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Selection principles and countable dimension

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Abstract. We consider player TWO of the game $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ when $\mathcal{A}$ and $\mathcal{B}$ are special classes of open covers of metrizable spaces. Our results give game-theoretic characterizations of the notions of a countable dimensional and of a strongly countable dimensional metric spaces.

Keywords: countable dimensional, strongly countable dimensional, selection principle, infinite game

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The selection principle $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ states: There is for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ a corresponding sequence $(b_n : n \in \mathbb{N})$ such that for each n we have $b_n \in A_n$, and $\{ b_n : n \in \mathbb{N} \}$ is an element of $\mathcal{B}$. There are many examples of this selection principle in the literature. One of the earliest examples of it is known as *the Rothberger property*, $\mathsf{S}_1(\mathcal{O},\mathcal{O})$. Here, $\mathcal{O}$ is the collection of all open covers of a topological space.

The following game, $\mathsf{G}_1(\mathcal{A},\mathcal{B})$, is naturally associated with $\mathsf{S}_1(\mathcal{A},\mathcal{B})$: Players ONE and TWO play an inning per positive integer. In the n-th inning ONE first chooses an element O_n of $\mathcal{A}$; TWO responds by choosing an element $T_n \in O_n$. A play

$$O_1, T_1, O_2, T_2, \dots, O_n, T_n, \dots$$

is won by TWO if $\{ T_n : n \in \mathbb{N} \}$ is in $\mathcal{B}$, else ONE wins.

$$\begin{array}{c}\text{TWO has a winning strategy in } \mathsf{G}_1(\mathcal{A},\mathcal{B})\\ \Downarrow \\ \text{ONE has no winning strategy in } \mathsf{G}_1(\mathcal{A},\mathcal{B})\\ \Downarrow \\ \mathsf{S}_1(\mathcal{A},\mathcal{B}).\end{array}$$

There are many known examples of $\mathcal{A}$ and $\mathcal{B}$ where neither of these implications reverse.

Several classes of open covers of spaces have been defined by the following schema: For a space X, and a collection $\mathcal{T}$ of subsets of X, an open cover $\mathcal{U}$ of X is said to be a $\mathcal{T}$-cover if X is not a member of $\mathcal{U}$, but there is for each $T \in \mathcal{T}$ a $U \in \mathcal{U}$ with $T \subseteq U$. The symbol $\mathcal{O}(\mathcal{T})$ denotes the collection of $\mathcal{T}$-covers of X. In this paper we consider only $\mathcal{A}$ which are of the form $\mathcal{O}(\mathcal{T})$ and $\mathcal{B} = \mathcal{O}$. Several examples of open covers of the form $\mathcal{O}(\mathcal{T})$ appear in the literature. To mention just a few: When $\mathcal{T}$ is the family of one-element subsets of X, $\mathcal{O}(\mathcal{T}) = \mathcal{O}$. When $\mathcal{T}$ is the family of finite subsets of X, then members of $\mathcal{O}(\mathcal{T})$ are called ω-covers in [3]. The symbol Ω denotes the family of ω-covers of X. When $\mathcal{T}$ is the collection of compact subsets of X, then members of $\mathcal{O}(\mathcal{T})$ are called k-covers in [5]. In [5] the collection of k-covers is denoted $\mathcal{K}$.

Though some of our results hold for more general spaces, in this paper "topological space" means separable metric space, and "dimension" means Lebesgue covering dimension. We consider only infinite-dimensional separable metric spaces. By classical results of Hurewicz and Tumarkin these are separable metric spaces which cannot be represented as the union of finitely many zerodimensional subspaces.

1 Properties of strategies of player TWO

1 Lemma. *Let F be a strategy of TWO in the game $\mathsf{G}_1(\mathcal{O}(\mathcal{T}),\mathcal{B})$. Then there is for each finite sequence $(\mathcal{U}_1,\dots,\mathcal{U}_n)$ of elements of $\mathcal{O}(\mathcal{T})$, an element $C \in \mathcal{T}$ such that for each open set $U \supseteq C$ there is a $\mathcal{U} \in \mathcal{O}(\mathcal{T})$ such that $U = F(\mathcal{U}_1,\dots,\mathcal{U}_n,\mathcal{U})$.*

Proof. For suppose on the contrary this is false. Fix a finite sequence $(\mathcal{U}_1,\dots,\mathcal{U}_n)$ witnessing this, and choose for each set $C \subset X$ which is in $\mathcal{T}$ an open set $U_C \supseteq C$ witnessing the failure of Claim 1. Then $\mathcal{U} = \{U_C : C \subset X \text{ and } C \in \mathcal{T}\}$ is a member of $\mathcal{O}(\mathcal{T})$, and as $F(\mathcal{U}_1,\dots,\mathcal{U}_n,\mathcal{U}) = U_C$ for some $C \in \mathcal{T}$, this contradicts the selection of U_C. QED

When $\mathcal{T}$ has additional properties, Lemma 1 can be extended to reflect that. For example: The family $\mathcal{T}$ is *up-directed* if there is for each A and B in $\mathcal{T}$, a C in $\mathcal{T}$ with $A \cup B \subseteq C$.

2 Lemma. *Let $\mathcal{T}$ be an up-directed family. Let F be a strategy of TWO in the game $\mathsf{G}_1(\mathcal{O}(\mathcal{T}),\mathcal{B})$. Then there is for each $D \in \mathcal{T}$ and each finite sequence $(\mathcal{U}_1,\dots,\mathcal{U}_n)$ of elements of $\mathcal{O}(\mathcal{T})$, an element $C \in \mathcal{T}$ such that $D \subseteq C$ and for each open set $U \supseteq C$ there is a $\mathcal{U} \in \mathcal{O}(\mathcal{T})$ such that $U = F(\mathcal{U}_1,\dots,\mathcal{U}_n,\mathcal{U})$.*

Proof. For suppose on the contrary this is false. Fix a finite sequence $(\mathcal{U}_1,\dots,\mathcal{U}_n)$ and a set $D \in \mathcal{T}$ witnessing this, and choose for each set $C \subset X$ which is in $\mathcal{T}$ and with $D \subset C$ an open set $U_C \supseteq C$ witnessing the failure of Claim 1. Then, as $\mathcal{T}$ is up-directed, $\mathcal{U} = \{U_C : D \subset C \subset X \text{ and } C \in \mathcal{T}\}$ is a member of $\mathcal{O}(\mathcal{T})$, and as $F(\mathcal{U}_1,\dots,\mathcal{U}_n,\mathcal{U}) = U_C$ for some $C \in \mathcal{T}$, this contradicts the selection of U_C. QED

We shall say that X is *$\mathcal{T}$-first countable* if there is for each $T \in \mathcal{T}$ a sequence $(U_n : n = 1,2,\dots)$ of open sets such that for all n, $T \subset U_{n+1} \subset U_n$, and for each open set $U \supset T$ there is an n with $U_n \subset U$. Let $\langle\mathcal{T}\rangle$ denote the subspaces which are unions of countably many elements of $\mathcal{T}$.

3 Theorem. *If F is any strategy for TWO in $\mathsf{G}_1(\mathcal{O}(\mathcal{T}),\mathcal{O})$ and if X is $\mathcal{T}$-first countable, then there is a set $S \in \langle\mathcal{T}\rangle$ such that: For any closed set $C \subset X\setminus S$, there is an F-play O_1, T_1, …, O_n, T_n … such that $\bigcup_{n=1}^{\infty} T_n \subseteq X\setminus C$.*

More can be proved for up-directed $\mathcal{T}$:

4 Theorem. *Let $\mathcal{T}$ be up-directed. If F is any strategy for TWO in $\mathsf{G}_1(\mathcal{O}(\mathcal{T}),\mathcal{O})$ and if X is $\mathcal{T}$-first countable, then there is for each set $T \in \langle\mathcal{T}\rangle$ a set $S \in \langle\mathcal{T}\rangle$ such that: $T \subseteq S$ and for any closed set $C \subset X \setminus S$, there is an F-play*

$$O_1,\ T_1,\ \dots,\ O_n,\ T_n\ \dots$$

such that $T \subseteq \bigcup_{n=1}^{\infty} T_n \subseteq X \setminus C$.

Proof. Let F be a strategy of TWO. Let T be a given element of $\langle\mathcal{T}\rangle$, and write $T = \bigcup_{n=1}^{\infty} T_n$, where each T_n is an element of $\mathcal{T}$.

Starting with T_1 and the empty sequence of elements of $\mathcal{O}(\mathcal{T})$, apply Lemma 2 to choose an element $S_\emptyset$ of $\mathcal{T}$ such that $T_1 \subset S_\emptyset$, and for each open set $U \supseteq S_\emptyset$ there is an element $\mathcal{U} \in \mathcal{O}(\mathcal{T})$ with $U = F(\mathcal{U})$. Since X is $\mathcal{T}$-first countable, choose for each n an open set U_n such that $U_n \supset U_{n+1}$, and for each open set U with $S_\emptyset \subset U$ there is an n with $U_n \subset U$. Using Lemma 2, choose for each n an element $\mathcal{U}_n$ of $\mathcal{O}(\mathcal{T})$ such that $U_n = F(\mathcal{U}_n)$.

Now consider T_2, and for each n the one-term sequence $(\mathcal{U}_n)$ of elements of $\mathcal{O}(\mathcal{T})$. Since $\mathcal{T}$ is up-directed, choose an element T of $\mathcal{T}$ with $S_\emptyset \cup T_2 \subset T$. Applying Lemma 2 to T and $(\mathcal{U}_n)$ choose an element $S_{(n)} \in \mathcal{T}$ such that for each open set $U \supseteq S_{(n)}$ there is a $\mathcal{U} \in \mathcal{O}(\mathcal{T})$ with $U = F(\mathcal{U}_n,\mathcal{U})$. Since X is $\mathcal{T}$-first countable, choose for each k an open set $U_{(n,k)} \supseteq S_{(n)}$ such that $U_{(n,k)} \supseteq U_{(n,k+1)} \supseteq S_{(n)}$, and for each open set $U \supset S_{(n)}$ there is a k with $U \supset U_{(n,k)}$. Then choose for each n and k an element $\mathcal{U}_{(n,k)}$ of $\mathcal{O}(\mathcal{T})$ such that $U_{(n,k)} = F(\mathcal{U}_{(n)},\mathcal{U}_{(n,k)})$.

In general, fix k and suppose we have chosen for each finite sequence $(n_1,\dots, n_k)$ of positive integers, sets $S_{(n_1,\dots,n_k)} \in \mathcal{T}$, open sets $U_{(n_1,\dots,n_k,n)}$ and elements $\mathcal{U}_{(n_1,\dots,n_k,n)}$ of $\mathcal{O}(\mathcal{T})$, $n < \infty$, such that:

(1) $T_1 \cup \cdots \cup T_k \subset S_{(n_1,\dots,n_k)}$;

(2) $\{\, U_{(n_1,\dots,n_k,n)} : n < \infty \,\}$ witnesses the $\mathcal{T}$-first countability of X at $S_{(n_1,\dots,n_k)}$;

(3) $U_{(n_1,\dots,n_k,n)} = F\left(\mathcal{U}_{(n_1)}, \dots, \mathcal{U}_{(n_1,\dots,n_k)}, \mathcal{U}_{(n_1,\dots,n_k,n)}\right)$;

Now consider a fixed sequence of length k, say $(n_1, \dots, n_k)$. Since $\mathcal{T}$ is up-directed choose an element T of $\mathcal{T}$ such that $T_{k+1} \cup S_{(n_1,\dots,n_k)} \subset T$. For each n apply Lemma 2 to T and the finite sequence $(\mathcal{U}_{(n_1)}, \dots, \mathcal{U}_{(n_1,\dots,n_k,n)})$: Choose a set $S_{(n_1,\dots,n_k,n)} \in \mathcal{T}$ such that $T \subseteq S_{(n_1,\dots,n_k,n)}$ and for each open set $U \supseteq S_{(n_1,\dots,n_k,n)}$ there is a $\mathcal{U} \in \mathcal{O}(\mathcal{T})$ such that $U = F\left(\mathcal{U}_{(n_1)}, \dots, \mathcal{U}_{(n_1,\dots,n_k,n)}, \mathcal{U}\right)$. Since X is $\mathcal{T}$-first countable, choose for each j an open set $U_{(n_1,\dots,n_k,n,j)}$ such that $U_{(n_1,\dots,n_k,j+1)} \subset U_{(n_1,\dots,n_k,n,j)}$, and for each open set $U \supset S_{(n_1,\dots,n_k,n)}$ there is a j with $U \supseteq U_{(n_1,\dots,n_k,j)}$. Then choose for each j an $\mathcal{U}_{(n_1,\dots,n_k,n,j)} \in \mathcal{O}(\mathcal{T})$ such that $U_{(n_1,\dots,n_k,n,j)} = F\left(\mathcal{U}_{(n_1)}, \dots, \mathcal{U}_{(n_1,\dots,n_k,n)}, \mathcal{U}_{(n_1,\dots,n_k,n,j)}\right)$.

This shows how to continue for all k the recursive definition of the items $S_{(n_1,\dots,n_k)} \in \mathcal{T}$, open sets $U_{(n_1,\dots,n_k,n)}$ and elements $\mathcal{U}_{(n_1,\dots,n_k,n)}$ of $\mathcal{O}(\mathcal{T})$, $n < \infty$ as above.

Finally, put $S = \cup_{\tau \in {}^{<\omega}\mathbb{N}} S_\tau$. It is clear that $S \in \langle \mathcal{T} \rangle$, and that $T \subset S$. Consider a closed set $C \subset X \setminus S$. Since $C \cap S_\emptyset = \emptyset$, choose an n_1 so that $U_{(n_1)} \cap C = \emptyset$. Then since $C \cap S_{(n_1)} = \emptyset$, choose an n_2 such that $U_{(n_1,n_2)} \cap C = \emptyset$. Since $C \cap S_{(n_1,n_2)} = \emptyset$ choose an n_3 so that $U_{(n_1,n_2,n_3)} \cap C = \emptyset$, and so on. In this way we find an F-play

$$\mathcal{U}_{(n_1)}, U_{(n_1)}, \mathcal{U}_{(n_1,n_2)}, U_{(n_1,n_2)}, \dots$$

such that $T \subset \bigcup_{k=1}^{\infty} U_{(n_1,\dots,n_k)} \subset X \setminus C$. QED

When $\mathcal{T}$ is a collection of compact sets in a metrizable space X then X is $\mathcal{T}$-first countable. Call a subset $\mathcal{C}$ of $\mathcal{T}$ *cofinal* if there is for each $T \in \mathcal{T}$ a $C \in \mathcal{C}$ with $T \subseteq C$. As an examination of the proof of Theorem 4 reveals, we do not need full $\mathcal{T}$-first countability of X, but only that X is $\mathcal{C}$-first countable for some cofinal set $\mathcal{C} \subseteq \mathcal{T}$. Thus, we in fact have:

5 Theorem. *Let $\mathcal{T}$ be up-directed. If F is any strategy for TWO in $\mathsf{G}_1(\mathcal{O}(\mathcal{T}), \mathcal{O})$ and if X is $\mathcal{C}$-first countable where $\mathcal{C} \subset \mathcal{T}$ is cofinal in $\mathcal{T}$, then there is for each set $T \in \langle \mathcal{T} \rangle$ a set $S \in \langle \mathcal{C} \rangle$ such that: $T \subseteq S$ and for any closed set $C \subset X \setminus S$, there is an F-play*

$$O_1, T_1, \dots, O_n, T_n \dots$$

such that $T \subseteq \bigcup_{n=1}^{\infty} T_n \subseteq X \setminus C$.

2 When player TWO has a winning strategy

Recall that a subset of a topological space is a G_δ-set if it is an intersection of countably many open sets.

6 Theorem. *If the family $\mathcal{T}$ has a cofinal subset consisting of G_δ subsets of X, then TWO has a winning strategy in $\mathsf{G}_1(\mathcal{O}(\mathcal{T}),\mathcal{O})$ if, and only if, the space is a union of countably many members of $\mathcal{T}$.*

Proof. 2 $\Rightarrow$ 1 is easy to prove. We prove 1 $\Rightarrow$ 2. Let F be a winning strategy for TWO. Let $\mathcal{C} \subseteq \mathcal{T}$ be a cofinal set consisting of G_δ-sets.
By Lemma 1 choose $C_\emptyset \in \mathcal{T}$ associated to the empty sequence. Since $\mathcal{C}$ is cofinal in $\mathcal{T}$, choose for $C_\emptyset$ a G_δ set $G_\emptyset$ in $\mathcal{C}$ with $C_\emptyset \subseteq G_\emptyset$. Choose open sets $(U_n : n \in \mathbb{N})$ such that for each n we have $G_\emptyset \subset U_{n+1} \subset U_n$, and $G_\emptyset = \cap_{n\in\mathbb{N}} U_n$.

For each n choose by Lemma 1 a cover $\mathcal{U}_n \in \mathcal{O}(\mathcal{T})$ with $U_n = F(\mathcal{U}_n)$. Choose for each n a $C_n \in \mathcal{T}$ associated to $(\mathcal{U}_n)$ by Lemma 1. For each n also choose a G_δ-set $G_n \in \mathcal{C}$ with $C_n \subseteq G_n$. For each n_1 choose a sequence $(U_{n_1 n} : n \in \mathbb{N})$ of open sets such that $G_{n_1} = \cap_{n\in\mathbb{N}} U_{n_1 n}$ and for each n, $U_{n_1 n+1} \subset U_{n_1 n}$. For each $n_1 n_2$ choose by Lemma 1 a cover $\mathcal{U}_{n_1 n_2} \in \mathcal{O}(\mathcal{T})$ such that $U_{n_1 n_2} = F(\mathcal{U}_{n_1}, \mathcal{U}_{n_1 n_2})$. Choose by Lemma 1 a $C_{n_1 n_2} \in \mathcal{T}$ associated to $(\mathcal{U}_{n_1}, \mathcal{U}_{n_1 n_2})$, and then choose a G_δ-set $G_{n_1 n_2} \in \mathcal{C}$ with $C_{n_1 n_2} \subset G_{n_1 n_2}$, and so on.

Thus we get for each finite sequence $(n_1 n_2 \cdots n_k)$ of positive integers

(1) a set $C_{n_1\cdots n_k} \in \mathcal{T}$,

(2) a G_δ-set $G_{n_1\cdots n_k} \in \mathcal{T}$ with $C_{n_1\cdots n_k} \subseteq G_{n_1\cdots n_k}$,

(3) a sequence $(U_{n_1\cdots n_k n} : n \in \mathbb{N})$ of open sets with $G_{n_1\cdots n_k} = \cap_{n\in\mathbb{N}} U_{n_1\cdots n_k n}$ and for each n $U_{n_1\cdots n_k n+1} \subseteq U_{n_1\cdots n_k n}$, and

(4) a $\mathcal{U}_{n_1\cdots n_k} \in \mathcal{O}_{(\mathcal{T})}$ such that for all n

$$U_{n_1\cdots n_k n} = F(\mathcal{U}_{n_1}, \ldots, \mathcal{U}_{n_1\cdots n_k n}).$$

Now X is the union of the countably many sets $G_\tau \in \mathcal{T}$ where τ ranges over ${}^{<\omega}\mathbb{N}$. For if not, choose $x \in X$ which is not in any of these sets. Since x is not in $G_\emptyset$, choose U_{n_1} with $x \notin U_{n_1}$. Now x is not in G_{n_1}, so choose $U_{n_1 n_2}$ with $x \notin U_{n_1 n_2}$, and so on. In this way we obtain the F-play

$$\mathcal{U}_{n_1}, U_{n_1}, \mathcal{U}_{n_1 n_2}, U_{n_1 n_2}, \ldots$$

lost by TWO, contradicting that F is a winning strategy for TWO. QED

Examples of up-directed families $\mathcal{T}$ include:

- $[X]^{<\aleph_0}$, the collection of finite subsets of X;

- $\mathcal{K}$, the collection of compact subsets of X;
- KFD, the collection of compact, finite dimensional subsets of X.
- CFD, the collection of closed, finite dimensional subsets of X.
- FD, the collection of finite dimensional subsets of X.

A subset of a topological space is said to be *countable dimensional* if it is a union of countably many zero-dimensional subsets of the space. A subset of a space is *strongly countable dimensional* if it is a union of countably many closed, finite dimensional subsets. Let X be a space which is not finite dimensional. Let $\mathcal{O}_{\mathsf{cfd}}$ denote $\mathcal{O}(\mathsf{CFD})$, the collection of CFD-covers of X. And let $\mathcal{O}_{\mathsf{fd}}$ denote $\mathcal{O}(\mathsf{FD})$, the collection of FD-covers of X.

7 Corollary. *For a metrizable space X the following are equivalent:*

(1) X is strongly countable dimensional.

(2) TWO has a winning strategy in $\mathsf{G}_1(\mathcal{O}_{\mathsf{cfd}}, \mathcal{O})$.

PROOF. $1 \Rightarrow 2$ is easy to prove. To see $2 \Rightarrow 1$, observe that in a metric space each closed set is a G_δ-set. Thus, $\mathcal{T} = \mathsf{CFD}$ meets the requirements of Theorem 6. QED

For the next application we use the following classical theorem of Tumarkin:

8 Theorem (Tumarkin). *In a separable metric space each n-dimensional set is contained in an n-dimensional G_δ-set.*

9 Corollary. *For a separable metrizable space X the following are equivalent:*

(1) X is countable dimensional.

(2) TWO has a winning strategy in $\mathsf{G}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$.

PROOF. $1 \Rightarrow 2$ is easy to prove. We now prove $2 \Rightarrow 1$. By Tumarkin's Theorem, $\mathcal{T} = \mathsf{FD}$ has a cofinal subset consisting of G_δ-sets. Thus the requirements of Theorem 6 are met. QED

Recall that a topological space is *perfect* if every closed set is a G_δ-set.

10 Corollary. *In a perfect space the following are equivalent:*

(1) TWO has a winning strategy in $\mathsf{G}_1(\mathcal{K}, \mathcal{O})$.

(2) The space is σ-compact.

PROOF. In a perfect space the collection of closed sets are G_δ-sets. Apply Theorem 6. QED

And when $\mathcal{T}$ is up-directed, Theorem 6 can be further extended to:

11 Theorem. *If $\mathcal{T}$ is up-directed and has a cofinal subset consisting of G_δ-subsets of X, the following are equivalent:*

(1) TWO has a winning strategy in $\mathsf{G}_1(\mathcal{O}(\mathcal{T}),\Gamma)$.

(2) TWO has a winning strategy in $\mathsf{G}_1(\mathcal{O}(\mathcal{T}),\Omega)$.

(3) TWO has a winning strategy in $\mathsf{G}_1(\mathcal{O}(\mathcal{T}),\mathcal{O})$.

PROOF. We must show that $3 \Rightarrow 1$. Since X is a union of countably many sets in $\mathcal{T}$, and since $\mathcal{T}$ is up-directed, we may represent X as $\bigcup_{n=1}^{\infty} X_n$ where for each n we have $X_n \subset X_{n+1}$ and $X_n \in \mathcal{T}$. Now, when ONE presents TWO with $O_n \in \mathcal{O}(\mathcal{T})$ in inning n, then TWO chooses $T_n \in O_n$ with $X_n \subset T_n$. The sequence of T_n's chosen by TWO in this way results in a γ-cover of X. QED

3 Longer games and player TWO

Fix an ordinal α. Then the game $\mathsf{G}_1^{\alpha}(\mathcal{A},\mathcal{B})$ has α innings and is played as follows. In inning β ONE first chooses an $O_\beta \in \mathcal{A}$, and then TWO responds with a $T_\beta \in O_\beta$. A play

$$O_0, T_0, \ldots, O_\beta, T_\beta, \ldots,\ \beta < \alpha$$

is won by TWO if $\{\, T_\beta : \beta < \alpha \,\}$ is in $\mathcal{B}$; else, ONE wins.

In this notation the game $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ is $\mathsf{G}_1^{\omega}(\mathcal{A},\mathcal{B})$. For a space X and a family $\mathcal{T}$ of subsets of X with $\cup\mathcal{T} = X$, define:

$$\mathsf{cov}_X(\mathcal{T}) = \min\{\, |\mathcal{S}| : \mathcal{S} \subseteq \mathcal{T} \text{ and } X = \cup\mathcal{S} \,\}.$$

When $X = \cup\mathcal{T}$, there is an ordinal $\alpha \leq \mathsf{cov}_X(\mathcal{T})$ such that TWO has a winning strategy in $\mathsf{G}_1^{\alpha}(\mathcal{O}(\mathcal{T}),\mathcal{O})$. In general, there is an ordinal $\alpha \leq |X|$ such that TWO has a winning strategy in $\mathsf{G}_1^{\alpha}(\mathcal{O}(\mathcal{T}),\mathcal{O})$.

$$\mathsf{tp}_{\mathsf{S}_1(\mathcal{O}(\mathcal{T}),\mathcal{O})}(X) = \min\{\, \alpha : \text{ TWO has a winning strategy in } \mathsf{G}_1^{\alpha}(\mathcal{O}(\mathcal{T}),\mathcal{O}) \,\}.$$

3.1 General properties

The proofs of the general facts in the following lemma are left to the reader.

12 Lemma.

(1) If Y is a closed subset of X then $\mathsf{tp}_{\mathsf{S}_1(\mathcal{O}(\mathcal{T}),\mathcal{O})}(Y) \le \mathsf{tp}_{\mathsf{S}_1(\mathcal{O}(\mathcal{T}),\mathcal{O})}(X)$.

(2) If α is a limit ordinal and if $\mathsf{tp}_{\mathsf{S}_1(\mathcal{O}(\mathcal{T}),\mathcal{O})}(X_n) \le \alpha$ *for each n, then* $\mathsf{tp}_{\mathsf{S}_1(\mathcal{O}(\mathcal{T}),\mathcal{O})}(\bigcup_{n<\infty} X_n) \le \alpha$.

We shall now give examples of ordinals α for which TWO has winning strategies in games of length α. First we have the following general lemma.

13 Lemma. *Let X be $\mathcal{T}$-first countable. Assume that:*

(1) $\mathcal{T}$ is up-directed;

(2) $X \notin \langle\mathcal{T}\rangle$;

(3) α is the least ordinal such that there is an element B of $\langle\mathcal{T}\rangle$ such that for any closed set $C \subset X \setminus B$ with $C \notin \mathcal{T}$, $\mathsf{tp}_{\mathsf{S}_1(\mathcal{O}(\mathcal{T}),\mathcal{O})}(C) \le \alpha$.

Then $\mathsf{tp}_{\mathsf{S}_1(\mathcal{O}(\mathcal{T}),\mathcal{O})}(X) = \omega + \alpha$.

Proof. We must show that TWO has a winning strategy for $\mathsf{G}_1^{\omega+\alpha}(\mathcal{O}(\mathcal{T}),\mathcal{O})$ and that there is no $\beta < \omega + \alpha$ for which TWO has a winning strategy in $\mathsf{G}_1^{\beta}(\mathcal{O}(\mathcal{T}),\mathcal{O})$.

To see that TWO has a winning strategy in $\mathsf{G}_1^{\omega+\alpha}(\mathcal{O}(\mathcal{T}),\mathcal{O})$, fix a B as in the hypothesis, and for each closed set F disjoint from B, fix a winning strategy τ_F for TWO in the game $\mathsf{G}_1^{\alpha}(\mathcal{O}(\mathcal{T}),\mathcal{O})$ played on F. Now define a strategy σ for TWO in $\mathsf{G}_1^{\omega+\alpha}(\mathcal{O}(\mathcal{T}),\mathcal{O})$ on X as follows: During the first ω innings, TWO covers B. Let $T_1, T_2, \dots$ be TWO's moves during these ω innings, and put $C = X \setminus \bigcup_{n=1}^{\infty} T_n$. Then C is a closed subset of X, disjoint from B. Now TWO follows the strategy τ_C in the remaining α innings, to also cover C.

To see that there is no $\beta < \omega + \alpha$ for which TWO has a winning strategy in $\mathsf{G}_1^{\beta}(\mathcal{O}(\mathcal{T}),\mathcal{O})$, argue as follows: Suppose on the contrary that $\beta < \omega + \alpha$ is such that TWO has a winning strategy σ for $\mathsf{G}_1^{\beta}(\mathcal{O}(\mathcal{T}),\mathcal{O})$ on X. We will show that there is a set $S \in \langle\mathcal{T}\rangle$ and an ordinal $\gamma < \alpha$ such that for each closed set C disjoint from S, TWO has a winning strategy in $\mathsf{G}_1^{\gamma}(\mathcal{O}(\mathcal{T}),\mathcal{O})$ on C. This gives a contradiction to the minimality of α in hypothesis 3.

We consider cases: First, it is clear that $\alpha \le \beta$, for otherwise TWO may merely follow the winning strategy on X and relativize to any closed set C to win on C in $\beta < \alpha$ innings, a contradiction. Thus, $\omega + \alpha > \alpha$. Then we have $\alpha < \omega^2$, say $\alpha = \omega \cdot n + k$. Since then $\omega + \alpha = \omega \cdot (n+1) + k$, we have that β with $\alpha \le \beta < \omega + \alpha$ has the form $\beta = \omega \cdot n + \ell$ with $\ell \ge k$. The other possibility, $\beta = \omega \cdot (n+1) + j$ for some $j < k$, does not occur because it would give $\alpha + \omega > \beta = \omega \cdot n + (\omega + j) = (\omega \cdot n + k) + (\omega + j) = \alpha + \omega + j$.

Let F be a winning strategy for TWO in $\mathsf{G}_1^{\beta}(\mathcal{O}(\mathcal{T}),\mathcal{O})$. By the second hypothesis and Theorem 6 we have $\beta > \omega$. By Theorem 4 fix an element

$S \in \langle T \rangle$ such that $B \subset S$, and for any closed set $C \subset X \setminus S$, there is an F-play $(O_1, T_1, \ldots, O_n, T_n, \ldots)$ with $S \subset (\bigcup_{n=1}^{\infty} T_n)$, and $C \cap (\bigcup_{n=1}^{\infty} T_n) = \emptyset$. Choose a closed set $C \subset X \setminus S$ with $C \notin \mathcal{T}$. This is possible by the second hypothesis. Choose an F-play $(O_1, T_1, \ldots, O_n, T_n, \ldots)$ with $S \subset (\bigcup_{n=1}^{\infty} T_n)$, and $C \cap (\bigcup_{n=1}^{\infty} T_n) = \emptyset$. This F-play contains the first ω moves of a play according to the winning strategy F for TWO in $\mathsf{G}_1^{\beta}(\mathcal{O}(\mathcal{T}), \mathcal{O})$, and using it as strategy to play this game on C, we see that it requires (an additional) $\gamma = \omega\cdot(n-1)+\ell < \alpha$ innings for TWO to win on C. Here, ℓ is fixed and the same for all such C. Thus: $\mathsf{tp}_{\mathsf{S}_1(\mathcal{O}(\mathcal{T}),\mathcal{O})}(C) \leq \gamma < \alpha$. This is in contradiction to the minimality of α. *QED*

3.2 Examples

For each n put $\mathbb{R}_n = \{ x \in \mathbb{R}^{\mathbb{N}} : (\forall m > n)(x(m) = 0) \}$. Then $\mathbb{R}_n$ is homeomorphic to $\mathbb{R}^n$ and thus is σ-compact, and n-dimensional. Thus $\mathbb{R}_\infty = \bigcup_{n=1}^{\infty} \mathbb{R}_n$ is a σ-compact strongly countable dimensional subset of $\mathbb{R}^{\mathbb{N}}$.

We shall now use the Continuum Hypothesis to construct for various infinite countable ordinals α subsets of $\mathbb{R}^{\mathbb{N}}$ in which TWO has a winning strategy in $\mathsf{G}_1^{\alpha}(\mathcal{O}(\mathcal{T}), \mathcal{O})$. The following is one of our main tools for these constructions:

14 Lemma. *If G is any G_δ-subset of $\mathbb{R}^{\mathbb{N}}$ with $\mathbb{R}_\infty \subset G$, then $G \setminus \mathbb{R}_\infty$ contains a compact nowhere dense subset C which is homeomorphic to* $[0,1]^{\mathbb{N}}$.

We call $[0,1]^{\mathbb{N}}$ the Hilbert cube. From now on assume the Continuum Hypothesis. Let $(F_\alpha : \alpha < \omega_1)$ enumerate all the finite dimensional G_δ-subsets of $\mathbb{R}^{\mathbb{N}}$, and let $(C_\alpha : \alpha < \omega_1)$ enumerate the G_δ-subsets which contain $\mathbb{R}_\infty$. Recursively choose compact sets $D_\alpha \subset \mathbb{R}^{\mathbb{N}}$, each homeomorphic to the Hilbert cube and nowhere dense, such that $D_0 \subset C_0 \setminus (\mathbb{R}_\infty \cup F_0)$, and for all $\alpha > 0$,

$$D_\alpha \subset (\cap_{\beta \leq \alpha} C_\beta) \setminus \left(\mathbb{R}_\infty \cup \left(\bigcup \{ D_\beta : \beta < \alpha \} \right) \cup \left(\bigcup_{\beta \leq \alpha} F_\beta \right) \right).$$

Version 1: For each α, choose a point $x_\alpha \in D_\alpha$ and put

$$B := \mathbb{R}_\infty \cup \{ x_\alpha : \alpha < \omega_1 \}.$$

Version 2: For each α, choose a strongly countable dimensional set $S_\alpha \subset D_\alpha$ and put

$$B := \mathbb{R}_\infty \cup \left(\bigcup \{ S_\alpha : \alpha < \omega_1 \} \right).$$

Version 3: For each α, choose a countable dimensional set $S_\alpha \subset D_\alpha$ and put

$$B := \mathbb{R}_\infty \cup \left(\bigcup \{ S_\alpha : \alpha < \omega_1 \} \right).$$

In all three versions, B is not countable dimensional: Otherwise it would be, by Tumarkin's Theorem, for some $\alpha < \omega_1$ a subset of $\bigcup_{\beta<\alpha} F_\beta$. Thus TWO has no winning strategy in the games $\mathsf{G}_1(\mathcal{O}_{\mathsf{cfd}}, \mathcal{O})$ and $\mathsf{G}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$. Also, in all three versions the elements of the family $\mathcal{C}$ of finite unions of the sets S_α are G_δ-sets in X, and in fact X is $\mathcal{C}$-first-countable. This is because the D_α's are compact and disjoint, and $\mathbb{R}^{\mathbb{N}}$ is $\mathcal{D}$-first countable, where $\mathcal{D}$ is the family of finite unions of the D_α's, and this relativizes to X.

For Version 1 TWO has a winning strategy in $\mathsf{G}_1^{\omega+1}(\mathcal{O}_{\mathsf{cfd}}, \mathcal{O})$ and in $\mathsf{G}_1^{\omega+1}(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$, and in $\mathsf{G}_1^{\omega+\omega}(\mathcal{K}, \mathcal{O})$. For Version 2 TWO has a winning strategy in $\mathsf{G}_1^{\omega+\omega}(\mathcal{O}_{\mathsf{cfd}}, \mathcal{O})$, and for Version 3 TWO has a winning strategy in $\mathsf{G}_1^{\omega+\omega}(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$.

To see this, note that in the first ω innings, TWO covers $\mathbb{R}_\infty$. Let $\{U_n : n \in \mathbb{N}\}$ be TWO's responses in these innings. Then $G = \bigcup_{n=1}^{\infty} U_n$ is an open set containing $\mathbb{R}_\infty$, and so there is an $\alpha < \omega_1$ such that:

Version 1: $B \setminus G \subseteq \{x_\beta : \beta < \alpha\}$ is a closed, countable subset of X and thus closed, zero-dimensional. In inning $\omega + 1$ TWO chooses from ONE's cover an element containing the set $B \setminus G$.

Version 2: $B \setminus G \subseteq \bigcup_{\beta<\alpha} S_\beta$. But $\bigcup_{\beta<\alpha} S_\alpha$ is strongly countable dimensional, and so TWO can cover this part of B in the remaining ω innings. By Lemma 13 TWO does not have a winning strategy in fewer then $\omega + \omega$ innings.

Version 3: $B \setminus G \subseteq \bigcup_{\beta<\alpha} S_\beta$. But $\bigcup_{\beta<\alpha} S_\alpha$ is strongly countable dimensional, and so TWO can cover this part of B in the remaining ω innings. By Lemma 13 TWO does not have a winning strategy in fewer then $\omega + \omega$ innings.

With these examples established, we can now upgrade the construction as follows: Let α be a countable ordinal for which we have constructed an example of a subspace S of $\mathbb{R}^{\mathbb{N}}$ for which $\mathsf{tp}_{\mathsf{S}_1(\mathcal{O}(\mathcal{T}),\mathcal{O})}(S) = \alpha$. Then choose inside each D_β a set C_β for which $\mathsf{tp}_{\mathsf{S}_1(\mathcal{O}(\mathcal{T}),\mathcal{O})}(C_\beta) = \alpha$. Then the resulting subset B constructed above has, by Lemma 13, $\mathsf{tp}_{\mathsf{S}_1(\mathcal{O}(\mathcal{T}),\mathcal{O})}(B) = \omega + \alpha$. In this way we obtain examples for each of the lengths $\omega \cdot n$ and $\omega \cdot n + 1$, for all finite n.

By taking topological sums and using part 2 of Lemma 12 we get examples for ω^2.

4 Conclusion

One obvious question is whether there is, under the Continuum Hypothesis, for each limit ordinal α subsets X_α and Y_α of $\mathbb{R}^{\mathbb{N}}$ such that $\mathsf{tp}_{\mathsf{S}_1(\mathcal{O}_{\mathsf{cfd}},\mathcal{O})}(X_\alpha) = \alpha$, and $\mathsf{tp}_{\mathsf{S}_1(\mathcal{O}_{\mathsf{cfd}},\mathcal{O})}(Y_\alpha) = \alpha + 1$. And the same question can be asked for $\mathsf{tp}_{\mathsf{S}_1(\mathcal{O}_{\mathsf{fd}},\mathcal{O})}$.

In [1] countable dimensionality of metrizable spaces were characterized in terms of the selective screenability game. A natural question is how $\mathsf{S}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$ and $\mathsf{S}_1(\mathcal{O}_{\mathsf{cfd}}, \mathcal{O})$ are related to selective screenability. It is clear that $\mathsf{S}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O}) \Rightarrow \mathsf{S}_1(\mathcal{O}_{\mathsf{cfd}}, \mathcal{O})$. The relationship among these two classes and selective screenability is further investigated in [2] where it is shown, for example, that $\mathsf{S}_1(\mathcal{O}_{\mathsf{cfd}}, \mathcal{O})$ implies selective screenability, but the converse does not hold. Thus, these two classes are new classes of weakly infinite dimensional spaces.

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