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On dense subspaces of $C_p(X)$

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Abstract. For a Tychonoff space X, we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence. We show the following: (1) if ω_1 is a caliber for every dense subspace of $C_p(X)$, then $C_p(X)$ is (ω_1, ω_1)-narrow; (2) if every dense subspace of $C_p(X)$ is compact-dense in $C_p(X)$, then every non-trivial countable ω-cover of open sets of X contains a γ-cover. The first result gives the positive answer to Problem 4.4 in [6], and the second one is a partial answer to Problem 4.3 in [6].

Keywords: function space, dense subspace, caliber, (κ, κ)-compact, (κ, κ)-narrow, linearly Lindelöf, ω-cover, γ-cover, Fréchet

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1 Introduction

All spaces considered in this paper are Tychonoff topological spaces. Unexplained notions and terminology are the same as in [3]. Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ will be used to denote the positive integers and the real line, respectively. For a set X, we denote by $[X]^{<\omega}$ the set of finite subsets of X. For a space X, we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence. The basic open sets of $C_p(X)$ are of the form

$$[x_0, \dots, x_k; O_0, \dots, O_k] = \{ f \in C_p(X) : f(x_i) \in O_i, i = 0, \dots, k \},$$

where $x_i \in X$ and each O_i is a non-empty open subset of $\mathbb{R}$. The symbol $\mathbf{0}$ is the constant function with the value 0.

Let us recall some definitions.

1 Definition ([3])**.** An infinite regular cardinal κ is called a *caliber* of a space X if for an arbitrary family $\mathcal{U}$ of non-empty open sets of X with $|\mathcal{U}| = \kappa$, there is a subfamily $\mathcal{V} \subset \mathcal{U}$ satisfying $|\mathcal{V}| = \kappa$ and $\bigcap\{ V : V \in \mathcal{V} \} \neq \emptyset$.

2 Definition ([1])**.** A point x in a space X is called a *complete accumulation point* of a set $E \subset X$ if for every neighborhood U of x we have $|U \cap E| = |E|$. For an infinite regular cardinal κ, a space X is called (κ, κ)-*compact* if every subset $E \subset X$ of cardinality κ has a complete accumulation point in X.

Obviously a space X is (ω,ω)-compact iff it is countably compact. The following is well-known.

3 Lemma ([1]). *Let κ be an infinite regular cardinal. Then the following are equivalent.*

(1) X is (κ,κ)-compact;

(2) every open cover $\mathcal{U}$ of X with $|\mathcal{U}|=\kappa$ has a subcover $\mathcal{V}$ of cardinality less than κ.

4 Definition ([6]). Let $\mathcal{U}$ be a family of open subsets of X. For $x \in X$, let $\mathrm{ord}(\mathcal{U},x) = |\{\, U \in \mathcal{U} : x \in U \,\}|$. For an infinite regular cardinal κ, a space X is called (κ,κ)-*narrow* if for every open family $\mathcal{U}$ of X, the set $\{\, x \in X : \mathrm{ord}(\mathcal{U},x) \geq \kappa \,\}$ is open in X.

Okunev and Tkachuk noted in [6, Remark 3.13] that: if ω_1 is a caliber of X and X is (ω_1,ω_1)-narrow, then every dense subspace of X has caliber ω_1, see Theorem 7. And they asked in [6, Problem 4.4] whether the converse is true among function spaces.

5 Problem ([6]). Suppose that ω_1 is a caliber for every dense subspace of $C_p(X)$. Is it true that $C_p(X)$ is (ω_1,ω_1)-narrow?

They posed one more problem [6, Problem 4.3] on dense subspaces of function spaces.

6 Problem ([6]). Suppose that every dense subspace A of $C_p(X)$ is compact-dense in $C_p(X)$ in the sense that for each $f \in C_p(X)$, there is a compact set $K \subset C_p(X)$ such that $f \in \overline{K \cap A}$. Must $C_p(X)$ be Fréchet?

In the second section we show that Problem 5 is positive, and in the third section we give a partial answer to Problem 6.

2 Calibers of dense subspaces of $C_p(X)$

7 Theorem. *Let κ be an uncountable regular cardinal. For a space X, the following are equivalent.*

(1) κ is a caliber of $C_p(X)$ and $C_p(X)$ is (κ,κ)-narrow;

(2) κ is a caliber for every dense subspace of $C_p(X)$.

Proof. The implication (1) $\to$ (2) is noted in [6, Remark 3.13]. For completeness, we give a proof. Let D be an arbitrary dense subspace of $C_p(X)$ and let $\mathcal{U} = \{\, U_\alpha : \alpha < \kappa \,\}$ be an arbitrary family of non-empty open subsets of D. For each $\alpha < \kappa$, take an open set $\tilde{U}_\alpha$ in $C_p(X)$ with $\tilde{U}_\alpha \cap D = U_\alpha$.

Since κ is a caliber of $C_p(X)$, there is a subset $J \subset \kappa$ satisfying $|J| = \kappa$ and $\bigcap\{ \tilde{U}_\alpha : \alpha \in J \} \neq \emptyset$. Let $\tilde{\mathcal{U}} = \{ \tilde{U}_\alpha : \alpha \in J \}$. Since $C_p(X)$ is (κ, κ)-narrow, the set $\{ f \in C_p(X) : \mathrm{ord}(\tilde{\mathcal{U}}, f) \geq \kappa \}$ is open in $C_p(X)$, and it is non-empty by

$$\emptyset \neq \bigcap\{ \tilde{U}_\alpha : \alpha \in J \} \subset \{ f \in C_p(X) : \mathrm{ord}(\tilde{\mathcal{U}}, f) \geq \kappa \}.$$

Hence there is $g \in D \cap \{ f \in C_p(X) : \mathrm{ord}(\tilde{\mathcal{U}}, f) \geq \kappa \}$. Therefore $g \in \bigcap\{ U_\alpha : \alpha \in J' \}$ for some $J' \subset J$ with $|J'| = \kappa$.

We show the implication (2) $\to$ (1). Okunev and Tkachuk proved in [6, Theorem 3.11] that $C_p(X)$ is (κ, κ)-narrow iff every finite power of X is (κ, κ)-compact. Therefore we prove that every finite power of X is (κ, κ)-compact. Fix an arbitrary $n \in \mathbb{N}$. In view of Lemma 3, we have only to show that every open cover $\mathcal{U}$ of X^n with $|\mathcal{U}| = \kappa$ has a subcover $\mathcal{V}$ of cardinality less than κ. Let $\mathcal{U}$ be an open cover of X^n with cardinality κ. We put $\mathcal{U} = \{ U_\alpha : \alpha < \kappa \}$ and $V_\alpha = \bigcup\{ U_\beta : \beta \leq \alpha \}$ for $\alpha < \kappa$. For each $F \in [X]^{<\omega}$, choose $\alpha_F < \kappa$ with $F^n \subset V_{\alpha_F}$. Then we can take an open set W_F of X such that $F \subset W_F$ and $W_F^n \subset V_{\alpha_F}$. Let

$$D = \{ f \in C_p(X) : \exists F \in [X]^{<\omega} \text{ such that } f(x) = 0 \text{ for } x \in X \setminus W_F \}.$$

The set D is dense in $C_p(X)$. Indeed, let $B = [x_0, \ldots, x_l; O_0, \ldots, O_l]$ be a non-empty basic open set of $C_p(X)$. Since $F = \{ x_0, \ldots, x_l \} \subset W_F$ holds, there is $g \in C_p(X)$ satisfying $g(x) = 1$ for $x \in F$ and $g(x) = 0$ for $x \in X \setminus W_F$. Take an arbitrary $h \in B$. Then the pointwise product $g \cdot h \in B \cap D$.

Suppose that for every $\alpha < \kappa$, there is a point $(z_1^\alpha, \ldots, z_n^\alpha) \in X^n \setminus V_\alpha$. For each $\alpha < \kappa$, let $F_\alpha = \{ z_1^\alpha, \ldots, z_n^\alpha \}$ and let

$$N_\alpha = \{ f \in C_p(X) : f(x) \neq 0 \text{ for all } x \in F_\alpha \}.$$

Since each N_α is a non-empty open set in $C_p(X)$, by our assumption there is a subset $J \subset \kappa$ such that $|J| = \kappa$ and

$$\bigcap\{ D \cap N_\alpha : \alpha \in J \} \neq \emptyset.$$

Let $f \in \bigcap\{ D \cap N_\alpha : \alpha \in J \}$. As $f \in D$, there is $E \in [X]^{<\omega}$ such that $f(x) = 0$ for every $x \in X \setminus W_E$. Since $f \in N_\alpha$ $(\alpha \in J)$ implies $F_\alpha \subset W_E$, $(z_1^\alpha, \ldots, z_n^\alpha) \in F_\alpha^n \subset W_E^n \subset V_{\alpha_E}$. Thus $\{ (z_1^\alpha, \ldots, z_n^\alpha) : \alpha \in J \} \subset V_{\alpha_E}$. This is a contradiction. Therefore $X^n = V_\alpha$ for some $\alpha < \kappa$. This implies that $\{ U_\beta : \beta \leq \alpha \}$ is a subcover of X^n. QED

A space X is called *linearly Lindelöf* [5] if every open cover of X that is linearly ordered by inclusion has a countable subcover of X. It is well-known that a space X is linearly Lindelöf iff X is (κ,κ)-compact for each uncountable regular cardinal κ.

A space X is called a *Šanin space* [8] if every uncountable regular cardinal is a caliber of X.

8 Corollary. *If every dense subspace of $C_p(X)$ is a Šanin space, then each finite power of X is linearly Lindelöf.*

Proof. Apply Theorem 7 and Theorem 3.11 in [6]. QED

9 Example. There is a space Y which is not (ω_1,ω_1)-narrow, but every dense subspace of Y is a Šanin space. Let X be a subspace of $\mathbb{R}^{\omega_1}$ which is homeomorphic to the compact space ω_1+1. Since $\mathbb{R}^{\omega_1}$ is separable [3, Theorem 2.3.15] and X is nowhere dense closed in $\mathbb{R}^{\omega_1}$, there is a countable dense subset D of $\mathbb{R}^{\omega_1}$ such that $D\cap X=\emptyset$. Let Y be the space obtained from the space $D\cup X\subset\mathbb{R}^{\omega_1}$ by isolating every point in D. Since each point of D is isolated in Y, every dense subspace of Y is separable. Note that every separable space is a Šanin space. Hence every dense subspace of Y is a Šanin space. But Y is not (ω_1,ω_1)-narrow. Indeed, consider the decreasing family $\mathcal{U}=\{\,(\alpha,\omega_1]\cup D:\alpha<\omega_1\,\}$ of open sets of Y. The set $\{\,y\in Y:\operatorname{ord}(\mathcal{U},y)\geq\omega_1\,\}=\{\omega_1\}\cup D$ is not open in Y.

3 Compact-dense subspaces in $C_p(X)$

10 Definition. A family $\mathcal{A}$ of subsets of a set X is called an *ω-cover* of X if every finite subset of X is contained in some member of $\mathcal{A}$. An ω-cover $\mathcal{A}$ of X is called *non-trivial* if $A\neq X$ for all $A\in\mathcal{A}$. A family $\mathcal{A}$ of subsets of a set X is called an *γ-cover* of X if it is countably infinite and every point of X is contained in all but finitely many members of $\mathcal{A}$. A space X is said to have *property (γ)* if every non-trivial ω-cover of open sets of X contains a γ-cover.

The terms of an ω-cover and a γ-cover are introduced in [4]. Note that a non-trivial ω-cover contains infinitely many members.

Gerlits and Nagy proved in [4] that $C_p(X)$ is Fréchet iff X satisfies property (γ).

11 Definition. A subset $A\subset X$ is called *sequentially dense* in X if for each $x\in X$, there is a sequence $\{\,x_n:n\in\omega\,\}\subset A$ converging to x.

In [6, Theorem 2.12], the authors proved that: if each dense subspace of $C_p(X)$ is sequentially dense, then $C_p(X)$ is Fréchet. This result is a motivation of Problem 6.

The following is a partial answer to Problem 6.

12 Theorem. *If every dense subspace of $C_p(X)$ is compact-dense in $C_p(X)$, then every non-trivial **countable** ω-cover of open sets of X contains a γ-cover.*

Proof. Let $\mathcal{U} = \{ U_n : n \in \mathbb{N} \}$ be a non-trivial ω-cover of open sets of X. For each $n \in \mathbb{N}$, let

$$A_n = \{ f \in C_p(X) : f(x) \geq n \text{ for } x \in X \setminus U_n \}.$$

Then it is easy to see that each A_n is closed in $C_p(X)$. The set $A = \bigcup \{ A_n : n \in \mathbb{N} \}$ is dense in $C_p(X)$. Indeed, for an arbitrary $f \in C_p(X)$ and a basic open neighborhood $B = [x_0, \dots, x_l; O_0, \dots, O_l]$ of f, choose $n \in \mathbb{N}$ with $\{ x_0, \dots, x_l \} \subset U_n$. Since X is Tychonoff, there is $g \in C_p(X)$ such that $g(x_i) = f(x_i)$ for $0 \leq i \leq l$ and $f(x) = n$ for $x \in X \setminus U_n$. Then $g \in B \cap A_n$.

Since $\mathcal{U}$ is non-trivial, $\mathbf{0} \notin A$. By our assumption, there is a compact set $K \subset C_p(X)$ such that $\mathbf{0} \in \overline{K \cap A}$. Let

$$S = \{ n \in \mathbb{N} : K \cap A_n \neq \emptyset \}.$$

Assume that there is $N \in \mathbb{N}$ such that for all $n \geq N$, $K \cap A_n = \emptyset$. Then $K \cap A \subset \bigcup \{ A_n : n \leq N \}$, and therefore $\mathbf{0} \in \overline{\bigcup \{ A_n : n \leq N \}} = \bigcup \{ A_n : n \leq N \} \subset A$, a contradiction. Hence S is infinite. For each $k \in S$, choose $f_k \in K \cap A_k$ and let $V_k = \{ x \in X : f_k(x) < k \}$. Obviously $V_k \subset U_k$ for $k \in S$. Let x be an arbitrary point in X. Since $\{ f_k : k \in S \}$ is contained in the compact set K, the set $\{ f_k(x) : k \in S \}$ is bounded in $\mathbb{R}$. Take $m \in \mathbb{N}$ such that $f_k(x) < m$ for $k \in S$. Then for each $k > m$ with $k \in S$, $x \in V_k$ holds. Therefore $\{ U_k : k \in S \}$ is a γ-cover of X. QED

A space X is called *countably tight* if $A \subset X$ and $x \in \overline{A}$ imply that there is a countable set $B \subset A$ such that $x \in \overline{B}$. It is known that $C_p(X)$ is countably tight iff every finite power of X is Lindelöf, see [2] and [7]. Also it is known in [4] that every finite power of X is Lindelöf iff every ω-cover of open sets of X contains a countable ω-cover of X.

13 Corollary. *$C_p(X)$ is Fréchet iff $C_p(X)$ is countably tight and every dense subspace of $C_p(X)$ is compact-dense in $C_p(X)$.*

So, Problem 6 is equivalent to: Assume that every dense subspace of $C_p(X)$ is compact-dense in $C_p(X)$. Must $C_p(X)$ be countably tight?

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