operators B and C, respectively. We shall focus on the well-posedness of the linear system

$$\begin{aligned} \dot{x}(t) &= Gx(t) + Bu(t), \quad x(0) = \eta, \quad t \geq 0, \\ y(t) &= Cx(t), \quad t \geq 0. \end{aligned} \tag{16}$$

We say that the system (16) is *well-posed* on Z, U, Y if there exists a family $\mathbb{F} := (\mathbb{F}(t))_{t\geq 0}$ of bounded linear operators from $L^2([0,t], U)$ to $L^2([0,t], Y)$, $t \geq 0$, satisfying

$$[\mathbb{F}(t+s)u](\tau) = [\mathbb{F}(t)(u(\cdot + s)|[0,t])](\tau - s) + [\Psi(t)\Phi(s)(u|[0,s])](\tau - s) \tag{17}$$

for $\tau \in [s, s+t],\ t, s \geq 0$, and $u \in L^2([0, s+t], U)$. In this case we say also that the quadruple $\Sigma := (T, \Phi, \Psi, \mathbb{F})$ is well-posed on Z, U, Y.

Let $\mathbf{P}_\tau$ be the operator of truncation to $[0, \tau]$, that is $(\mathbf{P}_\tau f)(t) = f(t)$ for $t \in [0, \tau]$, $f \in L_{2,loc}$, and zero otherwise. One shows that the operators $\mathbb{F}(\tau)$ are compatible in the sense that for $t > \tau$ we have $\mathbf{P}_\tau \mathbb{F}(t) = \mathbb{F}(t)$. This property provides a unique operator $\mathbb{F}_\infty : L^2_{loc}(\mathbb{R}_+, U) \to L^2_{loc}(\mathbb{R}_+, Y)$ called the *extended input-output map* and verifies $\mathbb{F}(\tau) = \mathbf{P}_\tau \mathbb{F}_\infty = \mathbf{P}_\tau \mathbb{F}_\infty \mathbf{P}_\tau$ for $\tau \geq 0$. We recall from [29, Theorem 3.6] that there exist $\alpha \in \mathbb{R}$ and a unique bounded and analytic function $H(\cdot) : \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > \alpha\} \to \mathcal{L}(U, Y)$ such that if $y = \mathbb{F}_\infty u$ then

$$\hat{y}(\lambda) = H(\lambda)\hat{u}(\lambda), \qquad \mathrm{Re}\lambda > \alpha.$$

The function H is called the *transfer function* of Σ.

We say that the well-posed system Σ is *regular* (with feedthrough zero) if the limit

$$\lim_{t\to 0} \frac{1}{t} \int_0^t (\mathbb{F}_\infty u_0)(\tau)\, d\tau = 0.$$

exists in Y for the constant input $u_0(s) = z,\ z \in U,\ s \geq 0$.

The following theorem (see [29]) gives a useful characterization of the regularity for well-posed systems. In fact, the reference [29] contains several equivalent conditions for regularity on Hilbert spaces.

1 Theorem. *Let $\Sigma := (T, \Phi, \Psi, \mathbb{F})$ be a well-posed system with control and observation operators B and C, respectively. Then the following statements are equivalent.*

(i) Σ *is regular.*

(ii) $R(\lambda, G_{-1})B \subset D(\tilde{C})$ *for some (hence for all)* $\lambda \in \rho(G)$.

(iii) For every $v \in U$, $\lim_{\lambda\to+\infty} H(\lambda)v = 0$, *where* H *is the transfer function of* Σ.

In this case the transfer function of Σ *is given by*

$$H(\lambda) = \tilde{C}R(\lambda, G_{-1})B, \qquad \mathrm{Re}\lambda > \omega_0(G). \tag{18}$$

The following definition will be used throughout this paper.

2 Definition. Let B and C be the control and observation operators issued from (T, Φ) and (T, Ψ), respectively. We say that the triple (G, B, C) generates a regular system Σ if there exists a bounded operator $\mathbb{F}_\infty : L^2_{loc}(\mathbb{R}_+, U) \to L^2_{loc}(\mathbb{R}_+, Y)$ such that $\Sigma = (V, \Phi, \Psi, \mathbb{F})$ is regular on Z, U, Y.

Let Σ be the regular system generated by (G,B,C). Then $\Phi(t)u \in D(\tilde{C})$ and $\mathbb{F}_\infty u := \tilde{C}\Phi(\cdot)u$ for all $u \in L^2_{loc}(\mathbb{R}_+,U)$.

The following theorem (see [29]) shows that the regularity of the system (16) allows us to extend its output function to a function in $L^2_{loc}(\mathbb{R}_+,Y)$.

3 Theorem. *Assume that (G,B,C) generates a regular system Σ in Z,U,Y with state trajectory $x(t)$ and output function $y(t)$. Then $x(t) \in D(\tilde{C})$ and*

$$y(t) = \tilde{C}x(t) = (\Psi_\infty x + \mathbb{F}_\infty u)(t) \tag{19}$$

for a.e. $t \geq 0$, $x \in X$ and $u \in L^2_{loc}(\mathbb{R}_+,U)$.

Let Σ be a regular system with the transfer function H and let $\Delta \in \mathcal{L}(Y,U)$. Then Δ is called an *admissible feedback* for Σ if $I - H(\cdot)\Delta$ has a uniformly bounded inverse in some right half plan.

The following perturbation theorem was proved by Weiss [30, Theorem 6.1, Theorem 7.2 and Proposition 7.10].

4 Theorem. *Assume that (G,B,C) generates a regular system*

$$\Sigma = (V,\Phi,\Psi,\mathbb{F})$$

with admissible feedback operator Δ. Then the operator defined by

$$G_\Delta = G_{-1} + B\Delta\tilde{C}$$
$$D(G_\Delta) := \{\eta \in D(\tilde{C}) : G_\Delta \eta \in Z\}$$

(the sum is defined in Z_{-1}) generates a $\mathcal{C}_0$–semigroup V_Δ on Z satisfying $V_\Delta(\sigma)\eta \in D(\tilde{C})$ for a. e. $\sigma \geq 0$ and

$$V_\Delta(t)\eta = V(t)\eta + \int_0^t V_{-1}(t-\sigma)B\Delta\tilde{C}V_\Delta(\sigma)\eta\, d\sigma \tag{20}$$

for $\eta \in Z,\ t \geq 0$.

2 A regular boundary control system

In this section we recall some known results on regular systems having the left shift as a C_0–semigroup (see e.g. [6, Sect. 4], [25, Chap. 2, Sect. 2.3]). The left shift semigroup $S := (S(t))_{t\geq 0}$ on L_2 is defined by

$$(S(t)f)(\theta) := 1\!\!\mathrm{I}_{[-r,0]}(t+\theta)f(t+\theta), \quad f \in L_2,\ t \geq 0,\ \theta \in [-r,0].$$

Here the symbol $1\!\!\mathrm{I}_J$ denotes the constant function equal to one in the interval $J \subset \mathbb{R}$ and zero otherwise. It can be seen that the generator of S is given by

$$Qf := f' \quad \text{for} \quad f \in D(Q) := \{f \in W^{1,2} : f(0) = 0\}.$$

Next, let $\Phi := (\Phi(t))_{t\geq 0}$ be the family of linear operators from $L^2([0,t],X)$ into L_2 defined by

$$(\Phi(t)x)(\theta) := 1\!\!\mathrm{I}_{\mathbb{R}_+}(t+\theta)x(t+\theta), \qquad t \geq 0,\ \theta \in [-r,0]. \tag{21}$$

It can be verified that (S,Φ) is a control system with input space X and state space L_2. Moreover, by taking the Laplace transform in (10) and (21) we deduce that

$$Bz = (\lambda - Q_{-1})\, e_\lambda z \tag{22}$$

holds for $z \in X$ and $\lambda \in \mathbb{C}$, where $e_\lambda : X \to L_2$ is a bounded linear operator given by

$$(e_\lambda z)(\theta) := e^{\lambda\theta} z, \qquad z \in X,\ \lambda \in \mathbb{C},\ \theta \in [-r,0].$$

Let $v(\cdot,\cdot) : \mathbb{R}_+ \times [-r,0] \to L^2$ with $v(0,\cdot) = \varphi$ be the state trajectory of (S,Φ). Then v is given by

$$v(t,\theta) = \begin{cases} x(t+\theta), & t+\theta \geq 0, \\ \varphi(t+\theta), & -r \leq t+\theta \leq 0. \end{cases}$$

Throughout this paper the function $v(t,\cdot)$ will be denoted by $x_t := x(\cdot + t)$ for $t \geq 0$.

We consider now the boundary control problem

$$\begin{aligned} \frac{\partial}{\partial t} v(t,\theta) = \frac{\partial}{\partial \theta} v(t,\theta), \quad v(0,\cdot) = \varphi, \quad t \geq 0,\ \theta \in [-r,0], \\ v(t,0) = x(t), \quad t \geq 0, \end{aligned} \tag{BCP}$$

for a control function $x \in L^2_{loc}(\mathbb{R}_+, X)$ and $\varphi \in L_2$. The state trajectory of this control equation is exactly the input function x_t. So the control system (S,Φ) is entirely determined by the abstract differential equation (BCP). Observe that the function x_t appears in our neutral differential equation $(\mathrm{NE})_f$. Moreover, the difference operator $\delta_0 - K$ and delay operator P are applied to this function, where δ_0 denotes the Dirac measure at 0. So it is convenient that these operators play the role of observation operators for (S,Φ). This remark will be important in the next section when we study equations with input delays.

We introduce now the set of regular triples

$$\mathrm{Reg}(X) := \{P \in \mathcal{L}(W^{1,2}, X) : (Q,B,P) \text{ generates a regular system}\}.$$

It is proved in [14, Proposition 3.1] that the set $\mathrm{Reg}(X)$ endowed with an appropriate norm is a Hilbert space. Observe, by Theorem 1 and (22), that if $P \in \mathrm{Reg}(X)$ then $\mathrm{rg}[e_\lambda] \subset D(\tilde{P})$ for all $\lambda \in \mathbb{C}$, where $\tilde{P}$ is the Yosida extension of P with respect to Q. In the sequel, we will characterize operators P satisfying this property. To that purpose we consider the following auxiliary operators (see [14, Section 3]).

5 Definition. The *mass operator* associated to $P \in \mathcal{L}(W^{1,2}, X)$ is defined by

$$\mathbb{P}z := \lim_{\sigma \to +\infty} Pe_\sigma z \quad \text{for} \quad z \in D(\mathbb{P}) := \{z \in X : \lim_{\sigma \to +\infty} Pe_\sigma z \,\text{exists in}\, X\}.$$

The following result has been proved in [14, Theorem 3.4].

6 Lemma. *Let P be a bounded linear operator from $W^{1,2}$ into X and $\mathbb{P}$ its mass operator. The following assertions are equivalent.*

(i) $\mathrm{rg}[e_\lambda] \subset D(\tilde{P})$ *for all* $\lambda \in \mathbb{C}$.

(ii) $D(\mathbb{P}) = X$.

In the case where $D(\mathbb{P}) = X$, we have $W^{1,2} \subset D(\tilde{P})$ and

$$Pg = \tilde{P}g + \mathbb{P}g(0) \quad \textit{for all} \quad g \in W^{1,2}. \tag{23}$$

We summarize the above results in the following theorem.

7 Theorem. *Let $P \in \mathrm{Reg}(X)$, let Σ_P be its associated regular system and let $x \in L^2(\mathbb{R}_+, U)$ be the control function of Σ_P. Then the input segment x_t is the state trajectory of Σ_P. So that, $x_t \in D(\tilde{P})$ for a.e. $t \geq 0$. Further, $[0, \infty) \ni t \mapsto \tilde{P}x_t \in X$ is an L^2– function. Finally, the transfer function of Σ_P is given by*

$$H(\lambda) = \tilde{P}e_\lambda = Pe_\lambda - \mathbb{P}, \qquad \lambda \in \mathbb{C}. \tag{24}$$

The following example gives a class of operators belonging to $\mathrm{Reg}(X)$.

8 Example. Let $\mu : [-r, 0] \to \mathcal{L}(X)$ be a function of bounded variation on $[-r, 0]$, i.e.

$$\gamma := \operatorname*{Var}_{[-r,0]}(\mu) := \sup_{n \in \mathbb{N}} \Big\{ \sum_{i=1}^{n} \|\mu(s_i) - \mu(s_{i-1})\|, \; -r = s_0 < \cdots < s_n = 0 \Big\}$$

is finite. Moreover, we shall assume that

$$\lim_{\varepsilon \to 0} \operatorname*{Var}_{[-\varepsilon,0]}(\mu) = 0. \tag{25}$$

Next, let $P : \mathcal{E} \to X$ be the linear bounded operator defined by

$$Pg = \int_{-r}^{0} d\mu(s)\, g(s), \qquad (g \in \mathcal{E}). \tag{26}$$

Then $P \in \mathrm{Reg}(X)$ and the transfer function of its associated regular system is $G(\lambda) = Pe_\lambda, \; \lambda \in \mathbb{C}$, (see [14, Theorem 3.3]).

9 Remark. Assume that $P \in \mathcal{L}(C([-r,0],\mathbb{R}^n),\mathbb{R}^n)$. From the Riesz representation theorem (cf. [2, p. 216]), there is an $n\times n$ matrix valued function $\mu(\cdot)$ whose entries are of bounded variation on $[-r,0]$ and such that P is represented by (26). In particular, if

$$Pg = g(0) - \sum_{j=1}^{m} \pi_j g(r_j) + \int_{-r}^{0} \Pi(s)g(s)ds,$$

where $0 < r_1 < r_2 < \cdots < r_m = r$, π_j are $n\times n$ matrices, and $\Pi(\cdot) \in L^2([-r,0],\mathbb{R}^{n\times n})$, then (Q,B,P) generates a regular system in $L_2,\mathbb{R}^n,\mathbb{R}^n$, by Example 8.

3 Feedback theory for linear systems with delay in control and observation variables

In this section we apply the feedback theory to the delay system given by $(\mathrm{LS})_f$. As it is well known (see [30]) the feedback theory is well established for distributed linear systems. So it is more convenient to convert $(\mathrm{LS})_0$ into a distributed linear system in a suitable state space and with suitable operators. This has been already done in [13]. In fact, it is shown that if $P,K \in \mathrm{Reg}(X)$ then $(\mathrm{LS})_0$ can be reformulated into the free delay system (1). In this case, the abstract differential system (1) determines a regular linear system $\mathbf{\Sigma} = (\mathbf{T},\mathbf{\Phi},\mathbf{\Psi},\mathbf{F})$ with input space X, state space $\mathcal{X} = X\times L_2$, output space X, where

$$\mathbf{T}(t) = \begin{pmatrix} T(t) & R(t) \\ 0 & S(t) \end{pmatrix}, \qquad t\geq 0, \tag{27}$$

and

$$\mathbf{\Phi}(t)u = \begin{pmatrix} \displaystyle\int_0^t T(t-\sigma)(\mathbb{P}u(\sigma) + \big[(\mathbb{F}_P)_\infty u\big](\sigma))\,d\sigma \\ \Phi_P(t)x \end{pmatrix}, \qquad t\geq 0, \tag{28}$$

where $\Sigma_P := (S,\Phi_P,\Psi_P,\mathbb{F}_P)$ is the regular system associated with P and $R(t): L_2 \to X,\ t\geq 0$, are the operators defined by

$$R(t)\varphi := \int_0^t T(t-\tau)\big[(\Psi_P)_\infty\varphi\big](\tau)\,d\tau.$$

Here $(\mathbb{F}_P)_\infty$ and $(\Psi_P)_\infty$ are the extended input-output operator and extended output operator of Σ_P, respectively. The C_0–semigroup $\mathbf{T}$ is generated by the operator $(\mathcal{A},D(\mathcal{A}))$ defined in the introduction (see [13]). On the other hand, due to [28, Theorem 3.9], there exists a unique control operator $\mathcal{B} \in \mathcal{L}(X, \mathcal{X}_{-1,\mathcal{A}})$ representing $\mathbf{\Phi}$ as in (10), where $\mathcal{X}_{-1,\mathcal{A}}$ denotes the extrapolation space associated with $\mathcal{X}$ and $\mathbf{T}$. Next, we propose to compute explicitly the control operator. Taking the Laplace transform on both sides of $\mathbf{\Phi}$ and using the definition of the transfer function, it follows from (24) that

$$R(\lambda, \mathcal{A}_{-1})\mathcal{B}z^0 = \begin{pmatrix} R(\lambda, A)Pe_\lambda z^0 \\ e_\lambda z^0 \end{pmatrix} \tag{29}$$

for large λ and $z^0 \in X$. It is proved in [13, Proposition 3.5] that $\mathcal{B} = (\mathbb{P}, B)^\top$ provided that P has a bounded extension to L_2, where B is the control operator of the control system (S, Φ_P). We prove here that this condition is not needed. Due to Lemma 6 (i) and (29) it suffices to compute the expression of $\mathcal{A}_{-1}$ on $X \times D(\tilde{P})$.

10 Lemma. *Assume that P is an admissible observation operator for Q. Then the operator $(\mathcal{A}, D(\mathcal{A}))$ coincides with the following one*

$$\mathit{\Delta}_P = \begin{pmatrix} A_{-1} & \tilde{P} \\ 0 & Q_{-1} \end{pmatrix},$$
$$D(\mathit{\Delta}_P) = \{\left(\begin{smallmatrix} \eta \\ \varphi \end{smallmatrix}\right) \in X \times D(\tilde{P}) \ : \ \mathit{\Delta}_P\left(\begin{smallmatrix} \eta \\ \varphi \end{smallmatrix}\right) \in \mathcal{X}\}.$$

Proof. Define the operator

$$\Lambda := \begin{pmatrix} A & 0 \\ 0 & Q \end{pmatrix}, \qquad D(\Lambda) := D(A) \times D(Q).$$

Clearly, Λ is the generator of a diagonal C_0–semigroup on $\mathcal{X}$. Next, we consider the operator

$$\Xi := \begin{pmatrix} 0 & P \\ 0 & 0 \end{pmatrix}, \qquad D(\Xi) := X \times D(Q).$$

Since P is an admissible observation operator for Q then it is obvious that Ξ is an admissible observation operator for Λ. Moreover, the Yosida extension of Ξ with respect to Λ is given by

$$\tilde{\Xi} = \begin{pmatrix} 0 & \tilde{P} \\ 0 & 0 \end{pmatrix}, \qquad D(\tilde{\Xi}) := X \times D(\tilde{P}). \tag{30}$$

On the other hand, one can see that the triple $(\Lambda, I_\mathcal{X}, \Xi)$ generates a regular system with $I_\mathcal{X}$ as an admissible feedback, since its control operator is bounded. Thus, by Theorem 4, the operator $\Lambda_{-1} + \tilde{\Xi}$ with domain

$$D(\Lambda_{-1} + \tilde{\Xi}) = \left\{\left(\begin{smallmatrix} \eta \\ \varphi \end{smallmatrix}\right) \in X \times D(\tilde{P}) \ : \ (\Lambda_{-1} + \tilde{\Xi})\left(\begin{smallmatrix} \eta \\ \varphi \end{smallmatrix}\right) \in \mathcal{X}\right\}$$

generates a C_0–semigroup on $\mathcal{X}$. On the other hand, it can be verified that

$$\Lambda_{-1} := \begin{pmatrix} A_{-1} & 0 \\ 0 & Q_{-1} \end{pmatrix}, \qquad D(\Lambda_{-1}) = \mathcal{X}.$$

Now, due to (30), we deduce that $\Lambda_{-1} + \tilde{\Xi} = \mathit{\Delta}_P$. Next, we shall prove that $\mathcal{A} = \mathit{\Delta}_P$. In fact, let $\binom{\eta}{\varphi} \in D(\mathit{\Delta}_P)$. Then $A_{-1}\eta + \tilde{P}\varphi \in X$ and $Q_{-1}\varphi \in L_2$. This implies that $\varphi \in D(Q)$ and $Q_{-1}\varphi = Q\varphi$. Hence, $\tilde{P}\varphi = P\varphi$ and $\eta \in D(A)$. This end the proof. QED

The following proposition generalizes the one given in [13, Proposition 3.5].

11 Proposition. *Assume that $P \in \mathrm{Reg}(X)$. Then the control system $(\mathbf{T}, \mathbf{\Phi})$ is represented by the admissible control operator $\mathcal{B} = (\mathbb{P}, B)^\top$.*

Proof. Let $H(\cdot)$ be the transfer function of the regular system associated to P. So that $H(\lambda)\eta = \tilde{P}e_\lambda\eta \to 0$ as $\lambda \to +\infty$ and $\eta \in X$ (see Theorem 7 and Theorem 1). According to (24), we have $Pe_\lambda\eta = \mathbb{P}\eta + H(\lambda)\eta$. Then (29) and (22) imply

$$\lambda R(\lambda, \mathcal{A}_{-1})\mathcal{B}\eta - \begin{pmatrix} \mathbb{P}\eta \\ B\eta \end{pmatrix} = \begin{pmatrix} (\lambda R(\lambda, A)\mathbb{P}\eta - \mathbb{P}\eta) + \lambda R(\lambda, A)H(\lambda)\eta \\ \lambda R(\lambda, Q_{-1})B\eta - B\eta \end{pmatrix}$$

for large λ. Set $\mathcal{B}_0 := \binom{\mathbb{P}}{B}$. It is clear that $\mathcal{B}_0\eta \in \mathcal{X}_{-1,\Lambda}$. Since $\|\lambda R(\lambda, A)H(\lambda)\eta\|_X$ goes to zero as $\lambda \to +\infty$ then

$$\lim_{\lambda \to +\infty} \|\lambda R(\lambda, \mathcal{A}_{-1})\mathcal{B}\eta - \mathcal{B}_0\eta\|_{\mathcal{X}_{-1,\Lambda}} = 0. \tag{31}$$

Next, we show that $\mathcal{B}\eta \in \mathcal{X}_{-1,\Lambda}$ and $\mathcal{B}\eta = \mathcal{B}_0\eta$ for $\eta \in X$. To this purpose we will use a technique similar to the one of Weiss [30]. From the proof of Lemma 10 and [30, Theorem 7.2], the triple $(\mathcal{A}, I_\mathcal{X}, \Xi_L)$ generates a regular linear system which is the closed-loop system (i.e. feedback system) associated to the regular system generated by $(\Lambda, I_\mathcal{X}, \Xi)$, where Ξ_L is the Lebesgue extension of Ξ with respect to Λ (see [27]). Moreover, if $\tilde{\Xi}_L$ is the Yosida extension of Ξ_L with respect to $\mathcal{A}$ then by [30, Proposition 7.1] (see [25, Theorem 7.5.3 (iii)]) we have $\tilde{\Xi}_L = \tilde{\Xi}$. As it is already mentioned by Weiss [30, Conjecture] $\mathcal{X}_{1,\mathcal{A}}$ is not in general dense in $\mathcal{D}(\tilde{\Xi})$. Then one defines the Banach space $\mathcal{W}_1$ as the closure of $\mathcal{X}_{1,\mathcal{A}}$ in $D(\tilde{\Xi})$. The norm in $\mathcal{W}_1$ is given by

$$\|g\|_{\mathcal{W}_1} := \|g\|_\mathcal{X} + \sup_{\lambda \geq \beta} \|\Xi\lambda R(\lambda, \mathcal{A})g\|_\mathcal{X},$$

where $\beta \in \mathbb{R}$ is such that $[\beta, \infty) \subset \rho(\mathcal{A})$. Due to [30, Proposition 5.3] (see [25, Theorem 5.4.3]) we have $\mathcal{X}_{1,\mathcal{A}} \subset D(\Xi_L) \subset \mathcal{W}_1 \subset \mathcal{X}$ densely and with continuous embedding. We now define another Banach space as follows:

$$\mathcal{W} := (\lambda_0 - \mathcal{A}_{-1})\mathcal{W}_1,$$

endowed with the norm $\|\binom{\eta}{\varphi}\|_{\mathcal{W}} := \|R(\lambda_0, \mathcal{A}_{-1})\binom{\eta}{\varphi}\|_{D(\tilde{\Xi})}$ for some $\lambda_0 \in \rho(A)$. By [30, page 54] we have $\mathcal{X} \subset \mathcal{W} \subset \mathcal{X}_{-1,\mathcal{A}}$ densely and with continuous embedding. From [30, page 55], we have $\mathcal{W} \subset \mathcal{X}_{-1,\Lambda} \cap \mathcal{X}_{-1,\mathcal{A}}$ and

$$\begin{aligned} g &= \lim_{\lambda\to+\infty} \lambda R(\lambda, \Lambda_{-1})g \qquad (\text{in} \mathcal{X}_{-1,\Lambda} \text{ and in } \mathcal{X}_{-1,\mathcal{A}}) \\ g &= \lim_{\lambda\to+\infty} \lambda R(\lambda, \mathcal{A}_{-1})g \qquad (\text{in } \mathcal{X}_{-1,\Lambda} \text{and in } \mathcal{X}_{-1,\mathcal{A}}) \end{aligned} \tag{32}$$

for all $g \in \mathcal{W}$. Let P_L be the Lebesgue extension of P with respect to Q. Since (Q, B, P) generates a regular system then by (22) and [29] we have $e_\lambda \eta \in D(P_L)$. So, by (29), and since $D(\Xi_L) = X \times D(P_L)$ it follows that $\mathcal{B}\eta \in \mathcal{W}$. This prove the first claim. For the second, by (32), we obtain

$$\lim_{\lambda\to+\infty} \|\lambda R(\lambda, \mathcal{A}_{-1})\mathcal{B}\eta - \mathcal{B}\eta\|_{\mathcal{X}_{-1,\Lambda}} = 0.$$

Thus, by (31),

$$\begin{aligned} \|\mathcal{B}\eta - \mathcal{B}_0\eta\|_{\mathcal{X}_{-1,\Lambda}} \leq & \lim_{\lambda\to+\infty} \|\lambda R(\lambda, \mathcal{A}_{-1})\mathcal{B}\eta - \mathcal{B}_0\eta\|_{\mathcal{X}_{-1,\Lambda}} + \\ & \lim_{\lambda\to+\infty} \|\lambda R(\lambda, \mathcal{A}_{-1})\mathcal{B}\eta - \mathcal{B}\eta\|_{\mathcal{X}_{-1,\Lambda}} = 0. \end{aligned}$$

QED

Let us consider the operator

$$\mathcal{C} := \begin{pmatrix} I & K \end{pmatrix} : D(\mathcal{A}) \to X. \tag{33}$$

Then $\mathcal{C} \in \mathcal{L}(D(\mathcal{A}), X)$ is the observation operator of $\mathbf{\Sigma}$. In what follows we denote by $\tilde{\mathcal{C}}$ its Yosida extension with respect to $\mathcal{A}$.

The following lemma can be proved as in [12, Theorem 4.5].

12 Lemma. *Assume that P and K belong to* Reg(X)*. Then $\mathcal{C}$ is an admissible observation operator for $\mathcal{A}$. Moreover,*

$$X \times [D(\tilde{P}) \cap D(\tilde{K})] \subset D(\tilde{\mathcal{C}}), \tag{34}$$

$$\tilde{\mathcal{C}} = \begin{pmatrix} I & \tilde{K} \end{pmatrix} \quad on \quad X \times [D(\tilde{P}) \cap D(\tilde{K})]. \tag{35}$$

Let us now deal with the non-homogeneous delay system $(\text{LS})_f$ (see Section iv). To this purpose we will consider the non-homogeneous term f as an additional control function of $(\text{LS})_0$. Then instead of the input space X, we shall work with a large one, namely $\mathcal{U} := X \times X$. Moreover, we introduce the operator

$$\mathfrak{B} = \begin{pmatrix} \mathbb{P} & I \\ B & 0 \end{pmatrix} : \mathcal{U} \longrightarrow \mathcal{X}_{-1,\mathcal{A}}. \tag{36}$$

Obviously, by Proposition 11, $\mathfrak{B}$ is an admissible control operator for $\mathcal{A}$. On the other hand, we shall modify the observation space X by the new one $\mathcal{Y} := \mathcal{U}$ and consider a new observation operator

$$\mathfrak{C} := \begin{pmatrix} I & K \\ 0 & 0 \end{pmatrix} : D(\mathcal{A}) \longrightarrow \mathcal{Y}. \tag{37}$$

According to Lemma 12 it can be verified that $\mathfrak{C}$ is an admissible observation operator for $\mathcal{A}$ and its Yosida extension $\tilde{\mathfrak{C}}$ with respect to $\mathcal{A}$ satisfies

$$\tilde{\mathfrak{C}} := \begin{pmatrix} I & \tilde{K} \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad X \times [D(\tilde{P}) \cap D(\tilde{K})]. \tag{38}$$

The main result of this section is given by the following theorem.

13 Theorem. *Assume that $P, K \in \mathrm{Reg}(X)$ with associated regular linear systems Σ_P and Σ_K, respectively. In addition we assume that the mass operator associated with K is identically null and that I_X is an admissible feedback operator for Σ_K. Then the triple $(\mathcal{A}, \mathfrak{B}, \mathfrak{C})$ generates a regular system $\mathbf{\Sigma}'$ with input space $\mathcal{U}$, state space $\mathcal{X}$, output space $\mathcal{Y}$ and $I_{\mathcal{X}}$ as an admissible feedback operator. Further, if $\mathbf{u} = (x, f)^\top$ is the control function of $\mathbf{\Sigma}'$ then its state trajectory is given by*

$$\mathbf{w}(t) = (z(t), x_t)^\top, \qquad t \geq 0, \tag{39}$$

while its observation equation satisfies

$$\mathbf{y}(t) = (z(t) + \tilde{K}x_t, 0)^\top \tag{40}$$

for almost every $t \geq 0$, where $z(\cdot) : [0, \infty) \to X$ is the continuous function given by (7).

Proof. A straightforward argument as in [13, Theorem 5.1] shows that $(\mathcal{A}, \mathfrak{B}, \mathfrak{C})$ generates a regular system $\mathbf{\Sigma}'$ with input space $\mathcal{U}$, state space $\mathcal{X}$, output space $\mathcal{Y}$. Moreover, by Theorem 1 and (29) (see also the proof of [13, Theorem 3.1]), the transfer function $\mathcal{H}$ of $\mathbf{\Sigma}'$ is given by

$$\mathcal{H}(\lambda) = \tilde{\mathfrak{C}}R(\lambda, \mathcal{A}_{-1})\mathfrak{B} \tag{41}$$

$$= \begin{pmatrix} I & \tilde{K} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R(\lambda, A)Pe_\lambda & R(\lambda, A) \\ e_\lambda & 0 \end{pmatrix} \tag{42}$$

$$= \begin{pmatrix} R(\lambda, A)Pe_\lambda + Ke_\lambda & R(\lambda, A) \\ 0 & 0 \end{pmatrix} \tag{43}$$

for large $\Re\lambda$, due to Lemma 6 and (38). Let us now set

$$\Upsilon(\lambda) := R(\lambda, A)Pe_\lambda + Ke_\lambda, \qquad \lambda \in \rho(A). \tag{44}$$

We claim that $I_X - \Upsilon(\lambda)$ is invertible and has a uniformly bounded inverse in some right half plan. In fact, let us denote by $\mathbb{C}_\gamma := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq \gamma\}$ for some $\gamma \in \mathbb{R}$. We now take, by assumption, $\gamma_1 \in \mathbb{R}$ such that $h(\lambda) := I - Ke_\lambda$ is invertible for $\lambda \in \mathbb{C}_{\gamma_1}$ and $\sup\{\|h(\lambda)^{-1}\| : \lambda \in \mathbb{C}_{\gamma_1}\} \leq \kappa_1 < +\infty$. Since, by Theorem 7, $Pe_\lambda - \mathbb{P}$ is a transfer function, there exists $\gamma_2 \in \mathbb{R}$ such that $\sup\{Pe_\lambda : \mathrm{Re}\lambda \geq \gamma_2\} \leq \kappa_2 < +\infty$. We now take $\omega > \omega_0(A)$, $M := \sup\{\|T(t)\| : t \in [0,1]\}$ and $\gamma := \max\{\gamma_1, \gamma_2, \omega + 2M\kappa_1\kappa_2\}$. Due to (44) we obtain $I - \Upsilon(\lambda) = \left[I - R(\lambda, A)Pe_\lambda h(\lambda)^{-1}\right] h(\lambda)$ for $\lambda \in \mathbb{C}_\omega \cap \mathbb{C}_{\gamma_1}$. Moreover,

$$\sup_{\lambda \in \mathbb{C}_\gamma} \|R(\lambda, A)Pe_\lambda h(\lambda)^{-1}\| < \frac{1}{2}.$$

Thus, $I - \Upsilon(\lambda)$ is invertible for $\lambda \in \mathbb{C}_\gamma$ and $\sup\{\|(I - \Upsilon(\lambda))^{-1}\| : \lambda \in \mathbb{C}_\gamma\} \leq 2\kappa_1$. So that, by (41), it is clear that $I_{\mathcal{X}}$ is an admissible feedback operator for $\mathbf{\Sigma}'$. Now if $(x, f)^\top \in L^2_{loc}(\mathbb{R}_+, \mathcal{U})$ is the control function of $\mathbf{\Sigma}'$ then its state trajectory is given by

$$\begin{aligned}\mathbf{w}(t) = \mathbf{T}(t)\left(\begin{smallmatrix}\eta\\ \varphi\end{smallmatrix}\right) + \int_0^t \mathbf{T}_{-1}(t-\tau)\mathfrak{B}\left(\begin{smallmatrix}x(\tau)\\ f(\tau)\end{smallmatrix}\right) d\tau \\ = \mathbf{T}(t)\left(\begin{smallmatrix}\eta\\ \varphi\end{smallmatrix}\right) + \mathbf{\Phi}(t)x + \int_0^t \mathbf{T}(t-\tau)\left(\begin{smallmatrix}f(\tau)\\ 0\end{smallmatrix}\right) d\tau\end{aligned}$$

for $\left(\begin{smallmatrix}\eta\\ \varphi\end{smallmatrix}\right) \in \mathcal{X}$ and $t \geq 0$, due to (36). Since (Q, B, P) generates a regular system $\Sigma_P = (S, \Phi_P, \Psi_P, \mathbb{F}_P)$ on L_2, X, X it follows from Theorem 7 that $x_t = S(t)\varphi + \Phi_P(t)x$, and by Theorem 3 we know that $\tilde{P}x_t = ((\Psi_P)_\infty\varphi + (\mathbb{F}_P)_\infty x)(t)$ for a.e. $t \geq 0$. Hence, by (27) and (28), we get

$$\mathbf{w}(t) = (z(t), x_t)^\top, \quad t \geq 0.$$

Finally, the observation of $\mathbf{\Sigma}'$ is given by $\tilde{\mathfrak{C}}\mathbf{w}(t)$ for a.e. $t \geq 0$, and hence (40) follows immediately from (38). *QED*

14 Remark. Let the assumptions of Theorem 13 be satisfied. As we have already recalled at the beginning of this section, the triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ generates a regular linear system $\mathbf{\Sigma}$ with input space X, state space $\mathcal{X}$ and output space X. Moreover, by using (29) and Lemma 12 one can see that the transfer function of $\mathbf{\Sigma}$ is exactly the function Υ given by (44), since by assumption $\mathbb{K} = 0$. Further, by the proof of Theorem 13, I_X is an admissible feedback operator for $\mathbf{\Sigma}$.

4 Well-posedness of neutral equations

In this section we study the well-posedness of the neutral equation $(\mathrm{NE})_f$. To this purpose we reformulate $(\mathrm{NE})_f$ as a non-homogeneous Cauchy problem in $\mathcal{X}$ (see [3], [4] for a similar technique).

Let us introduce some concept of well-posedness of $(\mathrm{NE})_f$ (see [4, Section 2]).

15 Definition. A classical solution of the initial value problem $(\mathrm{NE})_f$ is a function $x: \ [-r,\infty) \to X$ satisfying

(i) x, $x(t) - Kx_t \in W^{1,2}_{loc}([-r,\infty);X)$, $x(t) - Kx_t \in D(A)$,

(ii) $x_0(s) = \varphi(s)$ on $[-r,0]$, and $\varphi(0) - K\varphi = \eta$,

(iii) $\frac{d}{dt}(x(t) - Kx_t) = A(x(t) - Kx_t) + Px_t + f(t)$ a.e. on $[0,\infty)$.

16 Definition. The system $(\mathrm{NE})_f$ is well-posed in the strong sense if, given $\varphi \in W^{1,2}$ and η defined by $\varphi(0) - K\varphi = \eta$, there exists a unique classical solution of the homogeneous problem $(\mathrm{NE})_0$, and the solution depends continuously on the initial data φ, with respect to the respective topologies of $W^{1,2}$ and $W^{1,2}_{loc}$.

Next, we introduce the following definition of generalized solutions (see also [4] for a slightly modified definition).

17 Definition. A generalized solution of the initial value problem $(\mathrm{NE})_f$ is a pair of functions $z(\cdot): \ [0,\infty) \to X$, $x(\cdot): \ [-r,\infty) \to X$ such that

(i) $z(\cdot)$ is continuous and $x_t \in D(\tilde{K}) \cap D(\tilde{P})$ for a.e. $t \geq 0$,

(ii) the function $[0,\infty) \ni t \mapsto \mathbb{P}x(t) + \tilde{P}x_t \in X$ is measurable,

(iii) $x_0 = \varphi$ a.e. on $[-r,0]$, $z(0) = \eta \in X$,

(iv) The pair $(z(\cdot), x(\cdot))$ satisfies

$$z(t) = T(t)\eta + \int_0^t T(t-\tau)[\mathbb{P}x(\tau) + \tilde{P}x_\tau + f(\tau)]\,d\tau, \quad \text{a.e. } t \geq 0, \tag{45}$$

and

$$z(t) = x(t) - \tilde{K}x_t, \qquad \text{a.e.} \quad \text{on } [0,\infty). \tag{46}$$

The weak well-posedness is given by the following definition (see [4, Definition 2.4]).

18 Definition. The system $(\mathrm{NE})_f$ is well-posed in the weak sense, if for any given initial data $\binom{\eta}{\varphi} \in \mathcal{X}$, there exists a unique generalized solution which depends continuously on $\binom{\eta}{\varphi}$.

In this section we suppose the following assumptions:

(H1) $K \in \mathrm{Reg}(X)$ with null mass operator and I_X is an admissible feedback operator for Σ_K, the regular system associated with K.

(H2) $P \in \mathrm{Reg}(X)$.

19 Theorem. *Let (H1) and (H2) be satisfied. Then the operator* $(\mathfrak{A}, D(\mathfrak{A}))$ *defined by* (6) *generates a* C_0*-semigroup* $\mathfrak{T} := (\mathfrak{T}(t))_{t\geq 0}$ *on* $\mathcal{X}$ *satisfying:*

(i) For almost every $\sigma \geq 0$,

$$\mathrm{rg}[\mathfrak{T}(\sigma)] \subset X \times [D(\tilde{P}) \cap D(\tilde{K})] \subset D(\tilde{\mathcal{C}}). \tag{47}$$

(ii) For $t \geq 0$ *and* $\binom{\eta}{\varphi} \in \mathcal{X}$,

$$\mathfrak{T}(t)\binom{\eta}{\varphi} = \mathbf{T}(t)\binom{\eta}{\varphi} + \int_0^t \mathbf{T}_{-1}(t-\sigma)\mathcal{B}\tilde{\mathcal{C}}\mathfrak{T}(\sigma)\binom{\eta}{\varphi}\, d\sigma. \tag{48}$$

Proof. Remark 14 and Theorem 4 show that the operator defined by

$$\mathcal{J} := \mathcal{A}_{-1} + \mathcal{B}\tilde{\mathcal{C}},\ D(\mathcal{J}) := \left\{\binom{\eta}{\varphi} \in D(\tilde{\mathcal{C}}) : \mathcal{J}\binom{\eta}{\varphi} \in \mathcal{X}\right\} \tag{49}$$

(the sum is defined in $\mathcal{X}_{-1,\mathcal{A}}$) generates a C_0–semigroup $\mathfrak{T} := (\mathfrak{T}(t))_{t\geq 0}$ satisfying $\mathfrak{T}(\sigma)\binom{\eta}{\varphi} \in D(\tilde{\mathcal{C}})$ for all $\binom{\eta}{\varphi} \in \mathcal{X}$ and almost every $\sigma \geq 0$. Moreover, (48) holds. That is

$$\mathfrak{T}(t)\binom{\eta}{\varphi} = \mathbf{T}(t)\binom{\eta}{\varphi} + \mathbf{\Phi}(t)\tilde{\mathcal{C}}\mathfrak{T}(\cdot)\binom{\eta}{\varphi}, \quad t \geq 0,\ \binom{\eta}{\varphi} \in \mathcal{X}. \tag{50}$$

Now by (27), (28), Lemma 12 and the results of Section 1 one can see that the semigroup $\mathfrak{T}$ satisfies (i) as well.

Let us now prove that $\mathcal{J} = \mathfrak{A}$ and $D(\mathcal{J}) = D(\mathfrak{A})$. To that purpose we fix $\lambda > 0$ be sufficiently large. By taking the Laplace transform in both hand sides of (50) we get

$$R(\lambda, \mathcal{J})\mathcal{X} = R(\lambda, \mathcal{A})\mathcal{X} + R(\lambda, \mathcal{A}_{-1})\mathcal{B}\tilde{\mathcal{C}}R(\lambda, \mathcal{J})\mathcal{X} \subset X \times [D(\tilde{P}) \cap D(\tilde{K})],$$

due to (29) and Lemma 6. Hence,

$$D(\mathcal{J}) = \left\{\binom{\eta}{\varphi} \in X \times [D(\tilde{P}) \cap D(\tilde{K})] : \mathcal{J}\binom{\eta}{\varphi} \in \mathcal{X}\right\}. \tag{51}$$

It follows from (29), (22), Lemma 10, Proposition 11 and Lemma 6 that

$$(\lambda - \tilde{\mathcal{A}}_P)R(\lambda, \mathcal{J}) = I_{\mathcal{X}} + \mathcal{B}\tilde{\mathcal{C}}R(\lambda, \mathcal{J}),$$

where

$$\tilde{\mathcal{A}}_P = \begin{pmatrix} A_{-1} & \tilde{P} \\ 0 & Q_{-1} \end{pmatrix} \quad \text{with} \quad D(\tilde{\mathcal{A}}_P) := X \times D(\tilde{P}).$$

Thus, $\mathcal{A}_{-1}R(\lambda, \mathcal{J}) = \tilde{\mathcal{A}}_P R(\lambda, \mathcal{J})$ and therefore, by Lemma 12 and Proposition 11, we obtain

$$\mathcal{J} = \begin{pmatrix} A_{-1} + \mathbb{P} & \mathbb{P}\tilde{K} + \tilde{P} \\ B & Q_{-1} + B\tilde{K} \end{pmatrix},$$

where $B = -Q_{-1}e_0$ (see (22)).
For $\binom{\eta}{\varphi} \in D(\mathcal{J})$ we have

$$\mathcal{J}\binom{\eta}{\varphi} = \begin{pmatrix} A_{-1}\eta + \mathbb{P}(\eta + \tilde{K}\varphi) + \tilde{P}\varphi \\ Q_{-1}(\varphi - e_0(\eta + \tilde{K}\varphi)) \end{pmatrix} \in \mathcal{X}.$$

This implies that $\eta \in D(A)$ and $\varphi - e_0(\eta + \tilde{K}\varphi) \in D(Q)$, so that $\varphi \in W^{1,2}$ and $\varphi(0) = \eta + K\varphi$, due to (23) and since $\mathbb{K} = 0$. Again by (23) we have $\mathbb{P}\varphi(0) + \tilde{P}\varphi = P\varphi$. Therefore, $\binom{\eta}{\varphi} \in D(\mathfrak{A})$ and $\mathcal{J}\binom{\eta}{\varphi} = \mathfrak{A}\binom{\eta}{\varphi}$. The converse follows immediately by the fact that $W^{1,2} \subset [D(\tilde{P}) \cap D(\tilde{K})]$ (see Lemma 6) and (51), as claimed. This shows that $\mathfrak{A}$ is the generator of $\mathfrak{T}$. QED

20 Proposition. *Let (H1) and (H2) be satisfied. Then the following assertions hold.*

(i) If $\binom{\eta}{\varphi} \in \mathcal{X}$ *then*
$$\mathfrak{T}(t)\binom{\eta}{\varphi} = \binom{x(t) - \tilde{K}x_t}{x_t}, \quad t \geq 0.$$

(ii) If $\binom{\eta}{\varphi} \in D(\mathfrak{A})$*, then* $(\mathrm{NE})_f$ *is well-posed in the strong sense. In this case we have* $\tilde{K}x_t = Kx_t$ *for* $t \geq 0$.

Proof. Let, by Remark 14, $\mathbf{\Sigma}$ be the regular system generated by the triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ with input function x, state trajectory w and output function y. Due to (27) and (28) we have $w(t) = (z(t), x_t)^\top$ for $t \geq 0$, where the function $z : [0, \infty) \to X$ satisfies (4) (take $f = 0$ in Theorem 13). On the other hand, we denote by $\mathbf{\Sigma}^I$ the closed loop system associated with $\mathbf{\Sigma}$ and the admissible feedback operator I_X. As we have seen in the proof of Theorem 19, $\mathfrak{T}$ is the C_0–semigroup of $\mathbf{\Sigma}^I$. Thus it corresponds to the feedback law $x(t) = y(t) = \tilde{\mathcal{C}}w(t)$ for a.e. $t \geq 0$, so that $\mathfrak{T}(t)\binom{\eta}{\varphi} = (z(t), x_t)^\top$ for all $\binom{\eta}{\varphi} \in \mathcal{X}$ and $t \geq 0$. Now due to (35) we have $x(t) = \tilde{\mathcal{C}}w(t) = z(t) + \tilde{K}x_t$ for a.e. $t \geq 0$. Thus, Assertion (i) follows.
We show (ii). For $\binom{\eta}{\varphi} \in D(\mathfrak{A})$ we have $\mathfrak{T}(t)\binom{\eta}{\varphi} \in D(\mathfrak{A})$ for all $t \geq 0$. This implies that $x_t \in W^{1,2}$ for $t \geq 0$ and by (23) $\tilde{K}x_t = Kx_t$. So thus $x(t) - Kx_t \in D(A)$ and finally one can see that x satisfies the condition of Definition 15 (with $f \equiv 0$). QED

Let us now deal with the connection between the non-homogeneous neutral equation $(\mathrm{NE})_f$ and the Cauchy problem (nCP) (see Section iv).

Under the assumptions of Theorem 19, (nCP) has a unique mild solution given by

$$w(t) = \mathfrak{T}(t)\binom{\eta}{\varphi} + \int_0^t \mathfrak{T}(t-s)\binom{f(s)}{0}\, ds, \quad t \geq 0. \tag{52}$$

21 Proposition. *Let (H1) and (H2) be satisfied. Then* $(\mathrm{NE})_f$ *is well-posed in the weak sense. In particular, the function* w *defined by* (52) *satisfies* $w(t) = (z(t), x_t)^\top$, $t \geq 0$, *where* (z, x) *is the generalized solution of* $(\mathrm{NE})_f$.

PROOF. Let, by Theorem 13, $\mathbf{\Sigma}'$ be the regular system generated by the triple $(\mathcal{A}, \mathfrak{B}, \mathfrak{C})$. Further, we denote by $\mathbf{\Sigma}'^I$ the closed loop system associated with $\mathbf{\Sigma}'$ and the admissible feedback operator $I_{\mathcal{X}}$. Since $\mathbf{\Sigma}'$ and its closed loop have the same state trajectory then the function $\mathbf{w}$ given by (39) is the state trajectory of $\mathbf{\Sigma}'^I$, where $\mathbf{u} = (x, f)^\top$ is the control function of $\mathbf{\Sigma}'$. Note, by Theorem 4, that the semigroup generator of $\mathbf{\Sigma}'^I$ is the operator defined by

$$\mathcal{A}_I := \mathcal{A}_{-1} + \mathfrak{B}\tilde{\mathfrak{C}} \quad \text{with} \quad D(\mathcal{A}_I) := \left\{ \left(\begin{smallmatrix} \eta \\ \varphi \end{smallmatrix}\right) \in D(\tilde{\mathfrak{C}}) : \mathcal{A}_I \left(\begin{smallmatrix} \eta \\ \varphi \end{smallmatrix}\right) \in \mathcal{X} \right\}.$$

Now by (36), (38) we have $\mathfrak{B}\tilde{\mathfrak{C}} = \mathcal{B}\tilde{\mathcal{C}}$ then by the proof of Theorem 19 we deduce that $\mathcal{A}_I = \mathfrak{A}$. We now consider the feedback law $\mathbf{u} = \mathbf{y} + \mathbf{u}_c$, where $\mathbf{y}$ is the output function of $\mathbf{\Sigma}'$ (see (40)) and $\mathbf{u}_c = (0, f)$, so that $\mathbf{u}_c$ is the control function of $\mathbf{\Sigma}'^I$. Since $\mathfrak{B}$ is the control operator of $\mathbf{\Sigma}'^I$ then its state trajectory satisfies

$$\begin{aligned} \mathbf{w}(t) = \mathfrak{T}(t)\left(\begin{smallmatrix} \eta \\ \varphi \end{smallmatrix}\right) + \int_0^t \mathfrak{T}_{-1}(t-\tau)\mathfrak{B}u_c(\tau)\,d\tau \\ = \mathfrak{T}(t)\left(\begin{smallmatrix} \eta \\ \varphi \end{smallmatrix}\right) + \int_0^t \mathfrak{T}(t-\tau)\left(\begin{smallmatrix} f(\tau) \\ 0 \end{smallmatrix}\right)d\tau = w(t) \end{aligned}$$

for $t \geq 0$, where w is the mild solution of (nCP). Thus $w(t) = (z(t), x_t)^\top$ for $t \geq 0$, due to (39), where z satisfies (45). On the other hand, by (40) we have $(x(t), f(t))^\top = \mathbf{u}(t) = \mathbf{y}(t) + \mathbf{u}_c(t) = (z(t) + \tilde{K}x_t, f(t))^\top$, for a.e. $t \geq 0$. Hence $x(t) = z(t) + \tilde{K}x_t$ for a.e. $t \geq 0$, so that (46) is satisfied. This implies that (z, x) is the generalized solution of $(\mathrm{NE})_f$, as claimed. Finally, by (52), this solution depends continuously on $\left(\begin{smallmatrix} \eta \\ \varphi \end{smallmatrix}\right)$. QED

For the next section we need the following definition.

22 Definition. The semigroup $\mathfrak{T}$ generated by $\mathfrak{A}$ will be called the *neutral semigroup*.

5 Spectral theory for the neutral semigroup

In this section we shall work with the assumptions (H1) and (H2) (see Section 4), so the operator $\mathfrak{A}$ defined by (6) generates the C_0–semigroup $\mathfrak{T}$ (see Theorem 19), which solves the neutral equation $(\mathrm{NE})_0$. This motivated us to study the spectrum $\sigma(\mathfrak{A})$ of $\mathfrak{A}$. Since the domain $D(\mathfrak{A})$ is not diagonal then a direct computation of $\sigma(\mathfrak{A})$ will be a difficult task. To overcome this difficulty we shall use the theory of one-side coupled operator matrices, recently developed by Engel [7].

In this section we need the following operator

$$Q_K f := Q_m f \quad \text{for} \quad f \in D(Q_K) := \{f \in W^{1,2} : \ f(0) = Kf\}.$$

23 Lemma. *Let (H1) be satisfied. Then Q_K is the generator of a $\mathcal{C}_0$–semigroup $(S_K(t))_{t\geq 0}$ on L_2 satisfying $S_K(\sigma)\varphi \in D(\tilde{K})$ for $\varphi \in L_2$ and almost every $\sigma \geq 0$, and*

$$[S_K(t)\varphi](s) = \begin{cases} \varphi(s+t), & t+s \leq 0, \\ \tilde{K}[S_K(s+t)\varphi], & t+s \geq 0. \end{cases} \tag{53}$$

Moreover, we have

$$\lambda \in \rho(Q_K) \quad \Longleftrightarrow \quad 1 \in \rho(K e_\lambda). \tag{54}$$

In this case,

$$R(\lambda, Q_K) = (I - e_\lambda K)^{-1} R(\lambda, Q) \tag{55}$$

$$= R(\lambda, Q) + e_\lambda (I - K e_\lambda)^{-1} K R(\lambda, Q). \tag{56}$$

Proof. The condition (H1), Theorem 4 and (22) imply that the operator defined by

$$\tilde{Q}_K := Q_{-1} + B\tilde{K} = Q_{-1}(I - e_0 \tilde{K}) \ \text{with} \ D(\tilde{Q}_K) := \{\psi \in D(\tilde{K}) : \ \tilde{Q}_K \psi \in L_2\}$$

generates a $\mathcal{C}_0$–semigroup $S_K(\cdot)$ on L_2 satisfying $S_K(t)\psi \in D(\tilde{K})$ for all $\psi \in L_2$ and a.e. $t \geq 0$. By using Lemma 6 and a similar argument as in the proof of Theorem 19 one can easily prove that $Q_K = \tilde{Q}_K$. By (20), the semigroup S_K satisfies

$$S_K(t)\varphi = S(t)\varphi + \Phi(t)\tilde{K} S_K(\cdot)\varphi$$

for $t \geq 0$ and $\psi \in L_2$, where Φ is given by (21), so that the translation property (53) follows.

Due to (22) and (23), we obtain

$$(\lambda - Q_K)\psi = (\lambda - Q_{-1})\psi - B\tilde{K}\psi = (\lambda - Q_{-1})(I - e_\lambda K)\psi$$

for $\psi \in D(Q_K)$ and $\lambda \in \mathbb{C}$. This shows the equivalence (54). Applying (55) we get the remainder part of the lemma. QED

The following result is a consequence of [9, Lemma 1.2].

24 Lemma. *Let (H1) be satisfied and let $\lambda \in \rho(Q_K)$. Then the restriction $(\delta_0 - K)|_{\ker(\lambda - Q_m)}$ is invertible and its inverse*

$$\mathcal{D}_\lambda := [(\delta_0 - K)|_{\ker(\lambda - Q_m)}]^{-1} : X \to \ker(\lambda - Q_m) \subseteq L_2,$$

called the associated Dirichlet operator, is bounded. Moreover,

$$\mathcal{D}_\lambda = (I - (\lambda - \mu)\ R(\lambda, Q_K))\mathcal{D}_\mu \quad \text{for} \quad \mu \in \rho(Q_K).$$

The following lemma gives an explicit expression for the Dirichlet operator $\mathcal{D}_\lambda$.

25 Lemma. *For $\lambda \in \rho(Q_K)$ we have*

$$\mathcal{D}_\lambda = e_\lambda (I - Ke_\lambda)^{-1}.$$

Moreover,

$$R(\lambda, Q_K) = R(\lambda, Q) + \mathcal{D}_\lambda K R(\lambda, Q).$$

Proof. Let $\lambda \in \rho(Q_K)$. Since, by Lemma 24, $\mathcal{D}_\lambda$ is invertible, then for all $v \in X$ there exists a unique $f \in \ker(\lambda - Q_m)$ such that $(\delta_0 - K)f = v$. Observe that $\ker(\lambda - Q_m) = \{e_\lambda g(0) : g \in W^{1,2}\}$. Then $v = f(0) - Ke_\lambda f(0) = (I - Ke_\lambda)f(0)$. Hence, by (54), we have $f(0) = (I - Ke_\lambda)^{-1}v$. Thus, $\mathcal{D}_\lambda = e_\lambda(I - Ke_\lambda)^{-1}$. QED

We are now able to use the theory of one-sided coupled operator matrices (cf. [7], [18]). In fact, it is known from [18, Lemma 8.8] that, for $\lambda \in \rho(Q_K)$, the operator $\lambda - \mathfrak{A}$ can be factorized as

$$\lambda - \mathfrak{A} = \begin{pmatrix} \lambda - A - P\mathcal{D}_\lambda & -P \\ 0 & \lambda - Q_K \end{pmatrix} \mathfrak{D}_\lambda, \tag{57}$$

where the operator $\mathfrak{D}_\lambda$ is invertible and defined by

$$\mathfrak{D}_\lambda = \begin{pmatrix} Id_X & 0 \\ -\mathcal{D}_\lambda & I_{L_2} \end{pmatrix},$$
$$D(\mathfrak{D}_\lambda) := \left\{ \left(\begin{smallmatrix} x \\ \psi \end{smallmatrix}\right) \in D(A) \times L_2 : \ -\mathcal{D}_\lambda x + \psi \in D(Q_K) \right\}.$$

We mention here that $D(\mathfrak{D}_\lambda) = D(\mathfrak{A})$ (see [18, Lemma 8.3]).

The following proposition, which characterizes the spectrum of the generator of the neutral semigroup $\mathfrak{T}$, is a consequence of Lemma 25, (57) and [18, Theorem 5.1].

26 Proposition. *For every* $\lambda \in \rho(Q_K)$ *we have*

$$\lambda \in \rho(\mathfrak{A}) \Longleftrightarrow \lambda \in \rho(A + P\mathcal{D}_\lambda).$$

In this case we have

$$R(\lambda, \mathfrak{A}) = \begin{pmatrix} \mathcal{N}(\lambda) & \mathcal{N}(\lambda)PR(\lambda, Q_K) \\ \mathcal{D}_\lambda\mathcal{N}(\lambda) & [\mathcal{D}_\lambda\mathcal{N}(\lambda)P + Id]R(\lambda, Q_K) \end{pmatrix},$$

where $\mathcal{N}(\lambda) := R(\lambda, A + P\mathcal{D}_\lambda)$.

27 Remark. (a) If we assume that K is identically null then $Q_K = Q$ and the results obtained in Proposition 26 coincide with those in [1] for retarded functional differential equation with L^2–phase spaces.
(b) Suppose that $X = \mathbb{R}^n$. Then for $\lambda \in \rho(Q_K)$ one has

$$\lambda \in \sigma(\mathfrak{A}) \Leftrightarrow \lambda \in P\sigma(A + P\mathcal{D}_\lambda) \Leftrightarrow \det(\lambda - A - P\mathcal{D}_\lambda) = 0.$$

This equation is called the *characteristic equation* for the spectrum of $\mathfrak{A}$ (see e.g. [22] for more details on such equations).

28 Example. Let $b, h \in (0, r]$ and $a \in (-1, 1)$. We shall consider the neutral equation in $\mathbb{R}$,

$$\frac{d}{dt}(x(t) - ax(t-h)) = x(t-b), \quad t \geq 0.$$

According to our abstract framework, we have $K = a\delta_{-h}$, $P = \delta_{-b}$ and $A \equiv 0$. Then, using Remark 9, one can see that these operators satisfy the assumptions (H1) and (H2). Then, on $\mathbb{R} \times L^2([-r, 0])$, the neutral semigroup $\mathfrak{T}$ exists. Let denotes by $\mathfrak{A}$ its generator. By an easy computation one can see that $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq 0\} \subset \rho(Q_K)$. To compute $\sigma(\mathfrak{A})$ we use Remark 27 (b). So, for $\lambda \in \mathbb{C}_+$, we have

$$\lambda \in \sigma(\mathfrak{A}) \Leftrightarrow \lambda - PD_\lambda = 0 \Leftrightarrow \Theta(\lambda) := \lambda - \lambda a e^{-\lambda h} - e^{-\lambda b} = 0$$

Hence, $0 \in \rho(\mathfrak{A})$. Therefore, if Θ has no zero on $\mathbb{C}_+$, then $\sigma(\mathfrak{A}) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\}$. This implies that $s(\mathfrak{A}) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(\mathfrak{A})\} \leq 0$. If $a \in [0, 1)$, then one can prove that $\mathfrak{T}$ is a positive semigroup on $\mathbb{R}^n \times L^2([-r, 0], \mathbb{R}^n)$ (see Theorem 29 below). In this case we obtain $s(\mathfrak{A}) < 0$ and hence the solution of the neutral equation is exponentially stable.

The above example motivate us to study the positivity of the neutral semigroup $\mathfrak{T}$.

In what follow we shall work in *Hilbert lattices* and with positive operators. The reader is refereed to [21] for a detailed discussion on positive semigroups and Banach lattices.

29 Theorem. *Let X be a Hilbert lattice and suppose that (H1) and (H2) are satisfied. If the initial semigroup T, the delay operators K and P are positive, and the spectral radius $r(Ke_{\lambda_0}) < 1$ for some large λ_0, then the neutral semigroup $\mathfrak{T}$ is positive.*

Proof. By Lemma 25 and Proposition 26 one has only to show the positivity of $\mathcal{N}(\lambda)$ for all large λ. Since, by (H2) and Theorem 7, $Pe_\lambda = H(\lambda) + \mathbb{P}$, it follows from [25, Lemma 4.6.2] that there is $\gamma_0 > 0$ and $\kappa_0 > 0$ such that $\|Pe_\lambda\| \leq \kappa_0$ for all $\lambda > \gamma_0$. On the other hand, since K is positive, we deduce that

$$Ke_\lambda \leq Ke_{\lambda_0}, \quad \forall \lambda \geq \lambda_0.$$

Hence, $r(Ke_\lambda) \leq r(Ke_{\lambda_0}) < 1$. Thus, $(I - Ke_\lambda)^{-1} = \sum_{n=0}^{\infty}(Ke_\lambda)^n$ is positive and $(I-Ke_\lambda)^{-1} \leq (I-Ke_{\lambda_0})^{-1}$ for all $\lambda \geq \lambda_0$. This implies that $\|(I-Ke_\lambda)^{-1}\| \leq \|(I-Ke_{\lambda_0})^{-1}\|$ for all $\lambda > \lambda_0$. We now take $\gamma := \max\{\gamma_0, \gamma_1, \omega_0(A)+2M\kappa_0\|(I-Ke_{\lambda_0})^{-1}\|\}$. Then

$$\|P\mathcal{D}_\lambda R(\lambda, A)\| \leq \frac{1}{2}, \quad \forall \lambda > \gamma.$$

Thus, since $P\mathcal{D}_\lambda$ is positive it follows that

$$\mathcal{N}(\lambda) = R(\lambda, A)\sum_{n=0}^{\infty}(P\mathcal{D}_\lambda R(\lambda, A))^n$$

is positive for all large λ. QED

30 Remark. Analyzing in profile what will happen if one replaces throughout this paper the Hilbert space X by a general Banach space and the exponent 2 by an arbitrary real number $p \in (1, \infty)$. Certainly, Theorem 4 is the main key for the proof of many results in this paper. Fortunately, this theorem is still holds in the Banach setting (see [25, Chapter 7]). Further, all results of Section 1 are naturally translated to Banach spaces (see [25], [29, Remark 5.9]). May be a serious difference is in the level of the definition of admissible feedback operators. For Hilbert spaces we have used transfer functions. However, in the Banach setting one has to use input-output operators rather than transfer functions. Thus, we can say that all results of this paper are verified in the Banach setting.

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