On F-planar mappings of spaces with affine connections

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Abstract. In this paper we study F-planar mappings of n-dimensional or infinitely dimensional spaces with a torsion-free affine connection. These mappings are certain generalizations of geodesic and holomorphically projective mappings.
Here we make fundamental equations on F-planar mappings for dimensions \( n > 2 \) more precise.

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Introduction

In many papers geodesic mappings and their generalizations, like quasi-geodesic, holomorphically-projective, F-planar, 4-planar, mappings, were considered. One of the basic tasks was and is the derivation of the fundamental equations of these mappings. They were shown in the most various ways, see [1]-[7].

Unless otherwise specified, all spaces, connections and mappings under consideration are differentiable of a sufficiently high class. The dimension \( n \) of the spaces being considered is higher than two, as a rule. This fact is not specially stipulated. All spaces are assumed to be connected.

Here we show a method that simplifies and generalizes many of the results. Our results are valid also for infinite dimensional spaces with Banach bases (\( n = \infty \)).
1 F-planar curves

We consider an $n$-dimensional ($n > 2$) or infinite dimensional ($n = \infty$) space $A_n$ with a torsion-free affine connection $\nabla$, and an affinor structure $F$, i.e. a tensor field of type $(1,1)$.

If $n = \infty$ we assume that $A_n$ is locally homeomorphic to a Banach space $E_\infty$. In connection with local studies we assume the existence of a coordinate neighbourhood $U$ in the Euclidean space $E_n$, resp. $U \subset E_\infty$.

1 Definition (J. Mikeš, N.S. Sinyukov [4]). A curve $\ell$, which is given by the equations

$$\ell = \ell(t), \quad \lambda(t) = d\ell(t)/dt \neq 0, \quad t \in I$$

where $t$ is a parameter, is called $F$-planar, if its tangent vector $\lambda(t_0)$, for any initial value $t_0$ of the parameter $t$, remains, under parallel translation along the curve $\ell$, in the distribution generated by the vector functions $\lambda$ and $F\lambda$ along $\ell$.

In particular, if $F = \varrho I$ we obtain the definition of a geodesic parametrized by an arbitrary parameter, see [4]. Here $\varrho$ is a function and $I$ is the identity operator.

In accordance with this definition, $\ell$ is $F$-planar if and only if the following condition holds [4]:

$$\nabla_{\lambda(t)} \lambda(t) = \varrho_1(t) \lambda(t) + \varrho_2(t) F\lambda(t),$$

where $\varrho_1$ and $\varrho_2$ are some functions of the parameter $t$.

2 F-planar mappings between two spaces with affine connection

We suppose two spaces $A_n$ and $\tilde{A}_n$ with torsion-free affine connections $\nabla$ and $\tilde{\nabla}$, respectively. Affine structures $F$ and $\tilde{F}$ are defined on $A_n$, resp. $\tilde{A}_n$.

2 Definition (J. Mikeš, N.S. Sinyukov [4]). A diffeomorphism $f: A_n \to \tilde{A}_n$ between two manifolds with affine connections is called $F$-planar if any $F$-planar curve in $A_n$ is mapped onto an $\tilde{F}$-planar curve in $\tilde{A}_n$.

Important convention. Due to the diffeomorphism $f$ we always suppose that $\nabla$, $\tilde{\nabla}$, and the affinors $F$, $\tilde{F}$ are defined on $A_n$. Moreover, we always identify a given curve $\ell: I \to A_n$ and its tangent vector function $\lambda(t)$ with their images $\tilde{\ell} = f \circ \ell$ and $\tilde{\lambda} = f_*(\lambda(t))$ in $\tilde{A}_n$.

Two principally different cases are possible for the investigation:

a) $\tilde{F} = a F + b I$;
b) \( \bar{F} \neq aF + bI \), \hspace{1cm} (4)

where \( a, b \) are some functions.

Naturally, case a) characterizes \( F \)-planar mappings which preserve \( F \)-structures. In case b) the structures of \( F \) and \( \bar{F} \) are essentially distinct. The following holds.

3 Theorem. An \( F \)-planar mapping \( f \) from \( A_n \) onto \( \bar{A}_n \) preserve \( F \)-structures and is characterized by the following condition

\[
P(X,Y) = \psi(X)Y + \psi(Y)X + \varphi(X)FY + \varphi(Y)FX \hspace{1cm} (5)
\]

for any vector fields \( X, Y \), where \( P \) is the deformation tensor field of \( f \), \( \psi, \varphi \) are some linear forms.

Let us recall that on each tangent space \( T_xA_n \), \( P(X,Y) \) is a symmetric bilinear mapping \( T_xA_n \times T_xA_n \to T_xA_n \) and a tensor field of type \( \frac{1}{2} \).

Theorem 3 was proved by J. Mikeš and N. Sinyukov [4] for finite dimension \( n > 3 \). Here we can show a more rational proof of this Theorem for \( n > 3 \) and also a proof for \( n = 3 \). We show a counter example for \( n = 2 \).

3 F-planar mappings which preserve \( F \)-structures

First we prove the following proposition

4 Theorem. An \( F \)-planar mapping \( f \) from \( A_n \) onto \( \bar{A}_n \) which preserves \( F \)-structures is characterized by condition (5).

In the sequel we shall need the following lemma:

5 Lemma. Let \( V \) be an \( n \)-dimensional vector space, \( Q: V \times V \to V \) be a symmetric bilinear mapping and \( F: V \to V \) a linear mapping. If, for each vector \( \lambda \in V \)

\[
Q(\lambda, \lambda) = \varrho_1(\lambda)\lambda + \varrho_2(\lambda)F(\lambda) \hspace{1cm} (6)
\]

holds, where \( \varrho_1(\lambda), \varrho_2(\lambda) \) are functions on \( V \), then there are linear forms \( \psi \) and \( \varphi \) such that the condition

\[
Q(X,Y) = \psi(X)Y + \psi(Y)X + \varphi(X)F(Y) + \varphi(Y)F(X) \hspace{1cm} (7)
\]

holds for any \( X, Y \in V \).

PROOF. Formula (6) has the following coordinate expression

\[
Q^h_{ab}\lambda^a\lambda^b = \varrho_1(\lambda)\lambda^h + \varrho_2(\lambda)F^h_{\alpha}\lambda^\alpha, \hspace{1cm} (8)
\]

where \( \lambda^i, F^h_i, Q^h_{ij} \) are the components of \( \lambda, F, Q \).
By multiplying (8) with $\lambda^j F^i_\delta \lambda^\alpha$ and antisymmetrizing the indices $h$, $i$ and $j$ we obtain
\[
\{ Q^{[h}_{(\alpha\beta}, \delta], F^{j]}_{\delta}\} \lambda^\alpha \lambda^\beta \lambda^\gamma \lambda^\delta = 0,
\]
where square brackets denote the alternation of indices. The term in curly brackets does not depend on $\lambda$ and (9) holds for any vector $\lambda \in V$, therefore
\[
Q^{[h}_{(\alpha\beta}, \delta], F^{j]}_{\delta]} = 0 \quad (10)
\]
holds, where the round brackets denote symmetrization of indices.

It is natural to assume that $F^i_\delta \neq a \delta_i^h$ with $a = \text{const.}$ By virtue of this there exist some vectors $\xi^h$ such that $\xi^\alpha F^h_{\alpha} \neq b \xi^h$, $b = \text{const.}$ Introducing $P^h_\alpha \equiv P^h_{\alpha \xi^\alpha}$, $P^h_\alpha \equiv P^h_{\alpha \xi^\alpha}$ and $F^h_\alpha \equiv F^h_{\alpha \xi^\alpha}$, we contract (10) with $\xi^\alpha \xi^\beta \xi^\gamma \xi^\delta$. Since $F^h \neq b \xi^h$, we obtain $P^h = 2a \xi^h + 2b F^h$, where $a$, $b$ are certain constants. Contracting (10) with $\xi^\alpha \xi^\beta \xi^\gamma \xi^\delta$, and taking into account the preceding, we have $P^h = a \delta_i^h + b F^h_i + a_i \xi^h + b_i F^h_i$, where $a_i$, $b_i$ are some components of linear forms. Analogously, contracting (10) with $\xi^\gamma \xi^\delta$, we have
\[
Q^{[h}_{(i\beta}, \delta], F^{j]}_{\delta]} = 0 \quad (11)
\]
where $\psi_i$, $\varphi_i$ are components of a 1-form $\psi$, $\varphi$ defined on $V$, and $a_{ij}$, $b_{ij}$ are components of a symmetric 2-form defined on $V$.

In case that $a_{ij} = b_{ij} = 0$, evidently from (11) we obtain formula (7).

Now we will suppose that either $a_{ij} \neq 0$, or $b_{ij} \neq 0$. Since $\xi^h$ and $F^h$ are noncollinear, it is evident that
\[
\xi^h a_{ij} + F^h b_{ij} \neq 0. \quad (12)
\]

Formula (10) by virtue of (11) has the form
\[
\Omega^{[h}_{(i\beta}, \delta], F^{j]}_{\delta]} = 0, \quad (13)
\]
where $\Omega^{[h}_{(i\beta}, \delta], F^{j]}_{\delta]} \equiv (\xi^h a_{\alpha\beta} + F^h b_{\alpha\beta}) \delta_i^\gamma - (\xi^\alpha a_{\alpha\beta} + F^h b_{\alpha\beta}) \delta^h_\gamma$. It is possible to show that there exists some vector $\varepsilon^h$ for which $\Omega^{[h}_{(i\beta}, \varepsilon^\alpha \varepsilon^\beta \varepsilon^\gamma \varepsilon^\delta] = 0$, otherwise (12) would be violated.

Contracting (13) with $\varepsilon^\alpha \varepsilon^\beta \varepsilon^\gamma \varepsilon^\delta$, we have $F^h_{\alpha \varepsilon^\alpha} = a \xi^h + b F^h + c \xi^h$, with $a$, $b$, $c$ being constants. Analogously, contracting (13) with $\varepsilon^\beta \varepsilon^\gamma \varepsilon^\delta$, we obtain that $F^h_i$ is represented in the following manner:
\[
F^h_i = a \delta_i^h + a_i \xi^h + b_i F^h + c_i \xi^h, \quad (14)
\]
where $a_i$, $b_i$, $c_i$ are components of 1-forms.
Formula (13) by virtue of (14) has the form
\[ \omega^{[hi]}_{(\alpha\beta\gamma)} = 0, \]  
(15)

where
\[ \omega^{hi}_{\alpha\beta\gamma} \overset{\text{def}}{=} \xi^{[hi]} F^{i}(a_{(\alpha\beta)b_{\gamma}} - b_{(\alpha\beta)a_{\gamma}}) + \xi^{[hi]a_{(\alpha\beta)c_{\gamma}}} + F^{[hi]a_{(\alpha\beta)c_{\gamma}}}. \]

a) If \( n > 3 \) then \( \omega^{hi}_{\alpha\beta\gamma} = 0 \) follows from (13), and because \( \xi^{h}, F^{h} \) and \( \varepsilon^{h} \) are linear independent, we obtain \( a_{(\alpha\beta)c_{\gamma}} = 0 \) and \( b_{(\alpha\beta)c_{\gamma}} = 0. \) Therefore \( c_{i} = 0 \) and
\[ F^{h}_{i} = a_{i} \delta^{h} + a_{i} \xi^{h} + b_{i} F^{h}. \]
(16)

b) If \( n = 3 \) the matrix \( F^{h}_{i} \) has always the previous form (16) while \( \xi^{h}, F^{h} \) and \( \varepsilon^{h} \) are not linear dependent.

Then formula (13) becomes (15), whereas \( \omega^{hi}_{\alpha\beta\gamma} \overset{\text{def}}{=} \xi^{[hi]} F^{i}(a_{(\alpha\beta)b_{\gamma}} - b_{(\alpha\beta)a_{\gamma}}). \)

For \( n > 2 \) it follows \( \omega^{hi}_{\alpha\beta\gamma} = 0 \) and consequently
\[ a_{(\alpha\beta)b_{\gamma}} = b_{(\alpha\beta)a_{\gamma}}. \]  
(17)

If \( a_{\alpha} \) and \( b_{\alpha} \) are linear independent, then from (17) we obtain
\[ a_{ij} = a_{i}(\omega_{j}) \quad \text{and} \quad b_{ij} = b_{i}(\omega_{j}), \]
where \( \omega_{i} \) are components of a 1-form. Afterwards it is possible to show that on the basis of (16) formula (11) assumes the following form
\[ Q^{h}_{ij} = (\psi_{i} - a_{\omega_{j}})\delta^{h}_{j} + (\psi_{j} - a_{\omega_{j}})\delta^{h}_{i} + (\varphi_{i} + a_{\omega_{j}})F^{h}_{j} + (\varphi_{j} + a_{\omega_{j}})F^{h}_{i}, \]
i.e. formula (7) also holds.

Now there remains the case that \( a_{\alpha} \) and \( b_{\alpha} \) are linear dependent. For example, \( b_{\alpha} = \alpha a_{\alpha}, \alpha \neq 0. \) Then from (17) follows \( b_{\alpha\beta} = \alpha a_{\alpha\beta}. \) We denote \( \Lambda^{h} = \xi^{h} + \alpha F^{h}, \omega_{i} = \psi_{i} + \alpha \varphi_{i}, \omega_{ij} = a_{ij} + a_{i}(\varphi_{j}), \) from (11) and (16) we obtain that \( Q^{h}_{ij} \) and \( F^{h}_{i} \) are represented by
\[ Q^{h}_{ij} = \psi_{i}\delta^{h}_{j} + \psi_{j}\delta^{h}_{i} + \Lambda^{h}\omega_{ij} \quad \text{and} \quad F^{h}_{i} = a_{i}\delta^{h} + \Lambda^{h}a_{i}. \]
(18)

Then formula (8) appears in the following way
\[ \Lambda^{h} (\omega_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta} - \varrho_{2}(\lambda) a_{\alpha}\lambda^{\alpha}) = \lambda^{h} (\varrho_{1}(\lambda) + a \varrho_{2}(\lambda) - 2\psi_{\lambda}\lambda^{\alpha}). \]

From this it follows that
\[ \omega_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta} = \varrho_{2}(\lambda) a_{\alpha}\lambda^{\alpha}, \quad \forall \lambda^{h} \neq \alpha \Lambda^{h}. \]

By simple analysis we obtain that \( \omega_{ij} = a_{i}(\sigma_{j}), \) where \( \sigma_{i} \) are components of a 1-form.

Then due to (18) we have \( Q^{h}_{ij} = (\psi_{i} - a_{\sigma_{i}})\delta^{h}_{j} + (\psi_{j} - a_{\sigma_{j}})\delta^{h}_{i} + \sigma_{i}F^{h}_{j} + \sigma_{j}F^{h}_{i}. \)

Evidently Lemma 5 is proved.
Proof of Theorem 4. It is obvious that geodesics are a special case of \( F \)-planar curves. Let a geodesic in \( A_n \), which satisfies the equations (1) and \( \nabla_\lambda \lambda = 0 \), be mapped onto an \( F \)-planar curve in \( \tilde{A}_n \), which satisfies equations (1) and
\[
\tilde{\nabla}_\lambda \lambda = \tilde{\varrho}_1(t) \lambda + \tilde{\varrho}_2(t) F \lambda.
\]
Here \( \tilde{\varrho}_1, \tilde{\varrho}_2 \) are functions of the parameter \( t \).

Because the deformation tensor satisfies
\[
P(\lambda, \lambda) = \nabla_\lambda \lambda - \nabla_\lambda \lambda,
\]
we have
\[
P(\lambda(t), \lambda(t)) = \tilde{\varrho}_1(t) \lambda + \tilde{\varrho}_2(t) F \lambda.
\]
It follows from the previous formula that in each point \( x \in A_n \)
\[
P(\lambda, \lambda) = \varrho_1(\lambda) \lambda + \varrho_2(\lambda) F \lambda.
\]
for each tangent vector \( \lambda \in T_x; \varrho_1(\lambda), \varrho_2(\lambda) \) are functions dependent on \( \lambda \).

Based on Lemma 5 it follows, that there exist linear forms \( \psi \) and \( \varphi \), for which formula (5) holds. \( \square \)

4 F-planar mappings which do not preserve F-structures

We now assume that the structures \( F \) and \( \tilde{F} \) are essentially distinct, i.e.
\[
\tilde{F}^h_i \neq a \delta_i^h + b F_i^h.
\]

a) It is obvious, that geodesics are a special case of \( F \)-planar curves. Let a geodesic in \( A_n \), which satisfies the equations (1) and \( \nabla_\lambda \lambda = 0 \), be mapped onto an \( \tilde{F} \)-planar curve in \( \tilde{A}_n \), which satisfies the equations (1) and
\[
\tilde{\nabla}_\lambda \lambda = \tilde{\varrho}_1(t) \lambda + \tilde{\varrho}_2(t) F \lambda.
\]
Here \( \tilde{\varrho}_1, \tilde{\varrho}_2 \) are functions of the parameter \( t \).

For the deformation tensor we have
\[
P(\lambda(t), \lambda(t)) = \tilde{\varrho}_1(t) \lambda + \tilde{\varrho}_2(t) F \lambda.
\]
It follows from the previous formula that in each point \( x \in A_n \)
\[
P(\lambda, \lambda) = \varrho_1(\lambda) \lambda + \varrho_2(\lambda) F \lambda.
\]
for each tangent vector \( \lambda \in T_x; \varrho_1(\lambda), \varrho_2(\lambda) \) are functions dependent on \( \lambda \).

Based on Lemma 5 it follows, that there exist linear forms \( \psi \) and \( \varphi \), for which formula
\[
P(X, Y) = \psi(X) Y + \psi(Y) X + \varphi(X) \tilde{F} Y + \varphi(Y) \tilde{F} X
\]
(19)
holds.

b) Let a special $F$-planar curve in $A_n$, which satisfies the equations (1) and $\nabla_{\lambda} \lambda = F \lambda$, be mapped onto an $\bar{F}$-planar curve in $\bar{A}_n$, which satisfies the equations (1) and

$$\bar{\nabla}_{\lambda} \lambda = \bar{g}_1(t) \lambda + \bar{g}_2(t) \bar{F} \lambda.$$  

Here $\bar{g}_1, \bar{g}_2$ are functions of the parameter $t$.

For the deformation tensor we have $P(\lambda(t), \lambda(t)) = F \lambda + \bar{g}_1(t) \lambda + \bar{g}_2(t) \bar{F} \lambda$.

It follows from the previous formula that in each point $x \in A_n$

$$P(\lambda, \lambda) = F \lambda + \varrho_1(\lambda) \lambda + \varrho_2(\lambda) \bar{F} \lambda.$$ 

for each tangent vector $\lambda \in T_x$; $\varrho_1(\lambda), \varrho_2(\lambda)$ are functions dependent on $\lambda$.

Applying (19) we obtain

$$F \lambda = \tilde{\varrho}_1(\lambda) \lambda + \tilde{\varrho}_2(\lambda) \bar{F} \lambda.$$ 

Analyzing this expression like in Lemma 5 we convince ourselves that formula (3) holds. In this way we prove

6 Theorem. Any $F$-planar mapping of a space with affine connection $A_n$ onto $\bar{A}_n$ preserves $F$-structures.

5 $F$-planar mappings for dimension $n = 2$

It is easy to see that for $n = 2$ Theorems 3 and 4 do not hold. If they would hold, the functions $\varrho_1$ and $\varrho_2$, appearing in (6), would be linear in $\lambda$.

In the case

$$F_i^h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

for example, these functions have the forms

$$\varrho_1(\lambda) = \frac{\lambda^1 P^1_{\alpha \beta} \lambda^\alpha \lambda^\beta + \lambda^2 P^2_{\alpha \beta} \lambda^\alpha \lambda^\beta}{(\lambda^1)^2 + (\lambda^2)^2} \quad \text{and} \quad \varrho_2(\lambda) = \frac{\lambda^1 P^2_{\alpha \beta} \lambda^\alpha \lambda^\beta - \lambda^2 P^1_{\alpha \beta} \lambda^\alpha \lambda^\beta}{(\lambda^1)^2 + (\lambda^2)^2},$$

which are not linear in general.

On the other hand an arbitrary diffeomorphism from $A_2$ onto $\bar{A}_2$ is an $F$-planar mapping with (6) being valid for the above functions $\varrho_1$ and $\varrho_2$.

References

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