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A hodgepodge of sets of reals

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Abstract. We open up a grab bag of miscellaneous results and remarks about sets of reals. Results concern: Kysiak and Laver-null sets, Kočinac and γ_k-sets, Fleissner and square Q-sets, Alikhani-Koopaei and minimal Q-like-sets, Rubin and σ-sets, and Zapletal and the Souslin number. See the survey papers Brown, Cox [1], and Miller [17, 19].

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1 σ-sets are Laver null

A subtree $T \subseteq \omega^{<\omega}$ of the finite sequences of elements of $\omega = \{0, 1, 2, \dots\}$ is called a Laver tree [14] iff there exists $s \in T$ (called the root node of T) with the property that for every $t \in T$ with $s \subseteq t$ there are infinitely many $n \in \omega$ with tn in T. Here tn is the sequence of length exactly one more than t and ending in n. We use $[T]$ to denote the infinite branches of T, i.e.,

$$[T] = \{ x \in \omega^\omega \;:\; \forall n \in \omega \;\; x \restriction n \in T \}.$$

A set $X \subseteq \omega^\omega$ is Laver-null iff for every Laver tree T there exists a Laver subtree $T' \subseteq T$ such that

$$[T'] \cap X = \emptyset$$

This is analogous to the ideal of Marczewski null sets, $(s)_0$. For some background on this topic, see Kysiak and Weiss [12] and Brown [2].

A separable metric space X is a σ-set iff every G_δ in X is also F_σ. It is known to be relatively consistent (Miller [16]) with the usual axioms of set theory that every σ-set is countable.

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At the Second Lecce conference, Kysiak asked if it is consistent to have a σ-set which is not Laver-null. The answer is no.[1]

1 Theorem. *Every σ-set is Laver-null. In fact, the Borel hierarchy of a non-Laver-null set must have ω_1 levels.*

PROOF. Here we use a result of Recław that appears in Miller [19]. Recław proved that if X is a set of reals and there exists a continuous onto map $f : X \to 2^\omega$, then the Borel hierarchy on X has ω_1 levels, in particular, X is not a σ-set.

2 Lemma. *Every set not Laver-null can be continuously mapped onto 2^ω.*

Let $X \subseteq \omega^\omega$ be a set which is not Laver-null. Hence there exists a Laver tree T such that for every Laver subtree $T' \subseteq T$ we have that $[T']$ meets X.

To simplify our notation assume that $T = \omega^{<\omega}$. Define the following continuous function $f : \omega^\omega \to 2^\omega$:

$$f(x)(n) = \begin{cases} 0 & \text{if } x(n) \text{ is even} \\ 1 & \text{if } x(n) \text{ is odd.} \end{cases}$$

The function f is the parity function. Note that f maps X continuously onto 2^ω. This is because for any $y \in 2^\omega$ there is a Laver-tree T such that $f([T]) = \{ y \}$. But since $[T]$ meets X there is some $x \in X$ with $f(x) = y$.

In the more general case T is an arbitrary Laver-tree. In this case note that there is a natural bijection from the splitting nodes of T to $\omega^{<\omega}$. If $g : [T] \to \omega^\omega$ is the continuous function corresponding to this natural map, then $f \circ g$ will map $X \cap [T]$ continuously onto 2^ω. This proves the Lemma and hence the Theorem. $\boxed{QED}$

It follows from Lemma 2 that all S_1-type properties in the Scheepers diagram imply being Laver-null.

2 γ_k-sets

In Kočinac [11] the notion of a γ_k-set is defined. See also Caserta, Di Maio, Kočinac, and Meccariello [3]. A k-cover of topological space X is a family of open subsets with the property that every compact subset of X is subset of an element of the family. X is called γ_k-set iff for every k-cover $\mathcal{U}$ of X there exists a sequence $(U_n \in \mathcal{U} : n \in \omega)$ such that for every compact $C \subseteq X$ we have that $C \subseteq U_n$ for all but finitely many n.

[1] As I was writing this I learned from Jack Brown that M. Kysiak, A. Nowik, and T. Weiss [13], also solved this problem at about the same time. In fact, their solution is a little better as it also solves the analogous problem for Ramsey null sets.

This is a generalization of γ-sets which were first considered by Gerlits-Nagy [8] and studied in many papers.

A theorem of Galvin and Todorčević (see Galvin and Miller [7]) shows that it is consistent that the union of two γ-sets need not be a γ-set. Kočinac asked at the Lecce conference if such a counterexample exists for γ_k-sets. We show that it does.

3 Example. There exist disjoint subsets of the plane X and Y such that both X and Y are γ_k-sets but $X \cup Y$ is not.

Let X be the open disk of radius one, i.e., $X = \{ (x, y) \ : \ x^2 + y^2 < 1 \}$, and Y be any singleton on the boundary of X, e.g., $Y = \{ (1, 0) \}$. The result follows easily from the following:

4 Lemma. *Suppose that Z is a metric space. Then Z is a γ_k-set iff Z is locally compact and separable.*

Proof. First suppose that Z is locally compact and separable. Then we can write Z as an increasing union of compact subsets C_n whose interiors cover Z. Given a k-cover $\mathcal{U}$ we simply choose $U_n \in \mathcal{U}$ so that $C_n \subseteq U_n$. This works because for every compact set C there exists n with $C \subseteq C_n$.

Conversely, suppose that Z is not locally compact. This means that for some $x \in Z$ we have that x is not in the interior of any compact set. Define a sequence $(\mathcal{U}_n : n < \omega)$ as follows: For each n, let $\mathcal{U}_n$ be the set of all open subsets of Z such that U does not contain the open ball of radius $1/2^n$ around x, i.e. there exists $y \notin U$ such that $d(x, y) < 1/2^n$.

Note that each $\mathcal{U}_n$ is a k-cover of Z. To see this, suppose C is a compact subset of Z. Since x is not in the interior of C, the set C cannot contain an open ball centered at x. Choose $y \notin C$ with $d(x, y) < 1/2^n$. Now cover C with (finitely) many open balls not containing y. The union of this cover is in $\mathcal{U}_n$.

We can use the trick of Gerlits and Nagy to get a single k-cover from the sequence of k-covers, $(\mathcal{U}_n \ : \ n \in \omega)$. Since Z cannot be compact there must exist a sequence $(x_n : n \in \omega)$ with no limit point. Define

$$\mathcal{U} = \{ U \setminus \{ x_n \} \ : \ n < \omega, \ \ U \in \mathcal{U}_n \}.$$

Since any compact set can contain at most finitely many of the x_n, we see that $\mathcal{U}$ is a k-cover of Z.

For contradiction, suppose Z is γ_k-set and $(U_n \in \mathcal{U} : n \in \omega)$ eventually contains each compact set. Without loss, we may assume that $U_n \in \mathcal{U}_{l_n}$ with l_n distinct. This is because at most finitely many U_n can be "from" any $\mathcal{U}_l$ since they eventually must include x_l. Choose $y_n \notin U_n$ with $d(x, y_n) < 1/2^{l_n}$. Then

$$\{ y_n \ : \ n \in \omega \} \cup \{ x \}$$

is a convergent sequence, hence compact. But it is not a subset of any U_n.

It is easy to see that Z must be separable as we can take $\mathcal{U}_n$ to be the family of finite unions of open balls of radius less than $1/2^n$, then apply the Gerlits Nagy trick as above to obtain a countable basis for Z. $\boxed{QED}$

In Example 3 each of X and Y are locally compact metric spaces but $X \cup Y$ is not locally compact at the point $(0,1)$, so the result follows.

Kočniac also asked if $X \times Y$ is γ_k-set if both X and Y are. For metric spaces, this must be true by the Lemma, since the product of locally compact separable metric spaces is a locally compact separable metric space.

This is very different from the situation for usual γ-sets.

5 Proposition (Tsaban)**.** *If $X \times Y$ is a γ-set, then $X \cup Y$ is a γ-set.*

Proof. If $\mathcal{U}$ is an open ω-cover of $X \cup Y$, then $\mathcal{U}^2 = \{ U^2 \ : \ U \in \mathcal{U} \}$ is an open ω-cover of $X \times Y$, so there is $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V}^2 = \{ V^2 \ : \ V \in \mathcal{V} \}$ is a γ-cover of $X \times Y$, but then $\mathcal{V}$ is a γ-cover of $X \cup Y$. $\boxed{QED}$

This was not noticed in Galvin-Miller [7], where a direct argument for $X \times Y$ not being a γ-set is provided after it is shown that $X \cup Y$ is not a γ-set.

3 Q-sets

A Q-set is a separable metric space X such that every subset of X is a (relative) G_δ-set. It is easy to see that $2^{|X|} = 2^\omega$, hence, if there is an uncountable Q-set, then $2^{\aleph_1} = 2^{\aleph_0}$. So uncountable Q-sets might not exist. Martin's axiom (MA) implies that every separable metric space of size less than the continuum is a Q-set (see Martin and Solovay [15]).

The Rothberger cardinal, $\mathfrak{b}$, is defined to be the cardinality of the smallest family $\mathcal{F} \subseteq \omega^\omega$ such that for every $g \in \omega^\omega$ there is some $f \in \mathcal{F}$ with $f(n) \geq g(n)$ for infinitely many n. That is to say, $\mathfrak{b}$ is the size of the smallest unbounded family in the quasi-ordering $(\omega^\omega, \leq^*)$. Martin's Axiom implies that $\mathfrak{b}$ is the continuum.

A set U is a universal G_δ-set, if it is G_δ and for every G_δ-set $V \subseteq 2^\omega$ there exists $x \in 2^\omega$ such that

$$U(x) \stackrel{\text{def}}{=} \{ y \in 2^\omega \ : \ (x,y) \in U \} = V.$$

6 Theorem. *Suppose $\kappa < \mathfrak{b}$. Then the following are equivalent:*

(1) There exists a Q-set $X \subseteq 2^\omega$ with $|X| = \kappa$.

(2) There exists $(f_\alpha : \omega^\omega \to 2^\omega \ : \ \alpha < \kappa)$ continuous functions such that given any $(y_\alpha \in 2^\omega \ : \ \alpha < \kappa)$ there exists $x \in \omega^\omega$ with the property that $f_\alpha(x) =^* y_\alpha$ for every $\alpha < \kappa$.

(3) *There exists a sequence $(U_\alpha \subseteq 2^\omega \times 2^\omega \;:\; \alpha < \kappa)$ of G_δ-sets which is universal for κ sequences of G_δ-sets, i.e., for every sequence*

$$(V_\alpha \subseteq 2^\omega \;:\; \alpha < \kappa)$$

of G_δ-sets there exists $x \in 2^\omega$ such that for every $\alpha < \kappa$

$$V_\alpha = U_\alpha(x) \stackrel{def}{=} \{\, y \;:\; (x,y) \in U_\alpha \,\}.$$

Proof. We will need the following lemma and the details of its proof.

7 Lemma. *There exists $U \subseteq 2^\omega \times 2^\omega$ which is a universal G_δ-set such that for every $x_1, x_2 \in 2^\omega$ if $x_1 =^* x_2$, then $U(x_1) = U(x_2)$.*

Proof. Define

$$U = \{\, (A, y) \in P(2^{<\omega}) \times 2^\omega \;:\; \exists^\infty n \;\; y \upharpoonright n \in A \,\}$$

where $\exists^\infty$ stands for "there exists infinitely many". It is easy to see that U is G_δ. To see that it is universal, suppose that $V = \bigcap_{n<\omega} V_n$ where the $V_n \subseteq 2^\omega$ are open and descending, i.e., $V_{n+1} \subseteq V_n$ for each n. For $\sigma \in 2^{<\omega}$ nontrivial let $\sigma^* \subseteq \sigma$ be the initial segment of σ of length exactly one less than σ, i.e., $|\sigma^*| = |\sigma| - 1$. Define

$$A = \{\, \sigma \;:\; [\sigma] \subseteq V \text{ or } \exists n \;\; [\sigma] \subseteq V_n \text{ and } [\sigma^*] \not\subseteq V_n \,\}$$

Then $U(A) = V$. To see this, suppose $x \in U(A)$. If for some n we have that $x \upharpoonright n \in A$ because $[x \upharpoonright n] \subseteq V$ then clearly $x \in V$. On the other hand, if there are infinitely many k such that for some n, $[x \upharpoonright k] \subseteq V_n$ but $[x \upharpoonright (k-1)] \not\subseteq V_n$, then these n's must all be distinct and since the V_n were descending $x \in V$.

Conversely, if $x \in V$ then either x is in the interior of V and so $x \upharpoonright k \in A$ for all but finitely many k or it isn't in the interior of V and there are thus infinitely many n with $x \upharpoonright n \in A$. Hence $x \in U(A)$.

From the definition of U it is easy to check that if $A =^* A'$, then $U(A) = U(A')$. $\boxed{QED}$

2→3:
This follows immediately from the Lemma. Just define

$$(x, y) \in U_\alpha \text{ iff } (f_\alpha(x), y) \in U$$

and identify ω^ω with a G_δ subset of 2^ω.

3→1:

By the proof of Lemma 7 there exists $A_\alpha \subseteq 2^{<\omega} \times 2^{<\omega}$ such that for any (x, y) we have that $(x, y) \in U_\alpha$ iff $\exists^\infty n \ (x \upharpoonright n, y \upharpoonright n) \in A_\alpha$. We claim that

$$\{ A_\alpha \ : \ \alpha < \kappa \}$$

is a Q-set. Fix $y \in 2^\omega$ arbitrary. Consider any $\Gamma \subseteq \kappa$ and define the sequence of G_δ sets $(V_\alpha : \alpha < \kappa)$ by

$$V_\alpha = \begin{cases} \{ y \} & \text{if } \alpha \in \Gamma \\ \emptyset & \text{if } \alpha \notin \Gamma. \end{cases}$$

By assumption there exists $x \in 2^\omega$ such that $U_\alpha(x) = V_\alpha$ for every $\alpha < \kappa$. But then

$$\alpha \in \Gamma \text{ iff } y \in U_\alpha(x) \text{ iff } \exists^\infty n \ (x \upharpoonright n, y \upharpoonright n) \in A_\alpha \text{ iff}$$

$$A_\alpha \in \{ A \ : \ \exists^\infty n \ (x \upharpoonright n, y \upharpoonright n) \in A \}.$$

But this last set is G_δ. It follows that $\{ A_\alpha \ : \ \alpha \in \Gamma \}$ is relatively G_δ in the set $\{ A_\alpha \ : \ \alpha \in \kappa \}$.

1→2:
Let $\{ v_\alpha^n \in 2^\omega \ : \ n < \omega, \ \ \alpha < \kappa \}$ be a Q-set. Now for each $\alpha < \kappa$ define a continuous map $f_\alpha : \omega^\omega \to 2^\omega$ as follows. Suppose $x = (A, (I_n : n < \omega))$ where $A \subseteq 2^{<\omega}$ and each $I_n \subseteq 2^{<\omega}$ is finite. (We can easily identify the set of such x with ω^ω.) Define

$$f_\alpha((A, (I_n : n < \omega)))(n) = \begin{cases} 1 & \text{if } \exists k \ \ v_\alpha^n \upharpoonright k \in I_n \cap A \\ 0 & \text{otherwise} \end{cases}$$

Since the I_n are finite, the function f_α is continuous. We verify that it has the property required. Let $x_\alpha \in 2^\omega$ for $\alpha < \kappa$ be arbitrary. Since $\{ v_\alpha^n \in 2^\omega \ : \ n < \omega, \ \ \alpha < \kappa \}$ is a Q-set, there is a G_δ-set $U \subseteq 2^\omega$ with the property that for every $\alpha < \kappa$ and $n < \omega$ we have that $v_\alpha^n \in U$ iff $x_\alpha(n) = 1$. By the proof of Lemma 7 there exists $A \subseteq 2^{<\omega}$ such that for all α, n

$$v_\alpha^n \in U \text{ iff } A \cap \{ v_\alpha^n \upharpoonright k \ : \ k < \omega \} \text{ is infinite.}$$

Since $\mathfrak{b} > \kappa$ there exists a partition $(I_l : l < \omega)$ of $2^{<\omega}$ into finite sets such that for every $\alpha < \kappa$ and $n < \omega$ the set $A \cap \{ v_\alpha^n \upharpoonright k \ : \ k < \omega \}$ is infinite iff $I_l \cap A \cap \{ v_\alpha^n \upharpoonright k \ : \ k < \omega \} \neq \emptyset$ for all but finitely many $l < \omega$. But this implies that for $f_\alpha((A, (I_l : l < \omega))) =^* x_\alpha$ for each α. $\boxed{QED}$

Condition 3 is a kind of uncountable version of Luzin's doubly universal sets, see Kechris [10] page 171 22.15 iv. Luzin used a doubly universal set to prove that the classical properties of separation and reduction cannot hold on the same side of a reasonable point-class.

In condition 2, $u =^* v$ means that $u(n) = v(n)$ except for finitely many n. It is impossible to have the stronger condition with "=" in place of "=*" at least when κ is uncountable. To see this, fix $y_0 \in 2^\omega$ and define $E_\alpha = f_\alpha^{-1}(y_0)$ for $\alpha < \omega_1$. It is not hard to see that the $F_\alpha = \bigcap_{\beta<\alpha} E_\beta$ would have to be a strictly decreasing sequence of closed sets, which is impossible in a separable metric space.

We do not know if the condition $\kappa < \mathfrak{b}$ is needed for this result. There are several models of set theory where there is a Q-set and $\mathfrak{b} = \omega_1$, Fleissner and Miller [4], Judah and Shelah [6], and Miller [20].

We obtained this result while working on the square Q-set problem, see Fleissner [5]. Unfortunately, Fleissner's proof that it is consistent there is a Q-set whose square is not a Q-set contains a gap. In his paper, he claims to show that in his model of set theory:

(1) there is a Q-set $Y \subseteq 2^\omega$ of size ω_2, and

(2) for any set of $Z = \{\, z_\alpha \;:\; \alpha < \omega_2 \,\} \subseteq 2^\omega$ the set

$$\{\, (z_\alpha, z_\beta) \;:\; \alpha < \beta < \omega_2 \,\}$$

is not G_δ in $Z \times Z$.

But we have a fairly easy proof that (1) implies the negation of (2).

8 Theorem. *If there exists a Q-set $Y \subseteq 2^\omega$ with $|Y| = \omega_2$, then there exists $Z = \{\, z_\alpha : \alpha < \omega_2 \,\} \subseteq 2^\omega$ such that*

$$\{\, (z_\alpha, z_\beta) : \alpha < \beta < \omega_2 \,\}$$

is (relatively) G_δ in Z^2.

Proof. Let $Y = \{\, y_\alpha : \alpha < \omega_2 \,\}$ and let $U \subseteq 2^\omega \times 2^\omega$ be a universal G_δ-set. Choose for each $\beta < \omega_2$ a $u_\beta \in 2^\omega$ such that for every $\alpha < \omega_2$

$$y_\alpha \in U(u_\beta) \text{ iff } \alpha < \beta.$$

Since U is G_δ there are clopen $C_{n,m}, D_{n,m} \subseteq 2^\omega$ with

$$U = \bigcap_{n<\omega} \bigcup_{m<\omega} (C_{n,m} \times D_{n,m}).$$

Now let $z_\alpha = (y_\alpha, u_\alpha)$ and identify $2^\omega \times 2^\omega$ with 2^ω.

Then for any $\alpha, \beta < \omega_2$ we have that

$\alpha < \beta$
iff $(y_\alpha, u_\beta) \in U$
iff $(y_\alpha, u_\beta) \in \bigcap_{n<\omega} \bigcup_{m<\omega} (C_{n,m} \times D_{n,m})$
iff $(z_\alpha, z_\beta) = ((y_\alpha, u_\alpha), (y_\beta, u_\beta)) \in \bigcap_{n<\omega} \bigcup_{m<\omega} ((C_{n,m} \times 2^\omega) \times (2^\omega \times D_{n,m}))$.

$\boxed{QED}$

As far as we know, the problem of the consistency of a Q-set whose square is not a Q-set, is open. We do not know where the mistake in Fleissner's proof occurs. One way to connect this problem with Theorem 6 is the following:

9 Corollary. *Suppose there is a Q-set of size ω_2 and $\mathfrak{b} > \omega_2$. Then given any family $\Gamma \subseteq P(\omega_2 \times \omega_2)$ with $|\Gamma| = \omega_2$ there is a Q-set*

$$Z = \{ z_\alpha \in 2^\omega \ : \ \alpha < \omega_2 \}$$

such that for every $A \in \Gamma$ the set $\{ (z_\alpha, z_\beta) \ : \ (\alpha, \beta) \in A \}$ is G_δ in Z.

The corollary is also valid for any κ in place of ω_2.

4 Minimal Q-like-sets

At the Slippery-Rock conference in June 2004, Ali A. Alikhani-Koopaei asked me if the following Q-like example was possible. We show that it is.

10 Example. There exist a T_0 space Y such that Y is not a Q-set but for every $A \subseteq Y$ there is a minimal G_δ set Q with $A \subseteq Q$. By minimal we mean that for any G_δ set Q' if $A \subseteq Q'$, then $Q \subseteq Q'$.

Proof. Let X be any Q-set, i.e., every subset of X is G_δ and X at least T_0. For example, a discrete space. Now let X' be a disjoint copy of X and let $p \mapsto p'$ a bijection from X to X'. For each $A \subseteq X$ let $A' = \{ p' : p \in A \}$. Define the topology on $Y = X \cup X'$ by letting the open sets of Y be exactly those of the form $U \cup V'$ where $U, V \subseteq X$ are open in X and $U \subseteq V$. Then Y is T_0, e.g. X' is open in Y and separates any p and p'. $\boxed{QED}$

11 Claim. For $A, B \subseteq X$ the set $A \cup B'$ is G_δ in Y iff $A \subseteq B$. Furthermore, given any $A, B \subseteq X$ the set $A \cup (A \cup B)'$ is the minimal G_δ in Y containing $A \cup B'$.

Proof. Suppose that A is not a subset of B and let $p \in A \backslash B$. Then any open set in Y which contains A must also contain p'. The same is true for any G_δ and hence $A \cup B'$ is not G_δ.

On the other hand, suppose $A \subseteq B$. Let $A = \bigcap_{n<\omega} U_n$ and $B = \bigcap_{n<\omega} V_n$ where the U_n and V_n are open in X. Now since $A \subseteq B$ we may assume that $U_n \subseteq V_n$ (if not just replace U_n by $U_n \cap V_n$). But then

$$A \cup B' = \bigcap_{n<\omega} (U_n \cup V_n').$$

Furthermore, note that if $C \cup D'$ is G_δ and contains $A \cup B'$, then $A \subseteq C$ and $C \cup B \subseteq D$ and so $A \cup (A \cup B)' \subseteq C \cup D'$. $\boxed{QED}$

12 Problem. Can we get an example which is uncountable but contains no uncountable Q-set?

Yes. Let $X = \omega_1$ have the topology with $U \subseteq X$ is open iff $U = \emptyset$ or there exists α with

$$U = [\alpha, \omega_1) \stackrel{\text{def}}{=} \{\, \beta : \alpha \leq \beta < \omega_1 \,\}.$$

Given any $A \subseteq X$ the smallest G_δ containing A is $[\min(A), \omega_1)$.

5 σ-sets and retractive boolean algebras

The definition of thin set of reals is due to Rubin [22] who showed it equivalent to a certain construction yielding a retractive boolean algebra which is not the subalgebra of any interval algebra. Rubin asked whether or not there is always an uncountable thin set of reals. We show that every thin set is a σ-set and so by the results of Miller [16] that it is consistent there are no uncountable σ-sets, it is also consistent there are no uncountable thin sets.

A thin set of reals is defined as follows. An OIT (ordered interval tree) is a family of $(G_n : n \in \omega)$ such that each G_n is a family of pairwise disjoint open intervals such that for n and $I \in G_{n+1}$ there exists $J \in G_n$ with $I \subseteq J$. A set of reals Y is $(G_n : n \in \omega)$-small iff there exists $(F_n \in [G_n]^{<\omega} : n \in \omega)$ such that for every $x \in Y$ and $n \in \omega$ if $x \in \cup G_n$, then $x \in \cup F_n$. A set of reals X is thin iff for every OIT, $(G_n : n \in \omega)$, the set X is a countable union of $(G_n : n \in \omega)$-small sets.

13 Proposition. *If $X \subseteq \mathbb{R}$ is thin, then X is a σ-set.*

Proof. A thin set cannot contain an interval (see Rubin [22]) so we may suppose that X is disjoint from a countable dense set $D \subseteq \mathbb{R}$. Let $\mathcal{B}$ be the family of nonempty open intervals with end points from D. The following claim is easy to prove and left to the reader.

14 Claim. Given any open set $U \subseteq I$ where $I \in \mathcal{B}$ we can construct a family of pairwise disjoint intervals $G \subseteq \mathcal{B}$ so that

(1) $cl(J) \subseteq I$ for each $J \in G$ and

(2) $\bigcup G \subseteq U \subseteq \bigcup G \cup D$.

Now suppose that $\bigcap_{n<\omega} U_n$ is an arbitrary G_δ set of reals where the U_n are open sets. Using the claim it is easy to construct a sequence $G_n \subseteq \mathcal{B}$ of pairwise disjoint intervals such that:

(1) if $I \in G_{n+1}$, then for some $J \in G_n$ we have $cl(I) \subseteq J$ and

(2) $\bigcup G_n \subseteq U_n \subseteq \bigcup G_n \cup D$.

Since X is thin, we have that $X = \bigcup_{m<\omega} X_m$ where each X_m is $\{ G_n : n < \omega \}$-small. Fix m. There exists $F_{n,m} \in [G_n]^{<\omega}$ for $n < \omega$ which witness the smallness of X_m. Let

$$C_m = \bigcap_{n<\omega} (\cup F_{n,m}).$$

Note that we may assume that for each n and $I \in F_{n+1,m}$ there is a $J \in F_{n,m}$ with $cl(I) \subseteq J$. Hence

$$\bigcap_{n<\omega} (\cup F_{n,m}) = \bigcap_{n<\omega} \left(\bigcup_{I \in F_{n,m}} cl(I) \right)$$

and since each $F_{n,m}$ is finite, C_m is closed. Since X is disjoint from D we have that

$$X \cap \left(\bigcap_{n<\omega} U_n \right) = X \cap \left(\bigcup_{m<\omega} C_m \right).$$

Since we started with an arbitrary G_δ set we have that X is a σ-set. $\boxed{QED}$

Next we see that a set of reals is thin iff it is hereditarily Hurewicz. See Miller and Fremlin [18] for the definition of the Hurewicz property.

15 Proposition. *A set of reals X is thin iff it is hereditarily Hurewicz.*

PROOF. Suppose X is hereditarily Hurewicz and let $(G_n : n \in \omega)$ be an OIT. Let

$$Y = \{ x \in X \; : \; \forall n \;\; x \in \cup G_n \}.$$

Since Y has the Hurewicz property, there exists $(F_n \in [G_n]^{<\omega} : n < \omega)$ such that $\forall^\infty n \;\; x \in \cup F_n$ for each $x \in Y$. Let $(F_n^m \in [G_n]^{<\omega} : n < \omega)$ for $m < \omega$ list all sequences such that $F_n^m = F_n$ for all but finitely many n. Define

$$X_m = X \cap \left(\bigcap_n \cup F_n^m \right).$$

Note that each X_m is $(G_n : n \in \omega)$-small and

$$X = \bigcup_m X_m \cup (X \setminus \cup G_0)$$

because the G_n are refining.

Conversely, suppose X is thin, $Y \subseteq X$, and $(\mathcal{U}_n : n \in \omega)$ a sequence of open covers of Y. As in the above proof we can find an OIT, $(G_n : n \in \omega)$, such that each G_n refines $\mathcal{U}_n$ and covers Y. Since X is thin, we have that is the countable union of $(G_n : n \in \omega)$-small sets. Let $(F_n^m \in [G_n]^{<\omega} : n < \omega)$ for $m < \omega$ list the finite sets given by the notion of smallness. Define

$$F_n = F_n^0 \cup F_n^1 \cup \cdots \cup F_n^n$$

Choose $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ so that for each n and $I \in F_n$ there exists $V \in \mathcal{V}_n$ with $I \subseteq V$. For each $x \in Y$ we have that $x \in \cup F_n$ for all but finitely many n. $\boxed{QED}$

6 Souslin number and nonmeager sets

We obtained these results in March 2004. First we define the following small cardinal number:

$$\mathrm{non}(\mathcal{M}) = \min\{\,|X| \ : \ X \subseteq 2^\omega \text{ nonmeager}\,\}.$$

For $X \subseteq 2^\omega$ we define $\mathrm{ord}(X)$ (the Borel order of X) to be the smallest $\alpha < \omega_1$ such that every Borel subset A of 2^ω there exist a $\mathbf{\Sigma}^0_\alpha$ subset B of 2^ω such that $A \cap X = B \cap X$, if there is no such $\alpha < \omega_1$, we define $\mathrm{ord}(X) = \omega_1$.

To prove our main result (Theorem 18) we will use the following theorem:

16 Theorem. *There exists $X \subseteq 2^\omega$ with $|X| \le \mathrm{non}(\mathcal{M})$ and $\mathrm{ord}(X) = \omega_1$.*

Proof. This is similar to the proof of Miller [16] Theorem 18. Notice that it is enough to show that for each $\alpha < \omega_1$ there exists an $X_\alpha \subseteq 2^\omega$ with

$$|X_\alpha| \le \mathrm{non}(\mathcal{M})$$

and $\mathrm{ord}(X_\alpha) \ge \alpha$, since the ω_1 union of these sets would be the X we need.

So fix $\alpha_0 < \omega_1$ with $\alpha_0 > 1$. According to Miller [16] Theorem 13, there exists a countable subalgebra $\mathcal{G} \subseteq \mathbb{B}$ where $\mathbb{B}$ is the complete boolean algebra:

$$\mathbb{B} = \frac{\mathrm{Borel}(2^\omega)}{\mathrm{meager}(2^\omega)}$$

such that $\mathcal{G}$ countably generates $\mathbb{B}$ in exactly α_0 steps. This last statement means the following:

Define $\mathcal{G}_0 = \mathcal{G}$. For $\alpha > 0$ an even ordinal define $\mathcal{G}_\alpha$ to be the family of countable disjuncts of elements from $\bigcup_{\beta<\alpha} \mathcal{G}_\beta$ and for α an odd ordinal define $\mathcal{G}_\alpha$ to be the family of countable conjuncts of elements from $\bigcup_{\beta<\alpha} \mathcal{G}_\beta$. These classes are analogous to the $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ families of Borel sets. Then $\mathcal{G}$ has the property that $\mathcal{G}_{\alpha_0} = \mathbb{B}$ but for each $\beta < \alpha_0$, $\mathcal{G}_\beta \ne \mathbb{B}$.

Now let $Y \subseteq 2^\omega$ be such that $|Y| = \text{non}(\mathcal{M})$ and $Y \cap U$ is nonmeager for every nonempty open subset U of 2^ω. Note that Y has the property that for any Borel subsets A and B of 2^ω, if $A \cap Y = B \cap Y$, then the symmetric difference, $A \Delta B$ is meager.

Let $\mathcal{F} \subseteq \text{Borel}(2^\omega)$ be a family of representatives for $\mathcal{G}$, i.e.,

$$\mathcal{G} = \{ [A] : A \in \mathcal{F} \}$$

where $[A] \in \mathbb{B}$ is the equivalence class of A modulo the meager ideal in 2^ω. Assume $\mathcal{F}$ is chosen so that the map $A \mapsto [A]$ is one-to-one and 2^ω and $\emptyset$ are the representatives of 1 and 0. By throwing out a meager subset of Y we may assume that for any $A, B, C \in \mathcal{F}$

(1) $[A] \vee [B] = [C]$ iff $(A \cup B) \cap Y = C \cap Y$, and

(2) $[A] \wedge [B] = [C]$ iff $(A \cap B) \cap Y = C \cap Y$.

Define $\mathcal{F}^Y = \{ Y \cap A : A \in \mathcal{F} \}$. Then we have that $\mathcal{G}$ and $\mathcal{F}^Y$ are isomorphic as boolean algebras:

$$(\mathcal{G}, \vee, \wedge, 0, 1) \simeq \left(\mathcal{F}^Y, \cup, \cap, \emptyset, Y\right).$$

Define $\mathcal{F}_\beta$ and $\mathcal{F}^Y_\beta$ exactly as we did $\mathcal{G}_\beta$ but using countable unions and intersections instead of disjuncts and conjuncts as we do in a boolean algebra.

17 Claim.

(1) By induction on β

   a. $\mathcal{G}_\beta = \{ [B] : B \in \mathcal{F}_\beta \}$ and
   b. $\mathcal{F}^Y_\beta = \{ B \cap Y : B \in \mathcal{F}_\beta \}$.

(2) If $\beta < \alpha_0$, then $\mathcal{F}^Y_\beta \neq \mathcal{F}^Y_{\alpha_0}$.

Proof. Item (1) is an easy induction. To see (2) suppose that $[B] \in \mathcal{G}_{\alpha_0} \setminus \cup_{\beta < \alpha_0} \mathcal{G}_\beta$. Without loss $B \in \mathcal{F}_{\alpha_0}$ and we claim that $B \cap Y \in \mathcal{F}^Y_{\alpha_0} \setminus \cup_{\beta < \alpha_0} \mathcal{F}^Y_\beta$. Suppose for contradiction that $B \cap Y \in \mathcal{F}^Y_\beta$ for some $\beta < \alpha_0$. Then there would exist $C \in \mathcal{F}_\beta$ with $B \cap Y = C \cap Y$. But this would imply that $[B] = [C] \in \mathcal{G}_\beta$ which is a contradiction. This proves the claim. $\boxed{QED}$

Now let $\mathcal{F}^Y = \{ C_n : n < \omega \}$ and let $i : Y \to 2^\omega$ be the Marczewski characteristic function of the sequence, which is defined by

$$i(a)(n) = \begin{cases} 1 & \text{if } a \in C_n \\ 0 & \text{if } a \notin C_n. \end{cases}$$

Let $X = i(Y)$. The map i need not be one-to-one but by definition, it is onto X, so $|X| \le |Y| = \mathrm{non}(\mathcal{M})$. Note that

$$\{ C \cap X \ : \ C \text{ is clopen in } 2^\omega \} = \{ i(C) : C \in \mathcal{F}^Y \}.$$

Hence, since the Borel order of $\mathcal{F}^Y$ is at least α_0 we have that $\mathrm{ord}(X) \ge \alpha_0$. This proves Theorem 16. $\boxed{QED}$

We define the Souslin number $\mathfrak{sn}$:

$$\mathfrak{sn} = \min\{ |X| \ : \ X \subseteq 2^\omega, \ \ \exists A \in \mathbf{\Sigma}^1_1 \ \forall B \in \mathbf{\Pi}^1_1 \ A \cap X \ne B \cap X \}.$$

In Zapletal [23] Appendix C, it is shown that $\mathfrak{sn} \ge \mathfrak{b}$, where $\mathfrak{b}$ is the smallest cardinality of an unbounded family in ω^ω. In Miller [21] it is shown to be consistent to have $\mathfrak{sn} > \mathfrak{b}$.

Define the following variant of the Souslin number $\mathfrak{sn}^*$:

$$\mathfrak{sn}^* = \min\{ |X| \ : \ X \subseteq 2^\omega, \ \ \exists A \in \mathbf{\Sigma}^1_1 \ \forall B \in \text{Borel} \ \ A \cap X \ne B \cap X \}.$$

The following theorem partially confirms a conjecture of Zapletal that $\mathfrak{sn} \le \mathrm{non}(\mathcal{M})$, since $\mathfrak{sn}^* \le \mathfrak{sn}$. Zapletal was motivated by results in [23] Appendix C and [24], which roughly speaking show that it is impossible to force $\mathfrak{sn} > \mathrm{non}(\mathcal{M})$ using a countable support iteration of definable real forcing in the presence of suitable large cardinal axioms. Zapletal's conjecture remains open.

18 Theorem. $\mathfrak{sn}^* \le \mathrm{non}(\mathcal{M})$.

Proof. Let $U \subseteq 2^\omega \times 2^\omega$ be a universal $\mathbf{\Sigma}^1_1$ set and consider the set of reals X from Theorem 16. For each $\alpha < \omega_1$ let $B_\alpha \subseteq 2^\omega$ be a $\mathbf{\Sigma}^0_\alpha$ such that for every C which is $\mathbf{\Pi}^0_\alpha$ we have that

$$B_\alpha \cap X \ne C \cap X.$$

Since U is universal there exists $a_\alpha \in 2^\omega$ such that the cross section $U_{a_\alpha} = B_\alpha$. Let Z be defined by

$$Z = \{ a_\alpha : \alpha < \omega_1 \} \times X \subseteq 2^\omega \times 2^\omega.$$

Then $|Z| \le \mathrm{non}(\mathcal{M})$ and there is no Borel set $B \subseteq 2^\omega \times 2^\omega$ such that $Z \cap U = Z \cap B$. This is because if B is say $\mathbf{\Pi}^0_\alpha$, then every cross section of B is $\mathbf{\Pi}^0_\alpha$, but then

$$B_\alpha \cap X = U_{a_\alpha} \cap X = B_{a_\alpha} \cap X$$

which contradicts our choice of B_α. $\boxed{QED}$

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