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Angle between two complex lines

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Abstract. The chordal distance function on a complex projective space algebraically defines an angle between any two complex lines, which is known as the Hermitian angle. In this expository paper, we show that one can canonically construct a real line corresponding to each of these complex lines so that the real angle between these two real lines exactly agrees with the Hermitian angle between the complex lines. This way, the Hermitian angle is interpreted as a real angle, and some well known results pertaining Hermitian angles are proved using real geometry. As an example, we give a direct and elementary proof that the chordal distance function satisfies the triangle inequality.

Keywords: Hermitian Angle, Chordal Distance, Complex Line

MSC 2020 classification: 51M15, 51N15, 51N20

1 Introduction

In this expository article we give real interpretations and proofs of some known results of complex geometry, which students and researchers entering the area might find useful until they get used to manipulating complex entities. The usual complex proofs are much simpler thanks to the power of algebraic techniques which permeates the whole of complex geometry. The advantage gained by using algebraic tools in complex geometry has a heavy price for the beginner since algebra tends to overshadow the underlying real geometry. We expose here the real implementations of complex geometry in the context of the Hermitian angle between two complex lines.

Angles between vector subspaces of an Euclidean space have attracted attention as generalization of an angle between two vectors. It is believed that such a generalization was first discussed by Jordan in [13]. There are other approaches to understand the relative positions of subspaces as in the context of Schubert calculus but that approach is used to answer some cohomological questions which we do not enter here.

The Hermitian angle between two complex lines of the complex projective space $\mathbb{P}^n$ is well known. Each such line corresponds to a real plane in $\mathbb{R}^{2n+2}$. Thus the Hermitian angle between two complex lines is the angle between two real planes touching each other at the origin in $\mathbb{R}^{2n+2}$.

We are interested in *visualizing* this angle as the angle between two real rays in $\mathbb{R}^{2n+2}$. So the question we ask is how to construct two real lines corresponding to two complex lines so that the real angle between these two real lines is the Hermitian angle between the given complex lines. To this end we first recall in section 4 the usual definition of a real ray associated to a complex point. Then in theorems 1 and 2 we answer this question.

Our presentation is mostly expository in nature and contains more material than necessary to prove our main result. Since the material on this subject is somewhat scattered in the literature we hope that our manuscript will be useful to the students and researchers for reference purposes also.

The complex geometric description of the Hermitian angle and its consequences are well understood. Our aim is not provide extra explanation to these facts. We aim at showing that a complex line, which is not a real line, can produce in a somewhat canonical way real line and the Hermitian angle between these complex lines is equal to the real angle between these real lines, which we believe most geometers will find interesting and satisfying.

An excellent introduction to the background of the study of angles between vector spaces, real or complex, can be found in Scharnhorst's article [21]. For further details and for different approaches we refer the reader to [3, 5, 6, 9, 15, 12, 22]

In this article as a bonus we first recall the relation between the *Fubini-Study metric* and the *chordal distance function* on $\mathbb{P}^n$. For completeness, we show directly that this distance function is actually a metric using the subadditivity of the sine function. The notation and the perspective used in the proof is later used to construct a real line corresponding to a complex point in $\mathbb{P}^n$. Finally, we show that the algebraically defined angle between two complex points of the complex space can be realized as the angle between the two canonically constructed real lines. We close with an example which we believe carries the beauty of higher dimensional real geometry in at least the smallest dimension where we can actually *see* things.

2 From Fubini-Study to a distance function

In this section we recall some facts regarding the *Fubini-Study metric* and the *chordal distance*. The reader is referred to Rumely's book [19] for further details.

The Fubini-Study metric is a Riemannian metric on complex projective space $\mathbb{P}^n$, first introduced by Fubini [6] and Study [22]. It can be expressed in complex affine coordinates as

$$ds^2 = \frac{\sum dz_i d\bar{z}_i}{1 + \sum z_i \bar{z}_i} - \frac{(\sum z_i d\bar{z}_i)(\sum \bar{z}_i dz_i)}{(1 + \sum z_i \bar{z}_i)^2}. \tag{2.1}$$

This metric is invariant under the action of the unitary group $U(n+1)$ on $\mathbb{P}^n$. In fact it is the only metric on $\mathbb{P}^n$ on which the unitary group acts isometrically (and transitively) [14].

On the other hand, the *chordal distance* between two points $\xi = [\xi_0, \ldots, \xi_n]$ and $\eta = [\eta_0, \ldots, \eta_n]$ on $\mathbb{P}^n$ is defined by

$$d(\xi, \eta) = \left(\frac{\sum_{i<j} |\xi_i \eta_j - \xi_j \eta_i|^2}{|\xi|^2 |\eta|^2} \right)^{1/2} \tag{2.2}$$

(see [19]). As we will see in following sections, this distance function is indeed well-defined and invariant under the action of the unitary group, and moreover we will give an elementary proof that it satisfies the triangle-inequality.

To see the relation between the chordal distance and the Fubini-Study metric, let us take two points on $\mathbb{P}^n$, say ξ and η, lying in the same affine chart. By using the unitary invariance, we can assume, without loss of generality, that $\xi = [1, 0, \ldots, 0]$ and $\eta = [1, r, 0, \ldots, 0]$, where $r \geq 0$ is some real number. Therefore, in the first local affine chart, we can write $\xi = (0, 0, \ldots, 0)$ and $\eta = (r, 0, \ldots, 0)$ in $\mathbb{C}^n$. We will now compute two types of distances between ξ and η (see [19]).

(1) **Geodesic distance between ξ and η:** Geodesic distance is simply defined by the length of the geodesic joining two points. Let us denote the length of the geodesic joining ξ to η by $\theta_{\xi,\eta}$. In the local coordinates we described, the geodesics emanating from ξ (the origin) are indeed *real lines* (see [14], p.277). The geodesic joining ξ to η can thus be parametrized by $\gamma(s) = (s, 0, \ldots, 0) \in \mathbb{R}^{2n}$, $0 \leq s \leq r$. The length of the tangent vector of γ (computed with respect to the Fubini-Study metric (2.1)) is $\|\gamma'(s)\| = \frac{1}{1+s^2}$. Therefore, the length of the geodesic is

$$\theta_{\xi,\eta} = \int_0^r \|\gamma'(s)\| ds = \arctan(r).$$

Since $\tan \theta_{\xi,\eta} = r$, it follows that $\sin \theta_{\xi,\eta} = \frac{r}{\sqrt{1+r^2}}$.

(2) **Chordal distance between ξ and η:**. Using (2.2), we easily compute that

$$d(\xi, \eta) = \frac{r}{\sqrt{1 + r^2}}.$$

Hence, the conclusion is that the **chordal distance** $d(\xi, \eta)$ is precisely the **sine of the geodesic distance** $\theta_{\xi,\eta}$ between ξ and η, computed with respect to the Fubini-Study metric. As pointed out by Harbater [9], even though the chordal distance and the geodesic distance (with respect to the Fubini-Study metric) are not exactly the same, infinitesimally speaking, they do agree; that is, *the chordal distance induces the Fubini-Study metric on the tangent bundle* (see Corollary 1.1.4 in [9]).

In Section 3, we will give equivalent definitions for $d(\xi, \eta)$ and $\theta_{\xi,\eta}$, which will tell another geometric story.

3 The Hermitian angle between complex vectors

In this section we review some results from the literature. We start with the following function

$$d(\xi, \eta) = \left(\frac{|\xi|^2|\eta|^2 - |(\xi, \eta)|^2}{|\xi|^2|\eta|^2} \right)^{1/2}, \tag{3.3}$$

where

$$\xi = (\xi_1, \dots, \xi_{n+1}), \eta = (\eta_1, \dots, \eta_{n+1}) \in \mathbb{C}^{n+1} - \{0\}$$

and

$$(\xi, \eta) = \sum_{k=0}^{n} \xi_k \bar{\eta}_k.$$

This function is clearly invariant under the $\mathbb{C}^*$-action on each component of the argument. Hence, it induces a function $d : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{R}$. Moreover, because of the unitary invariance of the hermitian inner product (ξ, η), we see that d is also unitary-invariant. A short computation shows that this definition of d is in fact equivalent to the one defined in (2.2). For non-zero ξ and η, the Hermitian angle

$$\theta_{\xi,\eta} \in [0, \pi/2]$$

is defined by the relation

$$\cos \theta_{\xi,\eta} = \frac{|(\xi, \eta)|}{|\xi|\,|\eta|}$$

(see [21]). For the Hermitian angle $\theta_{\xi,\eta}$ it may be convenient to use the identity

$$d(\xi, \eta) = \sin \theta_{\xi,\eta}, \tag{3.4}$$

which justifies the notation used in the previous section.

The range of $\theta_{\xi,\eta}$ is justified by the relations

$$d(\pm\xi, \eta) = d(\xi, \pm\eta) = d(\xi, \eta), \tag{3.5}$$

so that if ξ and η are in $\mathbb{R}^{n+1} - \{0\} \subset \mathbb{C}^{n+1} - \{0\}$, then $\theta_{\xi,\eta}$ denotes the smaller angle between the real rays defined by ξ and η.

4 Real rays associated to complex vectors

In this section, corresponding to each point of $\mathbb{P}^n$ we will construct a real line in $\mathbb{R}^{2n+2}$ passing through the origin. We will then relate the angle $\theta_{\xi,\eta}$ to an angle between two real rays constructed using the complex vectors ξ and η.

Let $\xi = (\xi_1, \dots, \xi_{n+1}) \in \mathbb{C}^{n+1} - \{0\}$. Set

$$\xi_j = \xi_{R,j} + i\xi_{I,j},\ \xi_{R,j}, \xi_{I,j} \in \mathbb{R},\ j = 1, \dots, n+1.$$

We now define two real vectors

$$\xi_R = (\xi_{R,1}, \xi_{I,1}, \dots, \xi_{R,n+1}, \xi_{I,n+1}) \in \mathbb{R}^{2n+2}$$

and

$$\xi_I = (-\xi_{I,1}, \xi_{R,1}, \dots, -\xi_{I,n+1}, \xi_{R,n+1}) \in \mathbb{R}^{2n+2}.$$

Notice that while the numbers $\xi_{R,j}$ and $\xi_{I,j}$ are the real and imaginary parts of the number $\xi_j \in \mathbb{C}$, the vectors ξ_R and ξ_I are not the real and imaginary parts of the vector $\xi \in \mathbb{C}^{n+1}$. Instead, ξ_R and ξ_I are the vectors corresponding to the vectors ξ and $i\xi$ in the underlying real space $\mathbb{R}^{2n+2}$.

Moreover, in complex geometry one would usually write that $\xi_I = J\xi_R$, where J is the *complex structure*, an operator that has in $\mathbb{R}^{2n+2}$ the same effect multiplication by i has in $\mathbb{C}^{n+1}$. In fact J rotates vectors by $90°$, so that $J^2 = -I$.

For $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}$ with $\lambda_1, \lambda_2 \in \mathbb{R}$, we see that

$$(\lambda\xi)_R = \lambda_1\xi_R + \lambda_2\xi_I. \tag{4.6}$$

Hence to the point $[\xi] \in \mathbb{P}^n$ we assign canonically the plane

$$\mathrm{span}_{\mathbb{R}}\{\xi_R, \xi_I\} \subset \mathbb{R}^{2n+2}.$$

This is the real plane which corresponds to the complex line ξ in the underlying real space.

For two vectors $\xi, \eta \in \mathbb{C}^{n+1}$ we can write

$$(\xi, \eta) = \xi \cdot \bar{\eta} = \xi_R \cdot \eta_R + i\, \xi_R \cdot \eta_I, \tag{4.7}$$

where as usual $\cdot$ is the dot product.

In Scharnhorst's notation $\xi = a$, $\xi_R = A$ and $\xi_I = \tilde{A}$, see [21].

We now collect several useful identities.

Lemma 1. *For any* ξ, $\eta \in \mathbb{C}^{n+1} - \{0\}$ *we have the following identities.*

(1) $\xi_R \cdot \xi_I = 0$,

(2) $\xi_R \cdot \xi_R = \xi_I \cdot \xi_I = |\xi|^2$,

(3) $\xi_R \cdot \eta_R = \xi_I \cdot \eta_I$,

(4) $\xi_R \cdot \eta_I = -\xi_I \cdot \eta_R$,

(5) $|(\xi, \eta)|^2 = (\xi_R \cdot \eta_R)^2 + (\xi_R \cdot \eta_I)^2$.

Proof. The verification of these identities is straightforward and is left to the reader. □

We now define the angle $\theta_{\xi_R,\eta_R} \in [0, \pi/2]$ to be the smaller angle between the rays in $\mathbb{R}^{2n+2}$ defined by the points ξ_R and η_R. The angles θ_{ξ_R,η_I} etc are defined similarly.

Putting these together we now have

Lemma 2. *For any* ξ, $\eta \in \mathbb{C}^{n+1} - \{0\}$ *we have*

$$\cos^2 \theta_{\xi,\eta} = \cos^2 \theta_{\xi_R,\eta_R} + \cos^2 \theta_{\xi_R,\eta_I}.$$

Proof.

$$\begin{aligned} \cos^2 \theta_{\xi,\eta} &= \frac{|(\xi, \eta)|^2}{|\xi|^2 \, |\eta|^2} \\ &= \frac{(\xi_R \cdot \eta_R)^2 + (\xi_R \cdot \eta_I)^2}{|\xi|^2 \, |\eta|^2} \\ &= \frac{(\xi_R \cdot \eta_R)^2}{|\xi_R|^2 \, |\eta_R|^2} + \frac{(\xi_R \cdot \eta_I)^2}{|\xi_R|^2 \, |\eta_I|^2} \\ &= \cos^2 \theta_{\xi_R,\eta_R} + \cos^2 \theta_{\xi_R,\eta_I}. \end{aligned}$$

□

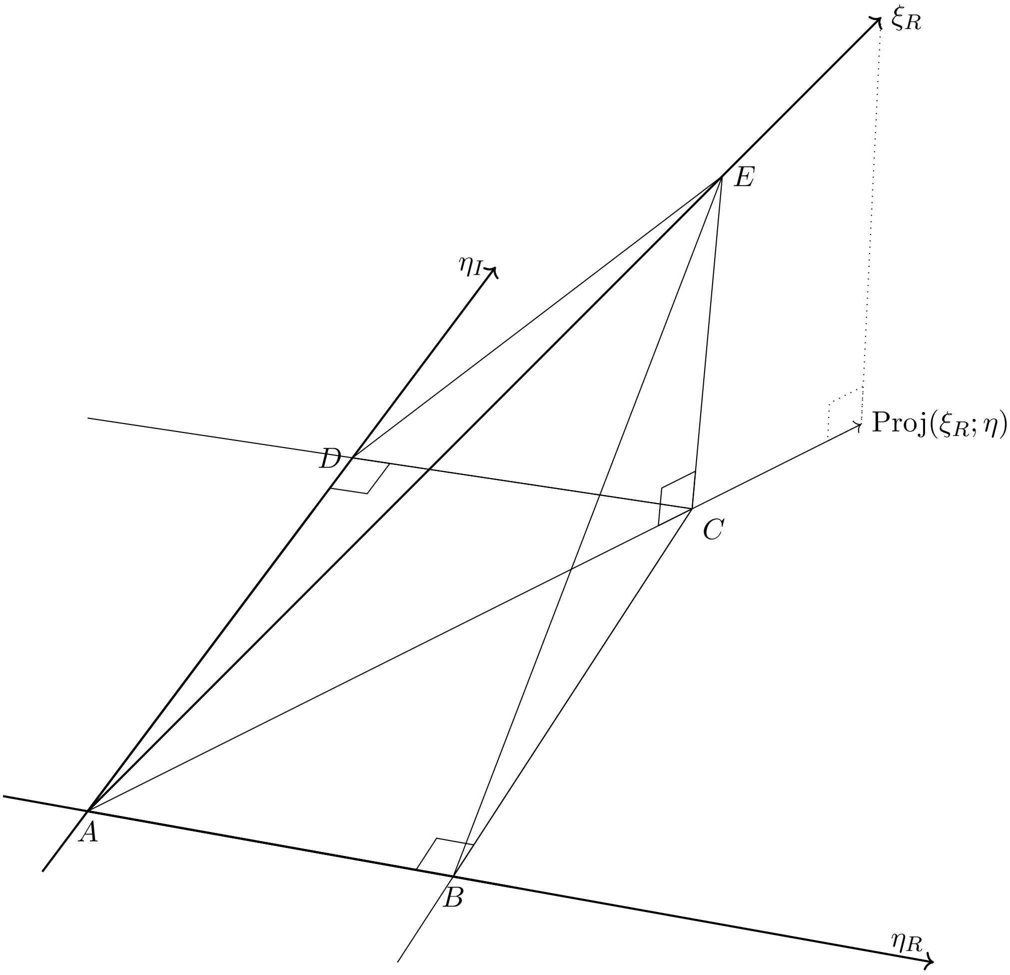
The angles θ_{ξ_R,η_R} and θ_{ξ_R,η_I} are angles between real lines in $\mathbb{R}^{2n+2}$, whereas the angle $\theta_{\xi,\eta}$ is an algebraically obtained angle between two complex points in $\mathbb{P}^n$. This lemma relates the cosine of the Hermitian angle with the cosines of two real angles.

We now proceed to show that the angle $\theta_{\xi,\eta}$ can also be visualized as an angle between two real lines in $\mathbb{R}^{2n+2}$.

5 Geometric Visualization of the Hermitian angle

The real rays ξ_R, η_R and η_I lie in some $\mathbb{R}^3$ of $\mathbb{R}^{2n+2}$. We consider them as vectors oriented as in the following figure.

This choice of orientation is possible since the lines η_R and η_I are orthogonal to each other and the direction of the vectors do not change the Hermitian angle, see (3.5). Moreover the correct orientation to obtain the set-up of the following figure may require switching places of η_R and η_I but this does not affect our calculations.



Let $AE = 1$ on the ray ξ_R. Let $\mathrm{Proj}(\xi_R;\eta)$ be the projection of ξ_R on the real plane spanned by η_R and η_I. On $\mathrm{Proj}(\xi_R;\eta)$ take C as the projection of the point E.

Let the plane through EC and perpendicular to η_R intersect η_R at B. Similarly let the plane through EC and perpendicular to η_I intersect η_I at D.

Then $ABCD$ is a rectangle and the triangles $\triangle EAB$, $\triangle EAD$, $\triangle EAC$ and $\triangle ABC$ are right triangles.

In the previous section we defined the angles

$$\angle EAB = \theta_{\xi_R,\eta_R} \quad \text{and} \quad \angle EAD = \theta_{\xi_R,\eta_I}.$$

Now define the angle

$$\theta_{\xi_R,\mathrm{Proj}(\xi_R;\eta)} := \angle EAC.$$

Note that if $\mathrm{Proj}(\xi_R;\eta) = 0$, then $EA \perp \mathrm{span}_{Rr}\{\eta_R, \eta_I\}$, so in this case we have $\theta_{\xi_R,\mathrm{Proj}(\xi_R;\eta)} := \frac{\pi}{2}$.

We can now realize the Hermitian angle as a real angle between two real lines. The following result can be found in [21, p. 97] but deriving it using only real terms may nonetheless be beneficial.

Theorem 1. *For any ξ, $\eta \in \mathbb{C}^{n+1} - \{0\}$, the Hermitian angle $\theta_{\xi,\eta}$ is equal to the real angle $\theta_{\xi_R,\mathrm{Proj}(\xi_R;\eta)}$.*

Proof. From the above right triangles and using the fact that we chose $AE = 1$ we have the following identities.

$$\begin{aligned} \cos\theta_{\xi_R,\eta_R} &= AB, \quad \text{in } \triangle EAB \\ \cos\theta_{\xi_R,\eta_I} &= AD = BC, \quad \text{in } \triangle EAD \\ \cos\theta_{\xi_R,\mathrm{Proj}(\xi_R;\eta)} &= AC, \quad \text{in } \triangle EAC \end{aligned}$$

Since we have $AB^2 + BC^2 = AC^2$ in $\triangle ABC$, we will have

$$\cos^2\theta_{\xi_R,\eta_R} + \cos^2\theta_{\xi_R,\eta_I} = \cos^2\theta_{\xi_R,\mathrm{Proj}(\xi_R;\eta)}.$$

From Lemma (2) of the previous section we now conclude that

$$\cos\theta_{\xi,\eta} = \cos\theta_{\xi_R,\mathrm{Proj}(\xi_R;\eta)},$$

and hence

$$\theta_{\xi,\eta} = \theta_{\xi_R,\mathrm{Proj}(\xi_R;\eta)},$$

since these angles are in $[0, \pi/2]$. ☐

In the next section we will show that the angle $\theta_{\xi_R,\mathrm{Proj}(\xi_R;\eta)}$ is independent of the choice of homogeneous coordinates for $[\xi]$.

6 Invariance of the angle

Since the homogeneous coordinates of $[\xi]$ are well defined only up to multiplication by a complex constant λ, the corresponding real line ξ_R is not well defined. However in this section we will show that the angle between the line $(\lambda\xi)_R$ and its projection on $\mathrm{span}_{\mathbb{R}}\{\eta_R, \eta_I\}$ is independent of the $\lambda \in \mathbb{C} - \{0\}$.

We first recall the definition of projection. In particular we have

$$\mathrm{Proj}(\xi_R;\eta) = \left(\frac{\xi_R \cdot \eta_R}{|\eta_R|^2}\right)\eta_R + \left(\frac{\xi_R \cdot \eta_I}{|\eta_I|^2}\right)\eta_I.$$

Now we once more collect some identities.

Lemma 3. *For any ξ, $\eta \in \mathbb{C}^{n+1} - \{0\}$, we have the following identities.*

$$\begin{aligned}
\xi_R \cdot \mathrm{Proj}(\xi_I;\eta) &= 0,\\
\xi_I \cdot \mathrm{Proj}(\xi_R;\eta) &= 0,\\
\mathrm{Proj}(\xi_R;\eta)\cdot\mathrm{Proj}(\xi_I;\eta) &= 0,\\
|\mathrm{Proj}(\xi_R;\eta)| = |\mathrm{Proj}(\xi_I;\eta)| &= \frac{|(\xi,\eta)|}{|\eta|} = |\xi|\cos\theta_{\xi,\eta},\\
\xi_R\cdot\mathrm{Proj}(\xi_R;\eta) = \xi_I\cdot\mathrm{Proj}(\xi_I;\eta) &= \frac{(\xi_R\cdot\eta_R)^2}{|\eta|^2} + \frac{(\xi_R\cdot\eta_I)^2}{|\eta|^2}.
\end{aligned}$$

Remark: In the literature such planes as $\mathrm{span}_{\mathbb{R}}\{\xi_R, \xi_I\}$ and $\mathrm{span}_{\mathbb{R}}\{\eta_R, \eta_I\}$ are called *isoclinic subspaces*, see [21, 20] and the references given there.

Remark: If a projection is zero, then the corresponding angle is $\pi/2$. For example if $\mathrm{Proj}(\xi_R;\eta) = 0$, then by the lemma $\cos\theta_{\xi,\eta} = 0$ and in the range from 0 to $\pi/2$ we have $\theta_{\xi,\eta} = \pi/2$. This is compatible with the geometric visualization of two planes being perpendicular.

Proof. The verification of these identities are mostly straightforward once the results of Lemma (1) and equation (4.7) are consulted. □

Using these identities we finally prove the invariance of the angle $\theta_{\xi_R,\mathrm{Proj}(\xi_R;\eta)}$.

Theorem 2. *For any ξ, $\eta \in \mathbb{C}^{n+1} - \{0\}$, the Hermitian angle $\theta_{\xi,\eta}$ is equal to the real angle between any non-zero vector in $\mathrm{span}_{\mathbb{R}}(\xi_r, \xi_I)$ and its projection on $\mathrm{span}_{\mathbb{R}}(\eta_R, \eta_I)$.*

Proof. Writing $(\lambda\xi)_R = \lambda_1\xi_R + \lambda_2\xi_I$, where $\lambda = \lambda_1 + i\lambda_2$ with $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$\mathrm{Proj}(\lambda_1\xi_R + \lambda_2\xi_I;\eta) = \lambda_1\,\mathrm{Proj}(\xi_R;\eta) + \lambda_2\,\mathrm{Proj}(\xi_I;\eta).$$

Writing

$$\alpha = \lambda_1\xi_R + \lambda_2\xi_I,$$

we want to show that

$$\theta_{\xi,\eta} = \theta_{\alpha,\mathrm{Proj}(\alpha;\eta)}.$$

For this we have the following set of equations which follow from Lemmas (1) and (3).

$$\begin{aligned}
\cos\theta_{\alpha,\mathrm{Proj}(\alpha;\eta)} &= \frac{(\lambda_1\xi_R+\lambda_2\xi_I)\cdot[\lambda_1\,\mathrm{Proj}(\xi_R;\eta)+\lambda_2\,\mathrm{Proj}(\xi_I;\eta)]}{|\lambda_1\xi_R+\lambda_2\xi_I|\,|\lambda_1\,\mathrm{Proj}(\xi_R;\eta)+\lambda_2\,\mathrm{Proj}(\xi_I;\eta))|}\\
&= \frac{\lambda_1^2(\xi_R\cdot\mathrm{Proj}(\xi_R;\eta))+\lambda_2^2(\xi_I\cdot\mathrm{Proj}(\xi_I;\eta))}{(\lambda_1^2+\lambda_2^2)|\xi_R|\,|\,\mathrm{Proj}(\xi_R;\eta)|}\\
&\quad+\frac{\lambda_1\lambda_2[\xi_R\cdot\mathrm{Proj}(\xi_I;\eta)+\xi_I\cdot\mathrm{Proj}(\xi_R;\eta)]}{(\lambda_1^2+\lambda_2^2)|\xi_R|\,|\,\mathrm{Proj}(\xi_R;\eta)|}\\
&= \frac{\xi_R\cdot\mathrm{Proj}(\xi_R;\eta)}{|\xi_R|\,|\,\mathrm{Proj}(\xi_R;\eta)|}\\
&= \cos\theta_{\xi_R,\mathrm{Proj}(\xi_R;\eta)}.
\end{aligned}$$

Since these angles are in $[0,\pi/2]$, we have

$$\theta_{\alpha,\mathrm{Proj}(\alpha;\eta)} = \theta_{\xi_R,\mathrm{Proj}(\xi_R;\eta)}.$$

But it now follows from Theorem (1) that

$$\theta_{\xi,\eta} = \theta_{\alpha,\mathrm{Proj}(\alpha;\eta)},$$

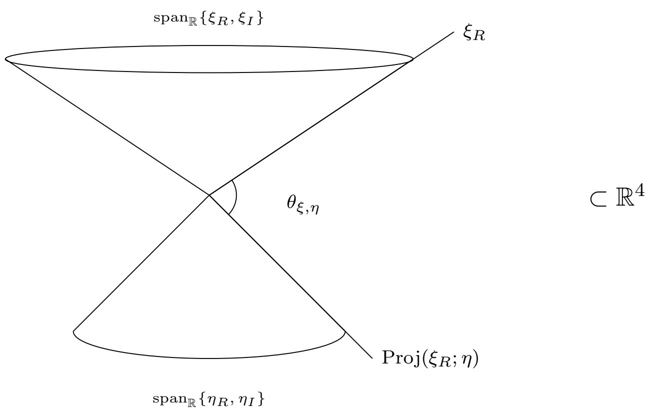
proving the invariance of the angle. □

Thus we have shown that the angle between $[\xi]$ and $[\eta]$ in $\mathbb{P}^n$ can be *visualized* in $\mathbb{R}^4$ as in the figure below.

Here the cones represent the two planes $\mathrm{span}_{\mathbb{R}}\{\xi_R,\xi_I\}$ and $\mathrm{span}_{\mathbb{R}}\{\eta_R,\eta_I\}$ in some $\mathbb{R}^4$ touching each other only at the origin. The angle between these two planes is well defined since the angle between any line through the origin on $\mathrm{span}_{\mathbb{R}}\{\xi_R,\xi_I\}$ and its projection on $\mathrm{span}_{\mathbb{R}}\{\eta_R,\eta_I\}$ is equal to the angle $\theta_{\xi,\eta}$ as shown in the figure.

7 The triangle inequality

It is known that the function $(\cdot,\cdot)$ given at the beginning in (3.3) is a metric on the space of $\mathbb{P}^n$ induced by the Fubini-Study metric of the tangent space



of $\mathbb{P}^n$. Therefore it follows indirectly that $d(\cdot,\cdot)$ satisfies the triangle inequality. However we will show in this section that this triangle inequality can be proved directly from the given definition of $d(\cdot,\cdot)$ using the techniques we developed so far.

We start with a lemma.

Lemma 4. *For $\xi, \eta, \zeta \in \mathbb{C}^{n+1} - \{0\}$ we have*

$$\theta_{\xi,\zeta} + \theta_{\zeta,\eta} \geq \theta_{\xi,\eta}.$$

Proof. Let $u := \mathrm{Proj}(\xi_R; \zeta)$ and $v := \mathrm{Proj}(u; \eta)$. First assume that $u \neq 0$ and $v \neq 0$. Then the angle between ξ_R and u is $\theta_{\xi,\zeta}$ by Theorem 1, and the angle between u and v is $\theta_{\zeta,\eta}$ by Theorem (2).

Let the angle between ξ_R and v be denoted by $\theta(\xi_R, v)$.

We have three real vectors, ξ_r, u and v emanating from the origin. Since each of the three angles $\theta_{\xi,\zeta}$, $\theta_{\zeta,\eta}$ and $\theta(\xi_R, v)$ is in $[0, \pi/2]$, their sum is less than 2π. Hence following Euclid we conclude that

$$\theta_{\xi,\zeta} + \theta_{\zeta,\eta} \geq \theta(\xi_r, v),$$

see [4, Book XI, Propositions 20, 21 and 23].

The vectors v and $\mathrm{Proj}(\xi_R; \eta)$ are both in $\mathrm{span}_{\mathbb{R}}(\eta_R, \eta_I)$. Therefore the angle between ξ_R and its projection on $\mathrm{span}_{\mathbb{R}}(\eta_R, \eta_I)$ is smaller or equal to the angle between ξ_R and any other vector in $\mathrm{span}_{\mathbb{R}}(\eta_R, \eta_I)$. This means

$$\theta(\xi_R, v) \geq \theta_{\xi,\eta}.$$

These two last inequalities give us the required inequality that

$$\theta_{\xi,\zeta} + \theta_{\zeta,\eta} \geq \theta_{\xi,\eta},$$

as claimed.

Next assume that $u = 0$ or $v = 0$. Then $\theta_{\xi,\zeta} = \frac{\pi}{2}$ or $\theta_{\zeta,\eta} = \frac{\pi}{2}$, respectively. Since the maximum value of $\theta_{\xi,\eta}$ is $\frac{\pi}{2}$, the claimed inequality holds without further work. □

Using theorem (1) as a starting proof, an alternate proof of this lemma can be found in [23, section 3].

The triangle inequality then reduces to the following.

Theorem 3. *Let $\theta_{\xi,\eta}, \theta_{\xi,\zeta}, \theta_{\zeta,\eta} \in [0, \pi/2]$ be such that $\theta_{\xi,\zeta} + \theta_{\zeta,\eta} \geq \theta_{\xi,\eta}$. Then*

$$\sin\theta_{\xi,\zeta} + \sin\theta_{\zeta,\eta} \geq \sin\theta_{\xi,\eta},$$

or in equivalent formulation, see (3.4),

$$d(\xi, \zeta) + d(\zeta, \eta) \geq d(\xi, \eta).$$

Proof. We first recall that $\sin x$ is sub-additive on $[0, \pi]$, i.e. for $x, y \in [0, \pi]$ we have

$$\sin x + \sin y \geq \sin(x + y). \tag{7.8}$$

Now without loss of generality we may assume that $\theta_{\xi,\zeta} \leq \theta_{\zeta,\eta}$.

We have two cases to consider.

Case 1: $\theta_{\xi,\eta} \leq \theta_{\xi,\zeta} + \theta_{\zeta,\eta} \leq \pi/2$.
In this case we have from (7.8)

$$\sin\theta_{\xi,\zeta} + \sin\theta_{\zeta,\eta} \geq \sin(\theta_{\xi,\zeta} + \theta_{\zeta,\eta}) \geq \sin\theta_{\xi,\eta}.$$

Case 2: $\theta_{\xi,\eta} \leq \pi/2 \leq \theta_{\xi,\zeta} + \theta_{\zeta,\eta}$.
Here we consider two subcases.

Subcase 2.1: $\theta_{\xi,\eta} \leq \theta_{\zeta,\eta} \leq \pi/2$. In this case since $\sin\theta_{\zeta,\eta} \geq \sin\theta_{\xi,\eta}$, the result follows trivially.

Subcase 2.2: $\theta_{\zeta,\eta} \leq \theta_{\xi,\eta} \leq \pi/2$. In this case we must have $\pi/4 \leq \theta_{\zeta,\eta}$ since $\pi/2 \leq \theta_{\xi,\zeta} + \theta_{\zeta,\eta}$ and by assumption $\theta_{\xi,\zeta} \leq \theta_{\zeta,\eta}$. Then $\theta_{\xi,\zeta} \geq \pi/2 - \theta_{\zeta,\eta}$, and hence $\sin\theta_{\xi,\zeta} \geq \cos\theta_{\zeta,\eta}$. We thus have

$$\sin\theta_{\xi,\zeta} + \sin\theta_{\zeta,\eta} \geq \cos\theta_{\zeta,\eta} + \sin\theta_{\zeta,\eta} \geq 1 \geq \sin\theta_{\xi,\eta}.$$

□

8 More on isoclinic planes

To be consistent with our notation we will consider the Grassmannian

$$G(2, \mathbb{R}^{2n+2}).$$

Let P and Q be two planes in $\mathbb{R}^{2n+2}$ both passing through the origin. By abuse of notation we will also use P and Q to denote the corresponding points in $G(2, \mathbb{R}^{2n+2})$.

For any non-zero $v \in \mathbb{R}^{2n+2}$, let $\theta_{v,\mathrm{Proj}(v;Q)}$ be the angle between v and its orthogonal projection on Q.

Whitehead in [23] defines the angle between P and Q as

$$\theta_{P,Q} = \max\{\theta_{v,\mathrm{Proj}(v;Q)} | v \in P - \{0\}\}.$$

If $P = \mathrm{span}_{\mathbb{R}}\{\xi_R, \xi_I\}$ and $Q = \mathrm{span}_{\mathbb{R}}\{\eta_R, \eta_I\}$ for some $\xi, \eta \in \mathbb{C}^{n+1} - \{0\}$, then P and Q are isoclinic subspaces by theorem (2). Hence the cardinality of the set $\{\theta_{v,\mathrm{Proj}(v;Q)} | v \in P - \{0\}\}$ is one. In other words as planes corresponding to to complex lines are isoclinic, all such angles for them are equal to the Hermitian angle between them.

9 Principal angles and singular values

The idea of finding several angles to measure how one subspace deviates from the other seems to have started with Jordan, [13, Section 37]. Then Hotelling [11, Section 7], and later Afriat [1, Section 5] worked on this idea. These angles are called principal angles and they are related to singular values of some matrices, see [2, 8, 7, 10, 18, 24].

We will define these concepts only for planes in $\mathbb{R}^{2n+2}$ since that is the only relevant medium for us.

The purpose of this section is to derive algebraically the equality of certain geometric identities.

Let P and Q be two planes in $\mathbb{R}^{2n+2}$ both passing through the origin. We will define the principal angles $[\theta_1, \theta_2]$ between P and Q recursively. $\theta_1 \in [0, \pi/2]$ is that angle for which

$$\cos\theta_1 = \max\{x \cdot y | x \in X, y \in Y, |x| = |y| = 1\}.$$

Let $x_1 \in X$ and $y_1 \in Y$ be two unit vectors such that

$$\cos\theta_1 = x_1 \cdot y_1.$$

For a non-zero $x \in X$ let $x^\perp$ denote the orthogonal complement of x in X. Now we define θ_2 as that angle for which

$$\cos\theta_2 = \max\{x \cdot y | x \in x_1^\perp, y \in y_1^\perp, |x| = |y| = 1\}.$$

Similarly let $x_2 \in x_1^\perp$ and $y_2 \in y_1^\perp$ be two unit vectors such that

$$\cos\theta_2 = x_2 \cdot y_2.$$

The vectors $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are called principal vectors.

One can likewise recover the principal angles from the principal vectors as follows.

Let A be the $2n+2$ by 2 matrix whose columns are the entries of the principal vectors x_1 and x_2. Similarly define B as the $2n+2$ by 2 matrix whose columns are the entries of the principal vectors y_1 and y_2. Then the singular values of the 2 by 2 matrix A^tB are $\cos\theta_1$ and $\cos\theta_2$.

In our setting $P = \mathrm{span}_{\mathbb{R}}\{\xi_R, \xi_I\}$ and $Q = \mathrm{span}_{\mathbb{R}}\{\eta_R, \eta_I\}$. The principal angles are $\theta_{\xi_r,\mathrm{Proj}(\xi_R;\eta)}$ and $\theta_{\xi_I,\mathrm{Proj}(\xi_I;\eta)}$.

The principal vectors then are

$$\left\{\frac{\xi_R}{|\xi|}, \frac{\xi_I}{|\xi|}\right\}, \left\{\frac{\mathrm{Proj}(\xi_R;\eta)}{|\mathrm{Proj}(\xi_R;\eta)|}, \frac{\mathrm{Proj}(\xi_I;\eta)}{|\mathrm{Proj}(\xi_I;\eta)|}\right\}.$$

Using the identities in lemma (3) we see that the matrix A^tB is already diagonal

$$A^tB = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

where we expect $a = \cos\theta_{\xi,\eta}$. In fact using lemmas (3) and (2) we see that

$$\begin{aligned}
a &= \frac{\xi_R \cdot \mathrm{Proj}(\xi_R;\eta)}{|\xi|\,|\mathrm{Proj}(\xi_R;\eta)|} \\
&= \frac{(\xi_R\cdot\eta_R)^2}{|\xi|\,|\eta|}\,\frac{1}{|(\xi,\eta)|} + \frac{(\xi_R\cdot\eta_I)^2}{|\xi|\,|\eta|}\,\frac{1}{|(\xi,\eta)|} \\
&= \left(\frac{\xi_R\cdot\eta_R}{|\xi|\,|\eta|}\right)^2 \frac{|\xi|\,|\eta|}{|(\xi,\eta)|} + \left(\frac{\xi_R\cdot\eta_I}{|\xi|\,|\eta|}\right)^2 \frac{|\xi|\,|\eta|}{|(\xi,\eta)|} \\
&= \frac{\cos^2\theta_{\xi_R,\eta_R} + \cos^2\theta_{\xi_R,\eta_I}}{\cos\theta_{\xi,\eta}} \\
&= \frac{\cos^2\theta_{\xi,\eta}}{\cos\theta_{\xi,\eta}} \\
&= \cos\theta_{\xi,\eta}.
\end{aligned}$$

There is a shorter way of obtaining the cosines of the principal angles. By using theorem 2.1 of [24, page 328] we can replace the matrix B above by any orthonormal base of $\mathrm{span}_{\mathbb{R}}(\eta_R, \eta_I)$ and still the singular values of A^tB will be the cosines of the principal angles. Hence we can use the usual orthonormal base for $\mathrm{span}_{\mathbb{R}}(\eta_R, \eta_I)$

$$\left\{\frac{\eta_R}{|\eta|}, \frac{\eta_I}{|\eta|}\right\}$$

and the columns of B consists of the entries of these base elements. Then we have

$$A^tB = \begin{pmatrix} u & v \\ -v & u \end{pmatrix},$$

where

$$u = \frac{\xi_R \cdot \eta_R}{|\xi|\,|\eta|} = \frac{\xi_R \cdot \eta_I}{|\xi|\,|\eta|}, \;\; v = \frac{\xi_R \cdot \eta_I}{|\xi|\,|\eta|}.$$

The two singular values of such a matrix are equal to each other and each is of the form

$$\left(u^2 + v^2\right)^{1/2}$$

which, using lemma (2), simplifies to

$$(\cos^2\theta_{\xi_R,\eta_R} + \cos^2\theta_{\xi_R,\eta_I})^{1/2} = \cos\theta_{\xi,\eta},$$

as expected.

10 An example

We take

$$\xi = (3 + 4i, 5\sqrt{3}) \in \mathbb{C}^2, \;\eta = (1, 0) \in \mathbb{C}^2.$$

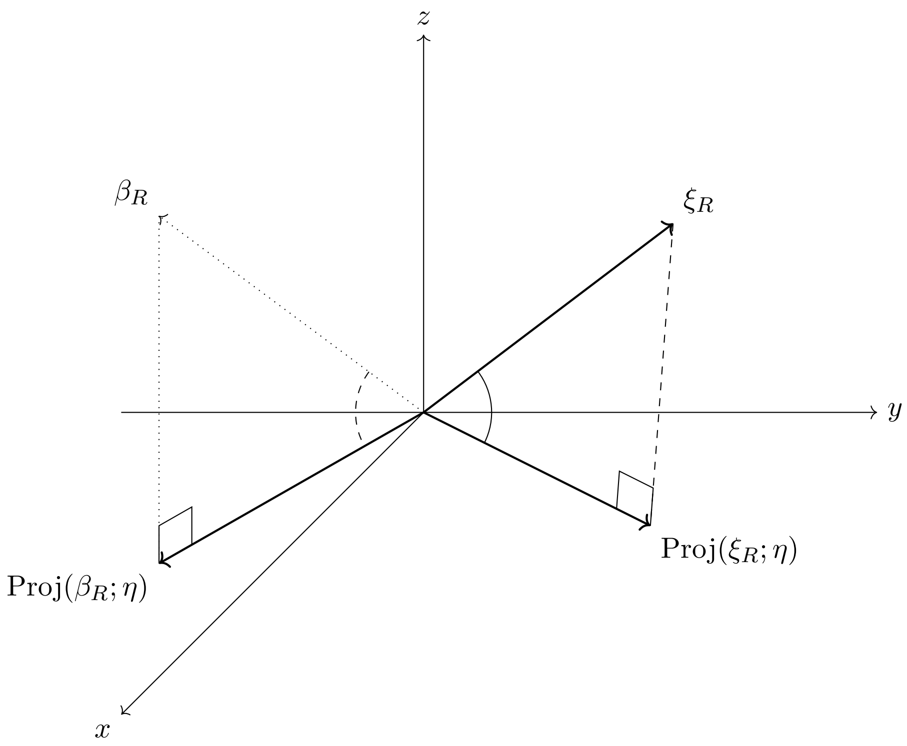
Then

$$\begin{aligned} \xi_R &= (3, 4, 5\sqrt{3}, 0) \\ \xi_I &= (-4, 3, 0, 5\sqrt{3}) \\ \eta_R &= (1, 0, 0, 0) \\ \eta_I &= (0, 1, 0, 0), \end{aligned}$$

all vectors being in $\mathbb{R}^4$. We use standard basis for $\mathbb{R}^4$ and consider ξ_R, η_R and η_I in the xyz-space as $\mathbb{R}^3 \subset \mathbb{R}^4$, see figure below.

Here $\mathrm{span}_{\mathbb{R}}(\eta_R, \eta_I)$ is the xy-plane and $\mathrm{Proj}(\xi_R;\eta) = (3, 4, 0)$. A simple calculation shows that the angle between ξ_R and $\mathrm{Proj}(\xi_R;\eta)$ is $\pi/3$.

Next we define $\beta = (11 - 2i, 5\sqrt{3}(1 - 2i))$. Note that $\beta_R = \xi_R - 2\xi_I$. This vector β_R now cannot be seen in our xyz-space except at the origin where it intersects this $\mathbb{R}^3$. Again the angle between β_R and $\mathrm{Proj}(\beta_R;\eta)$ can be easily seen to be $\pi/3$ as predicted by the theorem 2.



Here $\beta_R = \xi_R - 2\xi_I$ and is not visible in this xyz-space.

We conclude by showing that the above isocliny angles agree with the corresponding Hermitian angles.

$$\cos\theta_{\xi,\eta} = \frac{|(\xi,\eta)|}{|\xi|\,|\eta|} = \frac{1}{2}, \quad \theta_{\xi,\eta} = \frac{\pi}{3},$$

and similarly

$$\cos\theta_{\beta,\eta} = \frac{|(\beta,\eta)|}{|\beta|\,|\eta|} = \frac{1}{2}, \quad \theta_{\beta,\eta} = \frac{\pi}{3}.$$

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