Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **44** (2024) no. 1, 85–98. doi:10.1285/i15900932v44n1p85

(m, ρ)-quasi-Einstein metrics on (κ, μ)-almost coKähler manifolds

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Received: 25.11.2023; accepted: 07.06.2024.

Abstract. In this article, a criterion for non-existence of closed (m, ρ)-quasi-Einstein metrics on (κ, μ)-almost coKähler manifolds is established. A similar result is stated for gradient (m, ρ)-quasi-Einstein metrics on three-dimensional (κ, μ)-almost coKähler manifolds.

Keywords: (m, ρ)-quasi-Einstein metric, ρ-Einstein soliton, gradient ρ-Einstein metric, gradient (m, ρ)-quasi-Einstein metric, (κ, μ)-almost coKähler manifold, Lie group.

MSC 2020 classification: primary 53C15, secondary 53D25

1 Introduction

The concept of (m, ρ)-quasi-Einstein metric was firstly introduced by Huang and Wei[24]. Later on many geometers considered (m, ρ)-quasi-Einstein metrics in different contexts ([8],[21],[28],[30],[32]). The concept of (gradient) (m, ρ)-quasi-Einstein metric can be regarded as an extension of the one of (gradient) Einstein metric.

Though, recently, (m, ρ)-quasi-Einstein metrics on almost coKähler manifolds have been studied by Wang [32], then by De et al.([15]), still there are

[i]Urmila Biswas is financially supported by UGC, India, Ref. ID-201610057626.
[ii]Maria Falcitelli is partially supported by PRIN 2022MWPMAB " Interactions between Geometric Structures and Function Theories".

some points left out for further investigations. In this article we go on with a careful study of (κ,μ)-almost coKähler manifolds equipped with (m,ρ)-quasi-Einstein metrics.

In 1982, as an initiative to solve the famous Poincaré conjecture, Hamilton [23] introduced the concept of Ricci flow given by

$$\frac{\partial}{\partial t}\mathfrak{g} = -2\mathcal{S},$$

where $\mathfrak{g}$ and $\mathcal{S}$ denote a Riemannian metric and its Ricci tensor, respectively. Self similar solutions, up to diffeomorphisms and scalings, of a Ricci flow are known as Ricci solitons and are given by

$$\pounds_X\mathfrak{g} + 2\mathcal{S} = 2\lambda\mathfrak{g},$$

where $\pounds_X$ indicates the Lie-derivative operator along the potential vector field X on a manifold $\mathcal{N}$ and λ is a real number, called the soliton constant. If instead of a constant, λ is considered as a smooth function, then the Ricci solitons are named almost Ricci solitons[1].

The notion of Ricci soliton was generalized by Nurowski et al.[25] using the equation

$$(\pounds_X\mathfrak{g}) + 2a_1\mathcal{S} + 2b_1X^b \otimes X^b = 2\lambda\mathfrak{g}, \tag{1.1}$$

where a_1, b_1 are real constants and X^b is the 1-form $\mathfrak{g}$-associated with X, namely $X^b(G_1) = \mathfrak{g}(X, G_1)$, for any vector field G_1 on $\mathcal{N}$. Particular types of generalized Ricci solitons have been studied by several authors([19],[29]) in different perspectives. If $X = D\mathfrak{f}$, for a smooth function $\mathfrak{f}$ on $\mathcal{N}$, D being the gradient operator, then generalized Ricci solitons are called gradient generalized Ricci solitons.

We recall the definition of (m,ρ)-quasi-Einstein metric, also named (m,ρ)-quasi-Einstein soliton [15].

Definition 1.1. *The metric $\mathfrak{g}$ of a Riemannian manifold $\mathcal{N}$ is called an (m,ρ)-quasi-Einstein metric if there exist three real numbers m, ρ, λ, $m > 0$, and a vector field X on $\mathcal{N}$ such that*

$$(\pounds_X\mathfrak{g}) + 2\mathcal{S} = \frac{2}{m}X^b \otimes X^b + 2(\rho\mathfrak{r} + \lambda)\mathfrak{g}, \tag{1.2}$$

where $\mathfrak{r}$ denotes the scalar curvature. If X^b is closed, then the $\mathfrak{g}$ is called a closed (m,ρ)-quasi-Einstein metric.

In this article, a triplet $(\mathfrak{g}, X, \lambda)$ satisfying (1.2) is named an (m,ρ)-quasi-Einstein structure.

The metric $\mathfrak{g}$ *is called a* ρ*-Einstein soliton if there exist two real numbers* ρ, λ *and a vector field* X *such that*

$$(\pounds_X\mathfrak{g}) + 2\mathcal{S} = 2(\rho\mathfrak{r}+\lambda)\mathfrak{g}. \tag{1.3}$$

Roughly speaking, ρ-Einstein solitons are considered as (m,ρ)-quasi-Einstein metrics such that $m=\infty$.

Definition 1.2. *The metric* $\mathfrak{g}$ *of a Riemannian manifold* $\mathcal{N}$ *is called a gradient* (m,ρ)*-quasi-Einstein metric if there exist a smooth function* $\mathfrak{f}$ *on* $\mathcal{N}$ *and three real numbers* m, ρ, λ, $m>0$, *such that*

$$Hess\mathfrak{f} + \mathcal{S} = \frac{1}{m}d\mathfrak{f}\otimes d\mathfrak{f} + \beta\mathfrak{g}, \tag{1.4}$$

where $\beta=\rho\mathfrak{r}+\lambda$. *A triplet* $(\mathfrak{g},\mathfrak{f},\lambda)$ *satisfying* (1.4) *is named a gradient* (m,ρ)*-quasi-Einstein structure.*
A gradient ρ*-Einstein soliton* $(\mathfrak{g},\mathfrak{f},\lambda)$ *is a solution of the equation*

$$Hess\mathfrak{f} + \mathcal{S} = \beta\mathfrak{g}, \tag{1.5}$$

where $\beta=\rho\mathfrak{r}+\lambda$ *([14],[31]). If* $\rho=\frac{1}{2}$, *a gradient* ρ*-Einstein soliton is called gradient Einstein soliton [7].*

Cosymplectic manifolds, introduced by Blair, were studied by Goldberg and Yano and many others (see [3], [20], [22] and the references therein).
A cosymplectic manifold is also named a coKähler manifold. Almost cosymplectic manifolds, also named almost coKähler manifolds, were introduced in [22]. Then several authors developed the study of these manifolds, providing explicit examples and curvature properties ([11, 12, 16, 18, 26, 27]). We also refer to [5] for an exhaustive overview on the theory of (almost) coKähler manifolds.
The aim of this article is the study of (m,ρ)-quasi-Einstein metrics and of ρ-Einstein solitons on almost coKähler manifolds, assuming that the curvature satisfies the (κ,μ)-condition.

The paper is organized as follows. In Section 2 we synthesize well-known properties of almost coKähler manifolds. Particular attention to the (κ,μ)-condition is paid and recent results on (m,ρ)-quasi-Einstein metrics are recalled. These results help in proving that the metric tensor of a (κ,μ)-almost coKähler manifold $\mathcal{N}^{2n+1}$ cannot support neither a closed (m,ρ)-quasi-Einstein nor a ρ-Einstein structure, provided that $\kappa<0$. Finally we give an alternative proof of similar results dealing with gradient structures on three-dimensional manifolds.

Throughout this paper all the manifolds are assumed to be connected and of class $\mathcal{C}^\infty$, as well as tensor fields, in particular functions, are $\mathcal{C}^\infty$ smooth.

2 (κ,μ)-almost coKähler manifolds

A $(2n+1)$-dimensional smooth manifold $\mathcal{N}^{2n+1}$ is called an almost contact metric manifold if there exist a Riemannian metric $\mathfrak{g}$, a 1-form η, a $(1,1)$-tensor field ϕ and a vector field ξ such that[2]

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \mathfrak{g}(\phi G_1, \phi H_1) + \eta(G_1)\eta(H_1) = \mathfrak{g}(G_1, H_1),$$

where η is defined by $\mathfrak{g}(G_1,\xi) = \eta(G_1)$, for any vector field G_1 on $\mathcal{N}^{2n+1}$ and I is the identity endomorphism. Then $(\phi,\xi,\eta,\mathfrak{g})$ is named almost contact metric structure and ξ is the Reeb vector field. For this structure we have

$$\phi\xi = 0, \quad \eta\circ\phi = 0 \quad \text{and} \quad \mathfrak{g}(\phi G_1, H_1) = -\mathfrak{g}(G_1, \phi H_1),$$

for any vector fields G_1 and H_1 on $\mathcal{N}^{2n+1}$. On an almost contact metric manifold $\mathcal{N}^{2n+1}$, we can define the 2-form Φ by

$$\Phi(G_1, H_1) = \mathfrak{g}(G_1, \phi H_1),$$

for any vector fields G_1 and H_1 on $\mathcal{N}^{2n+1}$. If $d\eta = 0$ and $d\Phi = 0$, then $\mathcal{N}^{2n+1}$ is called an almost coKähler manifold [10, 11, 13]. These manifolds set up the Chinea-Gonzalez class $C_2 \oplus C_9$ [9]. Almost coKähler manifolds whose Reeb vector field ξ is ∇-parallel are also named K-cosymplectic manifolds and set up the class C_2.

Given an almost contact metric manifold $\mathcal{N}^{2n+1}$, one defines the $(1,1)$-tensor fields h, h' putting $h = \frac{1}{2}\pounds_\xi\phi$, $h' = h\circ\phi$.

If $\mathcal{N}^{2n+1}$ is almost coKähler, both the operators h, h' are symmetric and satisfy the properties

$$h\xi = 0, \quad tr(h) = 0, \quad tr(h') = 0, \quad h\phi = -\phi h, \quad \nabla\xi = h', \tag{2.1}$$

$$\phi l\phi = l + 2h^2, \quad trh^2 = -\mathcal{S}(\xi,\xi), \tag{2.2}$$

where $l = \mathcal{R}(.,\xi)\xi$ denotes the Jacobi operator and $\mathcal{S}$ indicates the Ricci tensor ([16], [26]).

In [4] Blair et al. introduced the notion of (κ,μ)-nullity distribution associated with an almost contact metric manifold, κ, μ being real numbers. In particular, a (κ,μ)-almost coKähler manifold is an almost coKähler manifold such that the Reeb vector field belongs to the (κ,μ)-nullity distribution, that is the curvature $\mathcal{R}$ satisfies

$$\mathcal{R}(G_1,H_1)\xi = \kappa\big(\eta(H_1)G_1 - \eta(G_1)H_1\big) + \mu\big(\eta(H_1)hG_1 - \eta(G_1)hH_1\big), \tag{2.3}$$

for any vector fields G_1, H_1 on $\mathcal{N}^{2n+1}$. If $\mu=0$, then ξ belongs to the κ-nullity distribution and $\mathcal{N}^{2n+1}$ is called an $N(\kappa)$-almost coKähler manifold. More generally, one can consider (κ,μ)-spaces, namely almost contact metric manifolds whose Reeb vector field belongs to the (κ,μ) distribution, κ, μ denoting smooth functions varying exclusively in the direction of ξ.

We refer to ([6], [12]) for a detailed study of these spaces. This paper deals with manifolds satisfying (2.3), with κ, μ constants.
So, let $\mathcal{N}^{2n+1}$ be a (κ,μ)-almost coKähler manifold. Then, one has

$$h^2=\kappa\phi^2. \tag{2.4}$$

It follows that $\kappa\leq 0$ and $\kappa=0$ if and only if ξ is ∇-parallel. Moreover, the Ricci operator Q satisfies

$$Q\xi=2n\kappa\xi \tag{2.5}$$

and if $\kappa<0$, Q acts as

$$Q(G_1)=2n\kappa\eta(G_1)\xi+\mu hG_1. \tag{2.6}$$

In particular, if $\kappa<0$, $\mathcal{N}^{2n+1}$ has constant scalar curvature $\mathfrak{r}=2n\kappa$ and the covariant derivative ∇h, ∇ denoting the Levi-Civita connection, satisfies

$$\begin{aligned}(\nabla_{G_1}h)H_1-(\nabla_{H_1})G_1=\kappa\big(&\eta(H_1)\phi G_1-\eta(G_1)\phi H_1\\&+2\mathfrak{g}(\phi G_1,H_1)\xi\big)+\mu\big(\eta(G_1)h'H_1-\eta(H_1)h'G_1\big),\end{aligned} \tag{2.7}$$

for every vector fields G_1,H_1 ([6], [18], [33]).
We also recall the following results, that will be utilized in section 3, 4.

Theorem 2.1. *([11]) An $N(\kappa)$-almost coKähler manifold with $\kappa<0$ is locally isomorphic to a solvable non-nilpotent Lie-group $\mathfrak{g}_\sigma$, $\sigma=\sqrt{-k}$, endowed with an almost coKähler structure.*

Theorem 2.2. *([15]) Let the metric $\mathfrak{g}$ of a (κ,μ)-almost coKähler manifold $\mathcal{N}^{2n+1}$ be a gradient (m,ρ)-quasi-Einstein metric. Then, if $\kappa<0$, $\mathcal{N}^{2n+1}$ reduces to an $N(\kappa)$-almost coKähler manifold.*

Theorem 2.3. *([15]) There does not exist a gradient (m,ρ)-quasi-Einstein structure $(\mathfrak{g},\mathfrak{f},\lambda)$ with $D\mathfrak{f}=(\xi\mathfrak{f})\xi$, for some non-constant smooth function $\mathfrak{f}$, on a compact (κ,μ)-almost coKähler manifold $\mathcal{N}^{2n+1}$ with $n>1$, $\kappa<0$.*

3 (m,ρ)-quasi-Einstein metrics on $(2n+1)$-dimensional (κ,μ)-almost coKähler manifolds

In this section, we prove the non-existence of closed (m,ρ)-quasi-Einstein metrics and of closed ρ-Einstein solitons on a (κ,μ)-almost coKähler manifold. In this connection, it should be mentioned that in [16] and [18], Endo proved the non-existence of (κ,μ)-almost coKähler Einstein manifolds.

Theorem 3.1. *Let $(\mathcal{N}^{2n+1},\phi,\xi,\eta,\mathfrak{g})$ be a (κ,μ)-almost coKähler manifold, $\kappa<0$. Then $\mathfrak{g}$ cannot be a closed (m,ρ)-quasi-Einstein metric.*

Proof. Assume that $(\mathfrak{g},X,\lambda)$ is a closed (m,ρ)-quasi-Einstein structure. By (1.2) and (2.6) for every $G_1\in\chi(\mathcal{N})$ one has

$$\begin{aligned}\nabla_{G_1}X=&-2n\kappa\eta(G_1)\xi-\mu hG_1+\frac{1}{m}\mathfrak{g}(X,G_1)X\\&+(\rho\mathfrak{r}+\lambda)G_1.\end{aligned}\tag{3.1}$$

Applying (3.1) and (2.7) one gets

$$\begin{aligned}R(G_1,H_1)X=&(2nk-\mu^2)(\eta(G_1)h'(H_1)-\eta(H_1)h'(G_1))\\&+\mu\kappa(\eta(G_1)\phi H_1-\eta(H_1)\phi G_1+2\mathfrak{g}(\phi H_i,G_1)\xi)\\&+\frac{1}{m}(\mathfrak{g}(X,H_1)\nabla_{G_1}X-\mathfrak{g}(X,G_1)\nabla_{H_1}X).\end{aligned}\tag{3.2}$$

In particular, by (3.2) and (3.1), we obtain

$$\begin{aligned}\mathfrak{g}(R(G_1,H_1)X,\xi)=&2\kappa\mu\mathfrak{g}(\phi H_1,G_1)\\&+\frac{1}{m}(\rho\mathfrak{r}+\lambda-2n\kappa)(\mathfrak{g}(X,H_1)\eta(G_1)\\&-\mathfrak{g}(X,G_1)\eta(H_1)).\end{aligned}$$

Moreover, by (2.3) one has

$$\begin{aligned}\mathfrak{g}(R(G_1,H_1)\xi,X)=&\kappa(\eta(H_1)\mathfrak{g}(G_1,X)-\eta(G_1)\mathfrak{g}(H_1,X))\\&+\mu(\eta(H_1)\mathfrak{g}(hG_1,X)-\eta(G_1)\mathfrak{g}(hH_1,X)).\end{aligned}$$

Being $\mathfrak{g}(R(G_1,H_1)\xi,X)=-\mathfrak{g}(R(G_1,H_1)X,\xi)$, we obtain

$$\begin{aligned}&2\kappa\mu\mathfrak{g}(\phi H_1,G_1)+(\frac{1}{m}(\rho\mathfrak{r}+\lambda-2n\kappa)-\kappa)(\mathfrak{g}(X,H_1)\eta(G_1)\\&-\mathfrak{g}(X,G_1)\eta(H_1))+\mu(\eta(H_1)\mathfrak{g}(hG_1,X)-\eta(G_1)\mathfrak{g}(hH_1,X)))\quad=0.\end{aligned}\tag{3.3}$$

Given a point $x \in \mathcal{N}$, we consider $H_1 \in T_x(\mathcal{N})$, $H_1 \neq 0$ such that $\eta(H_1) = 0$. Then by (3.3), we have

$$2\kappa\mu\mathfrak{g}(H_1, H_1) = 0.$$

It follows that $\mu = 0$. Hence $\mathcal{N}$ is an $N(\kappa)$-contact manifold and (3.1), (3.2) reduce to

$$\nabla_{G_1} X = -2n\kappa\eta(G_1)\xi + \frac{1}{m}\mathfrak{g}(X, G_1)X + (\rho\mathfrak{r} + \lambda)G_1. \tag{3.4}$$

$$\begin{aligned} R(G_1, H_1)X &= 2n\kappa(\eta(G_1)h'(H_1) - \eta(H_1)h'(G_1)) \\ &\quad + \frac{1}{m}(\mathfrak{g}(X, H_1)\nabla_{G_1} X - \mathfrak{g}(X, G_1)\nabla_{H_1} X). \end{aligned} \tag{3.5}$$

By (3.4) and (3.5) one has

$$\begin{aligned} S(G_1, X) = &-\frac{2n\kappa}{m}(\mathfrak{g}(X, G_1) - \eta(X)\eta(G_1)) \\ &+ \frac{2n}{m}(\rho\mathfrak{r} + \lambda)\mathfrak{g}(X, G_1). \end{aligned}$$

Being $\mathcal{N}$ an $N(\kappa)$-manifold, we have $S = 2n\kappa\eta \otimes \eta$. It follows, for every $G_1 \in \chi(\mathcal{N})$

$$(\rho\mathfrak{r} + \lambda - \kappa)\mathfrak{g}(X, G_1) + (1 - m)\kappa\eta(X)\eta(G_1) = 0. \tag{3.6}$$

Therefore, one has

$$(\rho\mathfrak{r} + \lambda - \kappa)(X - \eta(X)\xi) = 0. \tag{3.7}$$

Assume that $\rho\mathfrak{r} + \lambda = \kappa$. By (3.3), being $\mu = 0$, we get, for $G_1, H_1 \in \chi(\mathcal{N})$

$$(m + 2n - 1)\kappa(\mathfrak{g}(X, H_1)\eta(G_1) - \mathfrak{g}(X, G_1)\eta(H_1)) = 0,$$

and being $m + 2n - 1 > 0$, one has

$$X - \eta(X)\xi = 0.$$

Therefore, taking account of (3.7), we obtain $X = f\xi$, with $f = \eta(X)$. Thus (3.4) entails

$$fh'(G_1) + G_1(f)\xi = (-2n\kappa + \frac{1}{m}f^2)\eta(G_1)\xi + (\rho\mathfrak{r} + \lambda)G_1, \quad G_1 \in \chi(\mathcal{N}). \tag{3.8}$$

Taking the inner product with ξ, we obtain

$$df = \xi(f)\eta, \quad \xi(f) = -2n\kappa + \frac{1}{m}f^2 + \rho\mathfrak{r} + \lambda.$$

Now, (3.8) reduces to

$$fh'(G_1) = (\rho\mathfrak{r} + \lambda)(G_1 - \eta(G_1)\xi), \quad G_1 \in \chi(\mathcal{N}). \tag{3.9}$$

Taking the trace in (3.9), one has

$$\rho\mathfrak{r} + \lambda = 0, \quad fh' = 0.$$

Now it follows that $X = 0$, and $S = 0$. Thus we arrive at a contradiction. This proves the theorem. $\boxed{QED}$

Remark 3.1. *By Theorems 2.1, 3.1, it follows that the metric of a solvable non-nilpotent Lie group $\mathfrak{g}_\sigma$, $\sigma = \sqrt{-k}$, endowed with an almost coKähler structure defined in [11], cannot be (m, ρ)-quasi-Einstein metric.*

Proposition 3.1. *Let $(\mathcal{N}^{2n+1}, \phi, \xi, \eta, \mathfrak{g})$ be a (κ, μ)-almost coKähler manifold, $\kappa < 0$. Then $\mathfrak{g}$ cannot be a closed ρ-Einstein soliton.*

Proof. Assume that $\mathfrak{g}$ is a closed ρ-Einstein soliton with potential vector field X. By (1.3) and (2.6), for any $G_1 \in \chi(\mathcal{N})$ one gets

$$\nabla_{G_1} X = -\mu h G_1 - 2n\kappa\eta(G_1)\xi + (2n\kappa\rho + \lambda)G_1. \tag{3.10}$$

It follows

$$\begin{aligned} R(G_1, H_1)X = {} & (2nk - \mu^2)(\eta(G_1)h'(H_1) - \eta(H_1)h'(G_1)) \\ & + \mu\kappa(\eta(G_1)\phi H_1 - \eta(H_1)\phi G_1 + 2\mathfrak{g}(\phi H_i, G_1)\xi). \end{aligned} \tag{3.11}$$

Thus, by (2.6) and (3.11) we obtain

$$\mu h X + 2n\kappa\eta(X)\xi = 0.$$

It follows $\eta(X) = 0$, so X, ξ are orthogonal. Arguing as in Theorem 3.1 and applying (3.10) one obtains

$$-2n\kappa + 2n\kappa\rho + \lambda = g(\nabla_\xi X, \xi) = -g(\nabla_\xi \xi, X) = 0,$$

and

$$\begin{aligned} \mathfrak{g}(hX, \phi G_1) = \mathfrak{g}(h'X, G_1) = \mathfrak{g}(\nabla_{G_1}\xi, X) = {} & \\ - \mathfrak{g}(\nabla_{G_1} X, \xi) = (2n\kappa - 2n\kappa\rho - \lambda)\eta(G_1) = 0, & \end{aligned}$$

for any $G_1 \in \chi(\mathcal{N})$. It follows $hX = 0$, $\kappa X = -\kappa\phi^2 X = -h^2 X = 0$. Then, we have $X = 0$ and $(\mathcal{N}, \mathfrak{g})$ is an Einstein manifold. This contradicts the hypothesis $\kappa < 0$. Hence, the proposition is proved. $\boxed{QED}$

Remark 3.2. *It is known that any $(0,\mu)$-almost coKähler manifold is a K-cosymplectic manifold. The following result of Chen [8] can be regarded as the counterpart of Proposition 3.2.*

Theorem 3.2. *([8]) Let M be a K-cosymplectic manifold. Suppose that V is a closed vector field on M such that*

$$\frac{1}{2}\pounds_V\mathfrak{g} + \mathcal{S} - \frac{1}{m}V^b \otimes V^b = \lambda\mathfrak{g},$$

$\lambda \in \mathbb{R}$, $\lambda \neq 0$. *Then M is η-Einstein.*

4 Gradient (m,ρ)-quasi-Einstein metrics on three - dimensional (κ,μ)-almost coKähler manifolds

In this section we give an alternative proof of the non-existence of gradient $(m.\rho)$-quasi-Einstein structures $(\mathfrak{g},\mathfrak{f},\lambda)$ on a (κ,μ)-almost coKähler manifold $(\mathcal{N}^3,\phi,\xi,\eta,\mathfrak{g})$. We point up that this result can be regarded as the counterpart of Theorem 2.3 dealing with manifolds of dimension $2n+1>3$. Moreover, we prove the non-existence of gradient ρ-Einstein solitons on $\mathcal{N}^3$. The proofs are based on the property that the Weyl tensor of any Riemannian manifold $(\mathcal{N}^3,\mathfrak{g})$ vanishes. Equivalently, the curvature tensor $\mathcal{R}$ acts as

$$\begin{aligned}\mathcal{R}(G_1,H_1)F_1 =&\mathfrak{g}(H_1,F_1)QG_1 - \mathfrak{g}(G_1,F_1)QH_1 + \mathcal{S}(H_1,F_1)G_1\\ &- \mathcal{S}(G_1,F_1)H_1 - \frac{\mathfrak{r}}{2}\big(\mathfrak{g}(H_1,F_1)G_1 - \mathfrak{g}(G_1,F_1)H_1\big),\end{aligned} \tag{4.1}$$

for any vector fields G_1, H_1, F_1.
Let $(\mathcal{N}^3,\phi,\xi,\eta,\mathfrak{g})$ be a (κ,μ)-almost coKähler manifold. Then Q acts as

$$Q = (\frac{\mathfrak{r}}{2} - \kappa)I + (3\kappa - \frac{\mathfrak{r}}{2})\eta\otimes\xi + \mu h. \tag{4.2}$$

Moreover, according to Theorem 2.2, if $\mathcal{N}^3$ admits a gradient (m,ρ)-quasi-Einstein metric and $\kappa<0$, then $\mu=0$ and $\mathcal{N}^3$ is a an $N(\kappa)$-almost coKähler manifold. Hence if $\kappa<0$, we have

$$\mathfrak{r} = 2\kappa, \quad Q = 2\kappa\eta\otimes\xi. \tag{4.3}$$

$$\begin{aligned}\mathcal{R}(G_1,H_1)F_1 =& -\kappa\big(\mathfrak{g}(H_1,F_1)G_1 - \mathfrak{g}(G_1,F_1)H_1\big) - 2\kappa\big(\eta(G_1)\eta(F_1)H_1\\ &- \eta(H_1)\eta(F_1)G_1 + \mathfrak{g}(G_1,F_1)\eta(H_1)\xi - \mathfrak{g}(H_1,F_1)\eta(G_1)\xi\big).\end{aligned} \tag{4.4}$$

In particular, $(\mathcal{N}^3,\mathfrak{g})$ cannot be an Einstein manifold.

Firstly, we state a preliminary result.

Proposition 4.1. *Let* $(\mathcal{N}^3, \phi, \xi, \eta, \mathfrak{g})$ *be a* (κ, μ)*-almost coKähler manifold,* $\kappa < 0$*, and* $\mathfrak{f}$ *be a smooth function such that* $D\mathfrak{f} = \xi(\mathfrak{f})\xi$*. Then* $(\mathfrak{g}, \mathfrak{f}, \lambda)$ *cannot be a gradient* (m, ρ)*-quasi-Einstein metric.*

Proof. Assume that $(\mathfrak{g}, \mathfrak{f}, \lambda)$ is a gradient (m, ρ)-quasi-Einstein structure and $D\mathfrak{f} = \xi(\mathfrak{f})\xi$. By (1.4), (4.4), for any vector fields G_1, H_1, we get

$$\begin{aligned} &G_1(\xi(\mathfrak{f}))\eta(H_1) + \xi(\mathfrak{f})\mathfrak{g}(h'G_1, H_1) = -2\kappa\eta(G_1)\eta(H_1) \\ &+ \frac{1}{m}\xi(\mathfrak{f})^2\eta(G_1)\eta(H_1) + (2\kappa\rho + \lambda)\mathfrak{g}(G_1, H_1). \end{aligned} \tag{4.5}$$

Let p be a point of $\mathcal{N}^3$ and consider $X \in T_p\mathcal{N}^3$, such that $hX = \sqrt{-\kappa}X$ and $||X|| = 1$. Putting $G_1 = X$, $H_1 = \phi X$ in (4.5), we have, at p,

$$\xi(\mathfrak{f})\mathfrak{g}(h'X, \phi X) = 0, \quad \text{namely} \quad \sqrt{-\kappa}\xi(\mathfrak{f})(p) = 0.$$

It follows $D\mathfrak{f} = \xi(\mathfrak{f})\xi = 0$ and by (4.5) we obtain

$$(2\kappa\rho + \lambda)\mathfrak{g} - 2\kappa\eta \otimes \eta = 0.$$

It follows $\kappa = 0$, $\lambda = 0$, and this contradicts the hypothesis $\kappa < 0$. QED

Theorem 4.1. *Let* $(\mathcal{N}^3, \phi, \xi, \eta, \mathfrak{g})$ *be a* (κ, μ)*-almost coKähler manifold. If* $\kappa < 0$*,* $\mathfrak{g}$ *cannot be a gradient* (m, ρ)*-quasi-Einstein metric.*

Proof. We assume that $(\mathfrak{g}, \mathfrak{f}, \lambda)$ is a gradient (m, ρ)-quasi-Einstein structure. By (1.4), (4.3), for any vector fields G_1, one has

$$\nabla_{G_1} D\mathfrak{f} = -2\kappa\eta(G_1)\xi + \frac{1}{m}G_1(\mathfrak{f})D\mathfrak{f} + (2\kappa\rho + \lambda)G_1. \tag{4.6}$$

Taking the covariant derivative along any vector field H_1, we get

$$\begin{aligned} &\nabla_{H_1}(\nabla_{G_1} D\mathfrak{f}) = -2\kappa H_1(\eta(G_1))\xi - 2\kappa\eta(G_1)h'H_1 \\ &+ \frac{1}{m}\big(H_1(G_1(\mathfrak{f}))D\mathfrak{f} + G_1(\mathfrak{f})\nabla_{H_1} D\mathfrak{f}\big) + (2\kappa\rho + \lambda)\nabla_{H_1} G_1. \end{aligned} \tag{4.7}$$

It follows

$$\begin{aligned} &\mathcal{R}(G_1, H_1)D\mathfrak{f} = -2\kappa\big(\eta(H_1)h'G_1 - \eta(G_1)h'H_1\big) \\ &- \frac{2\kappa}{m}\big(H_1(\mathfrak{f})\eta(G_1) - G_1(\mathfrak{f})\eta(H_1)\big)\xi + \frac{1}{m}(2\kappa\rho + \lambda)\big(H_1(\mathfrak{f})G_1 - G_1(\mathfrak{f})H_1\big). \end{aligned} \tag{4.8}$$

In particular, taking the inner product with ξ, we get

$$\mathfrak{g}(\mathcal{R}(G_1, H_1)D\mathfrak{f}, \xi) = \frac{1}{m}\big(2\kappa(\rho - 1) + \lambda\big)\big(H_1(\mathfrak{f})\eta(G_1) - G_1(\mathfrak{f})\eta(H_1)\big). \tag{4.9}$$

On the other hand, applying Theorem 2.2, $\mathcal{N}^3$ is an $N(\kappa)$-almost coKähler manifold, so we have

$$\mathfrak{g}(\mathcal{R}(G_1,H_1)D\mathfrak{f},\xi)=-\kappa\big(\eta(H_1)G_1(\mathfrak{f})-\eta(G_1)H_1(\mathfrak{f})\big).$$

Then, comparing with (4.9), one obtains

$$\Big(\frac{1}{m}\big(2\kappa(\rho-1)+\lambda\big)-\kappa\Big)\big(H_1(\mathfrak{f})\eta(G_1)-G_1(\mathfrak{f})\eta(H_1)\big)=0. \tag{4.10}$$

Equation (4.10) implies

$$\Big(\frac{1}{m}\big(2\kappa(\rho-1)+\lambda\big)-\kappa\Big)\big(D\mathfrak{f}-\xi(\mathfrak{f})\xi\big)=0.$$

If $2\kappa(\rho-1)+\lambda-m\kappa\neq 0$, then $D\mathfrak{f}=\xi(\mathfrak{f})\xi$ and by Prop. 4.1, this case cannot occur. Hence, we have

$$2\kappa\rho+\lambda=(2+m)\kappa. \tag{4.11}$$

By direct computation, applying (4.8), (4.11) one obtains, for any vector field G_1

$$\mathcal{S}(G_1,D\mathfrak{f})=\frac{2\kappa}{m}\big(\xi(\mathfrak{f})\eta(G_1)+(m+1)G_1(\mathfrak{f})\big).$$

Moreover, applying (4.3), we have

$$\mathcal{S}(G_1,D\mathfrak{f})=2\kappa\eta(G_1)\xi(\mathfrak{f}).$$

It follows

$$d\mathfrak{f}=\frac{m-1}{m+1}\xi(\mathfrak{f})\eta.$$

This entails $\xi(\mathfrak{f})=0$, $d\mathfrak{f}=0$ and (1.4) reduces to $\mathcal{S}=(2\kappa\rho+\lambda)\mathfrak{g}=(m+2)\kappa\mathfrak{g}$. Hence $(\mathcal{N}^3,\mathfrak{g})$ is an Einstein manifold and we obtain a contradiction. QED

An analogous statement holds in the case of ρ-Einstein solitons. In fact, we prove the next result.

Theorem 4.2. *Let $(\mathcal{N}^3,\phi,\xi,\eta,\mathfrak{g})$ be a (κ,μ)-almost coKähler manifold. If $\kappa<0$, then $\mathfrak{g}$ is not a gradient ρ-Einstein soliton.*

Proof. Being $\mathcal{N}^3$ a (κ,μ)-almost coKähler manifold, the Ricci operator acts as ([18], [6])

$$Q=2\kappa\eta\otimes\xi+\mu h. \tag{4.12}$$

Assume that $(\mathfrak{g},\mathfrak{f},\lambda)$ is a gradient ρ-Einstein soliton. By (1.5), (4.12) the covariant derivative acts as

$$\nabla_{G_1}D\mathfrak{f}=-2\kappa\eta(G_1)\xi-\mu hG_1+(2\kappa\rho+\lambda)G_1. \tag{4.13}$$

Then, by direct computation, for any vector fields G_1, H_1 one gets

$$\mathcal{R}(G_1,H_1)D\mathfrak{f} = -2\kappa\big(\eta(H_1)h'G_1 - \eta(G_1)h'H_1\big) + \mu\big((\nabla_{H_1}h)G_1 - (\nabla_{G_1}h)H_1\big). \tag{4.14}$$

Taking the inner product by ξ, we have

$$\begin{aligned}\mathfrak{g}(\mathcal{R}(G_1,H_1)D\mathfrak{f},\xi) &= -\mu\mathfrak{g}(\nabla_{H_1}\xi, hG_1) + \mu\mathfrak{g}(\nabla_{G_1}\xi, hH_1)\\ &= -2\mu\mathfrak{g}(\phi H_1, h^2G_1) = 2\kappa\mu\mathfrak{g}(G_1,\phi H_1).\end{aligned}$$

Hence, applying the (κ,μ)-condition, one obtains

$$\begin{aligned}&2\kappa\mu\mathfrak{g}(G_1,\phi H_1) + \kappa\big(\eta(H_1)G_1(\mathfrak{f}) - \eta(G_1)H_1(\mathfrak{f})\big)\\ &+ \mu(\eta(H_1)\mathfrak{g}(hG_1,D\mathfrak{f}) - \eta(G_1)\mathfrak{g}(hH_1,D\mathfrak{f})) = 0.\end{aligned}$$

In particular, putting $G_1 = \phi H_1$, with H_1 orthogonal to ξ, we get $\kappa\mu\mathfrak{g}(H_1,H_1) = 0$. It follows $\mu = 0$, so $\mathcal{S} = 2\kappa\eta\otimes\eta$. On the other hand, by (4.14) we obtain $\mathcal{S}(G_1,D\mathfrak{f}) = 0$, for any vector field G_1. It follows $\mathcal{S}(\xi,D\mathfrak{f}) = 2\kappa\xi(\mathfrak{f}) = 0$, and then $\xi(\mathfrak{f}) = 0$.
Therefore, for any vector field G_1, we have

$$\mathfrak{g}(\nabla_{G_1}D\mathfrak{f},\xi) = -\mathfrak{g}(\nabla_{G_1}\xi, D\mathfrak{f}) = -\mathfrak{g}(h'G_1, D\mathfrak{f}).$$

By (4.13) we have $\mathfrak{g}(\nabla_{G_1}D\mathfrak{f},\xi) = \big(2\kappa(\rho-1)+\lambda\big)\eta(G_1)$. Hence, for any G_1 orthogonal to ξ one gets $\mathfrak{g}(h'G_1,D\mathfrak{f}) = 0$, so $D\mathfrak{f}$ is orthogonal to the distribution $<\xi>^{\perp}$, namely $D\mathfrak{f} = \xi(\mathfrak{f})\xi = 0$. Applying (1.5), $\mathfrak{g}$ is an Einstein metric and we obtain a contradiction. QED

Acknowledgment. The authors are thankful to the referee for his/her valuable suggestions towards the improvement of the paper.

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