Several comments about the combinatorics of \( \tau \)-covers

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Abstract. In a previous work with Mildenberger and Shelah, we showed that the combinatorics of the selection hypotheses involving \( \tau \)-covers is sensitive to the selection operator used. We introduce a natural generalization of Scheepers' selection operators, and show that:

(1) A slight change in the selection operator, which in classical cases makes no difference, leads to different properties when \( \tau \)-covers are involved.

(2) One of the newly introduced properties sheds some light on a problem of Scheepers concerning \( \tau \)-covers.

Improving an earlier result, we also show that no generalized Luzin set satisfies \( U_{\text{fin}}(\Gamma, T) \).

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1 Introduction

Topological properties defined by diagonalizations of open or Borel covers have a rich history in various areas of general topology and analysis, and they are closely related to infinite combinatorial notions, see \([8, 12, 5, 13]\) for surveys on the topic and some of its applications and open problems.

Let \( X \) be an infinite set. By a cover of \( X \) we mean a family \( U \) with \( X \not\in U \) and \( X = \cup U \). A cover \( U \) of \( X \) is said to be

(1) a large cover of \( X \) if: \( (\forall x \in X) \{ U \in U : x \in U \} \) is infinite.

(2) an \( \omega \)-cover of \( X \) if: \( (\forall \text{ finite } F \subseteq X) (\exists U \in U) F \subseteq U. \)

(3) a \( \tau \)-cover of \( X \) if: \( U \) is a large cover of \( X \), and \( (\forall x, y \in X) \{ U \in U : x \in U \text{ and } y \not\in U \} \) is finite, or \( \{ U \in U : y \in U \text{ and } x \not\in U \} \) is finite.

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(4) a $\gamma$-cover of $X$ if: $\mathcal{U}$ is infinite and $(\forall x \in X) \{ U \in \mathcal{U} : x \not\in U \}$ is finite.

Let $X$ be an infinite, zero-dimensional, separable metrizable topological space (in other words, a set of reals). Let $\Omega$, $T$ and $\Gamma$ denote the collections of all open $\omega$-covers, $\tau$-covers and $\gamma$-covers of $X$, respectively. Additionally, denote the collection of all open covers of $X$ by $\mathcal{O}$. Similarly, let $C_\Omega$, $C_T$, $C_\Gamma$, and $C$ denote the corresponding collections of clopen covers. Our restrictions on $X$ imply that each member of any of the above classes contains a countable member of the same class [11]. We therefore confine attention in the sequel to countable covers, and restrict the above four classes to contain only their countable members. Having this in mind, we let $\mathcal{B}_\Omega$, $\mathcal{B}_T$, $\mathcal{B}_\Gamma$, and $\mathcal{B}$ denote the corresponding collections of countable Borel covers.

Let $\mathcal{A}$ and $\mathcal{B}$ be any of the mentioned classes of covers (but of the same descriptive type, i.e., both open, or both clopen, or both Borel). Scheepers [7] introduced the following selection hypotheses that $X$ might satisfy:

- $S_1(\mathcal{A}, \mathcal{B})$: For each sequence $\langle U_n : n \in \mathbb{N} \rangle$ of members of $\mathcal{A}$, there exist members $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{ U_n : n \in \mathbb{N} \} \in \mathcal{B}$.

- $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$: For each sequence $\langle U_n : n \in \mathbb{N} \rangle$ of members of $\mathcal{A}$, there exist finite (possibly empty) subsets $F_n \subseteq U_n$, $n \in \mathbb{N}$, such that $\bigcup_{n \in \mathbb{N}} F_n \in \mathcal{B}$.

- $U_{\text{fin}}(\mathcal{A}, \mathcal{B})$: For each sequence $\langle U_n : n \in \mathbb{N} \rangle$ of members of $\mathcal{A}$ which do not contain a finite subcover, there exist finite (possibly empty) subsets $F_n \subseteq U_n$, $n \in \mathbb{N}$, such that $\{ \bigcup F_n : n \in \mathbb{N} \} \in \mathcal{B}$.

Some of the properties are never satisfied, and many equivalences hold among the meaningful ones. The surviving properties appear in Figure 1, where an arrow denotes implication [10]. It is not known whether any other implication can be added to this diagram – see [6] for a summary of the open problems concerning this diagram.

Below each property $P$ in Figure 1 appears its critical cardinality, $\text{non}(P)$, which is the minimal cardinality of a space $X$ not satisfying that property. The definitions of most of the cardinals appearing in this figure can be found in [2, 1], whereas $o_\delta$ is defined in [6], and the results were established in [4, 10, 9, 6].

A striking observation concerning Figure 1 is, that in the top plane of the figures, the critical cardinality of $\Pi(\Gamma, \mathcal{B})$ for $\Pi \in \{ S_1, S_{\text{fin}}, U_{\text{fin}} \}$ is independent of $\Pi$ in all cases except for that where $\mathcal{B} = T$. We demonstrate this anomaly further in Section 2, where we also give a partial answer to a problem of Scheepers. In Section 3 we show that no Luzin set satisfies $U_{\text{fin}}(\Gamma, T)$, improving a result from [10].
2 Generalized selection hypotheses

1 Definition. Let $\kappa < \lambda$ be any (finite or infinite) cardinal numbers. Denote

- $S_{[\kappa, \lambda]}(\mathcal{A}, \mathcal{B})$: For each sequence $\langle U_n : n \in \mathbb{N} \rangle$ of members of $\mathcal{A}$, there exist subsets $\mathcal{F}_n \subseteq U_n$ with $\kappa \leq |\mathcal{F}_n| < \lambda$ for each $n \in \mathbb{N}$, and $\bigcup_n \mathcal{F}_n \in \mathcal{B}$.

- $U_{[\kappa, \lambda]}(\mathcal{A}, \mathcal{B})$: For each sequence $\langle U_n : n \in \mathbb{N} \rangle$ of members of $\mathcal{A}$ which do not contain subcovers of size less than $\lambda$, there exist subsets $\mathcal{F}_n \subseteq U_n$ with $\kappa \leq |\mathcal{F}_n| < \lambda$ for each $n \in \mathbb{N}$, and $\{ \cup \mathcal{F}_n : n \in \mathbb{N} \} \in \mathcal{B}$.

So that $S_{[1,2]}(\mathcal{A}, \mathcal{B})$ is $S_1(\mathcal{A}, \mathcal{B})$, $S_{[0,\omega_0]}(\mathcal{A}, \mathcal{B})$ is $S_{\infty}(\mathcal{A}, \mathcal{B})$, and $U_{[0,\omega_0]}(\mathcal{A}, \mathcal{B})$ is $U_{\infty}(\mathcal{A}, \mathcal{B})$.

2 Definition. Say that a family $\mathcal{A} \subseteq \mathcal{A}$ of members of $\mathcal{A}$ is semi $\tau$-diagonalizable if there exists a partial function $g : \mathbb{N} \to \mathbb{N}$ such that:

1. For each $A \in \mathcal{A}$: $(\exists \infty n \in \text{dom}(g)) A(n, g(n)) = 1$;

2. For each $A, B \in \mathcal{A}$:
   - Either $(\forall \infty n \in \text{dom}(g)) A(n, g(n)) \leq B(n, g(n))$,
   - or $(\forall \infty n \in \text{dom}(g)) B(n, g(n)) \leq A(n, g(n))$.

   In the following theorem, note that $\min\{ s, b, \omega \} \geq \min\{ s, b, \text{cov}(\mathcal{M}) \} = \min\{ s, \text{add}(\mathcal{M}) \}$.

3 Theorem.
(1) \(X\) satisfies \(S_{[0,2]}(\mathcal{B}_T, \mathcal{B}_T)\) if, and only if, for each Borel function \(\Psi : X \rightarrow \{0,1\}^{\mathbb{N} \times \mathbb{N}}\): If \(\Psi[X]\) is a \(\tau\)-family, then it is semi \(\tau\)-diagonalizable (Definition 2). The corresponding clopen case also holds.

(2) The minimal cardinality of a \(\tau\)-family that is not semi \(\tau\)-diagonalizable is at least \(\min\{a, b, od\}\).

(3) \(\min\{a, b, od\} \leq \text{non}(S_{[0,2]}(\mathcal{B}_T, \mathcal{B}_T)) = \text{non}(S_{[0,2]}(T, T)) = \text{non}(S_{[0,2]}(C_T, C_T))\).

Proof. (1) is proved as usual, (2) is shown in the proof of Theorem 4.1 of [6], and (3) follows from (1) and (2).

4 Definition ([9]). For functions \(f, g, h \in \mathbb{N}^\mathbb{N}\), and binary relations \(R, S\) on \(\mathbb{N}\), define subsets \([f R g]\) and \([f R g S h]\) of \(\mathbb{N}^\mathbb{N}\) by:

\[[f R g] = \{n : f(n) R g(n)\}, \quad [f R g S h] = [f R g] \cap [g S h]\].

For a subset \(Y\) of \(\mathbb{N}^\mathbb{N}\) and \(g \in \mathbb{N}^\mathbb{N}\), we say that \(g\) avoids middles in \(Y\) with respect to \(\langle R, S \rangle\) if:

1. for each \(f \in Y\), the set \([f R g]\) is infinite;
2. for all \(f, h \in Y\) at least one of the sets \([f R g S h]\) and \([h R g S f]\) is finite.

\(Y\) satisfies the \(\langle R, S \rangle\)-excluded middle property if there exists \(g \in \mathbb{N}^\mathbb{N}\) which avoids middles in \(Y\) with respect to \(\langle R, S \rangle\).

In [10] it is proved that \(U_{\text{fin}}(\mathcal{B}_T, \mathcal{B}_T)\) is equivalent to having all Borel images in \(\mathbb{N}^\mathbb{N}\) satisfying the \(\langle \leq, < \rangle\)-excluded middle property (the statement in [10] is different but equivalent).

5 Theorem. For a set of reals \(X\), the following are equivalent:

1. \(X\) satisfies \(U_{[1,\aleph_0]}(\mathcal{B}_T, \mathcal{B}_T)\).
2. Each Borel image of \(X\) in \(\mathbb{N}^\mathbb{N}\) satisfies the \(\langle \leq, < \rangle\)-excluded middle property.

The corresponding assertion for \(U_{[1,\aleph_0]}(C_T, C_T)\) holds when “Borel” is replaced by “continuous”.

Proof. The proof is similar to the one given in [10] for \(U_{\text{fin}}(\mathcal{B}_T, \mathcal{B}_T)\), but is somewhat simpler.

1 \(\Rightarrow\) 2: Assume that \(Y \subseteq \mathbb{N}^\mathbb{N}\) is a Borel image of \(X\). Then \(Y\) satisfies \(U_{[1,\aleph_0]}(\mathcal{B}_T, \mathcal{B}_T)\). For each \(n\), the collection \(\mathcal{U}_n = \{U_m : m \in \mathbb{N}\}\), where \(U_m = \{f \in \mathbb{N}^\mathbb{N} : f(n) \leq m\}\), is a clopen \(\gamma\)-cover of \(\mathbb{N}^\mathbb{N}\). By standard arguments (see (1 \(\Rightarrow\) 2) in the proof of Theorem 2.3 of [6]) we may assume that no \(\mathcal{U}_n\)
contains a finite cover. For all \( n \), the sequence \( \{ U^n_m : m \in \mathbb{N} \} \) is monotonically increasing with respect to \( \subseteq \), therefore—as large subcovers of \( \tau \)-covers are also \( \tau \)-covers—we may use \( S_1(\mathcal{B}_\Gamma, \mathcal{B}_T) \) instead of \( U_{[1,\aleph_0)}(\mathcal{B}_\Gamma, \mathcal{B}_T) \) to get a \( \tau \)-cover \( U = \{ \Psi^{-1}[U^n_m] : n \in \mathbb{N} \} \) for \( X \). Let \( g \in \mathbb{N}^\mathbb{N} \) be such that \( g(n) = m_n \) for all \( n \). Then \( g \) avoids middles in \( Y \) with respect to \( \langle \leq, < \rangle \).

2 \( \Rightarrow \) 1: Assume that \( U_n = \{ U^n_m : m \in \mathbb{N} \} \), \( n \in \mathbb{N} \), are Borel covers of \( X \) which do not contain a finite subcover. Replacing each \( U^n_m \) with the Borel set \( U_{k \leq m} U^n_m \) we may assume that the sets \( U^n_m \) are monotonically increasing with \( m \). Define \( \Psi : X \to \mathbb{N}^\mathbb{N} \) by: \( \Psi(x)(n) = \min \{ m : x \in U^n_m \} \). Then \( \Psi \) is a Borel map, and so \( \Psi[X] \) satisfies the \( \langle \leq, < \rangle \)-excluded middle property. Let \( g \in \mathbb{N}^\mathbb{N} \) be a witness for that. Then \( U = \{ U^n_g(n) : n \in \mathbb{N} \} \) is a \( \tau \)-cover of \( X \).

The proof in the clopen case is similar.

**6 Corollary.** The critical cardinalities of \( U_{[1,\aleph_0)}(\mathcal{B}_\Gamma, \mathcal{B}_T) \), \( U_{[1,\aleph_0)}(\Gamma, T) \), and \( U_{[1,\aleph_0)}(\mathcal{C}_T, \mathcal{C}_T) \) are all equal to \( b \).

**Proof.** This follows from Theorem 5 and the corresponding combinatorial assertion, which was proved in [9].

Recall from Figure 1 that the critical cardinality of \( U_{\text{fin}}(\Gamma, T) = U_{[0,\aleph_0)}(\Gamma, T) \) is \( \max \{ s, b \} \). Contrast this with Corollary 6.

According to Scheepers [12, Problem 9.5], one of the more interesting problems concerning Figure 1 is whether \( S_1(\Omega, T) \) implies \( U_{\text{fin}}(\Gamma, \Gamma) \). If \( U_{[1,\aleph_0)}(\Gamma, T) \) is preserved under taking finite unions, then we get a positive solution to Scheepers’ Problem. (Note that \( S_1(\Omega, T) \) implies \( S_1(\Gamma, T) \).)

**7 Corollary.** If \( U_{[1,\aleph_0)}(\Gamma, T) \) is preserved under taking finite unions, then it is equivalent to \( U_{\text{fin}}(\Gamma, \Gamma) \) and \( S_1(\Gamma, T) \) implies \( U_{\text{fin}}(\Gamma, \Gamma) \).

**Proof.** The last assertion of the theorem follows from the first since \( S_1(\Gamma, T) \) implies \( U_{[1,\aleph_0)}(\Gamma, T) \).

Assume that \( X \) does not satisfy \( U_{\text{fin}}(\Gamma, \Gamma) \). Then, by Hurewicz’ Theorem [3], there exists an unbounded continuous image \( Y \) of \( X \) in \( \mathbb{N}^\mathbb{N} \). For each \( f \in Y \), define \( f_0, f_1 \in \mathbb{N}^\mathbb{N} \) by \( f_i(2n + i) = f(n) \) and \( f_i(2n + (1 - i)) = 0 \). For each \( i \in \{ 0, 1 \} \), \( Y_i = \{ f_i : f \in Y \} \) is a continuous image of \( Y \). It is not difficult to see that \( Y_0 \cup Y_1 \) does not satisfy the \( \langle \leq, < \rangle \)-excluded middle property [9]. By Theorem 5, \( Y_0 \cup Y_1 \) does not satisfy \( U_{[1,\aleph_0)}(\Gamma, T) \), thus, by the theorem’s hypothesis, one of the sets \( Y_i \) does not satisfy that property. Therefore \( Y \) (and therefore \( X \)) does not satisfy \( U_{[1,\aleph_0)}(\Gamma, T) \) either.

We do not know whether \( U_{[1,\aleph_0)}(\Gamma, T) \) is preserved under taking finite unions. We also do not know the situation for \( U_{\text{fin}}(\Gamma, T) \). The following theorem is only interesting when \( s < b \).
8 Theorem. If there exists a set of reals \( X \) satisfying \( \mathcal{U}_{\text{fin}}(\Gamma, \Gamma) \) but not \( \mathcal{U}_{\text{fin}}(\Gamma, T) \), then \( \mathcal{U}_{\text{fin}}(\Gamma, T) \) is not preserved under taking unions of \( s \) many elements.

Proof. The proof is similar to the last one, except that here we define \( s \) many continuous images of \( Y \) as we did in [9] to prove that the critical cardinality of \( \mathcal{U}_{\text{fin}}(\Gamma, T) \) is \( \max\{ s, b \} \).

3 Luzin sets

A set of reals \( L \) is a generalized Luzin set if for each meager set \( M \), \(|L \cap M| < |L|\). In [10] we constructed (assuming a portion of the Continuum Hypothesis) a generalized Luzin set which satisfies \( S_1(\mathcal{B}_0, \mathcal{B}_1) \) but not \( \mathcal{U}_{\text{fin}}(\Gamma, T) \). We now show that the last assertion always holds.

9 Theorem. Assume that \( L \subseteq \mathbb{N} \) is a generalized Luzin set. Then \( L \) does not satisfy the \( (<, \leq) \)-excluded middle property. In particular, \( L \) does not satisfy \( \mathcal{U}_{\text{fin}}(C_T, CT) \).

Proof. We use the following easy observation.

10 Lemma ([10]). Assume that \( A \) is an infinite set of natural numbers, and \( f \in \mathbb{N} \). Then the sets

\[
M_{f,A} = \{ g \in \mathbb{N} : [g \leq f] \cap A \text{ is finite} \}
\]

\[
\tilde{M}_{f,A} = \{ g \in \mathbb{N} : [f < g] \cap A \text{ is finite} \}
\]

are meager subsets of \( \mathbb{N} \).

Fix any \( f \in \mathbb{N} \). We will show that \( f \) does not avoid middles in \( Y \) with respect to \( (<, \leq) \). The sets \( M_{f,N} = \{ g \in \mathbb{N} : [g \leq f] \text{ is finite} \} \) and \( \tilde{M}_{f,N} = \{ g \in \mathbb{N} : [f < g] \text{ is finite} \} \) are meager, thus there exists \( g_0 \in L \setminus (M_{f,N} \cup \tilde{M}_{f,N}) \). Now consider the meager sets \( M_{f,[f < g_0]} = \{ g \in \mathbb{N} : [g \leq f < g_0] \text{ is finite} \} \) and \( \tilde{M}_{f,[g_0 \leq f]} = \{ g \in \mathbb{N} : [g_0 \leq f < g] \text{ is finite} \} \), and choose \( g_1 \in L \setminus (M_{f,[f < g_0]} \cup \tilde{M}_{f,[g_0 \leq f]}) \). Then both sets \([g_0 < f \leq g_1]\) and \([g_1 < f \leq g_0]\) are infinite.

References


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