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The uniqueness of a fixed degree singular plane model

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Abstract. Let X be the normalization of an integral degree $d \geq 9$ plane curve Y. We prove that X has a unique g^2_d if $h^1(\mathbb{P}^2, \mathcal{I}_Z(\lceil d/2 \rceil - 3)) = 0$, where Z is the conductor of Y. Moreover, Y is the unique plane model of X of degree at most d.

Keywords: plane curve, uniqueness of a plane model

MSC 2020 classification: 14H50

Introduction

Let $Y \subset \mathbb{P}^2$ be an integral plane curve of degree $d \geq 2$. Let $v : X \to Y$ denote the normalization map and let $Z \subset \mathbb{P}^2$ be the conductor of v. We have $\deg(Z) = (d-1)(d-2)/2 - g(X)$.

Under what assumptions the map induced by $v^*(\mathcal{O}_Y(1))$ is the only g^2_d on X and X has no base point free g^2_t, $t < d$?

If Y is smooth, this is true if $d \geq 4$, but not for $d = 3$. For very singular Y, this is certainly false, e.g. for X of genus $g \geq 4$ we at least need to exclude the Brill-Noether range for curves with general moduli and at least 2 different g^2_d's.

In this paper we give the following result which only uses a cohomological assumption on Z.

Theorem 1. *Fix an integer $d \geq 9$ and a degree d integral plane curve $Y \subset \mathbb{P}^2$. Let $v : X \to Y$ denote the normalization map and let $Z \subset \mathbb{P}^2$ be the conductor of v. Assume $h^1(\mathbb{P}^2, \mathcal{I}_Z(\lceil d/2 \rceil - 3)) = 0$. Then $v^*(\mathcal{O}_Y(1))$ is the unique g^2_d on X, it is a complete linear series and X has no g^2_t with $t < d$.*

The following example shows that Theorem 1 is sometimes sharp.

Example 1. Fix integers $d \geq 4$ and m such that $d/2 \leq m < d$. Let $Y \subset \mathbb{P}^2$ be an integral degree d curve with a unique singular point, p, which is an ordinary m-point. Let $v : X \to Y$ be the normalization map. The pencil of lines through p induces a g^1_{d-m} on X. Thus $2g^1_{d-m}$ induces a g^2_{2d-2m} on X,

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perhaps not complete. Even when $d = 2m$, the line bundle giving this g^2_d is not $v^*(\mathcal{O}_Y(1))$, because $h^0(\mathbb{P}^2, v^*(\mathcal{O}_Y(1))) = 3$ (Lemma 2) and the elements of the minimal sum $2g^1_{d/2}$ are not contained in lines. The conductor of the singularity of Y is $Z := (m-1)p$, which has degree $\binom{m}{2}$. We have $h^i(\mathbb{P}^2, \mathcal{I}_{(m-1)p}(m-2)) = 0$, $i = 0, 1$, and $h^1(\mathbb{P}^2, \mathcal{I}_{(m-1)p}(m-3)) = \binom{m}{2} - \binom{m-1}{2} = m-1$. Taking $m = \lceil d/2 \rceil$ we get $h^1(\mathbb{P}^2, \mathcal{I}_Z(\lceil d/2 \rceil - 3)) = \lceil d/2 \rceil - 1$ and $h^1(\mathbb{P}^2, \mathcal{I}_Z(\lceil d/2 \rceil - 2)) = 0$.

For a smooth plane curve $C \subset \mathbb{P}^2$ of degree $d \geq 4$ the usual way to see that it has a unique g^2_d is to prove that for each g^1_{d-1} on C there is a unique $p \in C$ such that this g^1_{d-1} is induced by the pencil of lines through p. If Y is singular, we do not know how to get the uniqueness of the g^2_d from a description of the pencils on X evincing the gonality of X. Of course, it would be sufficient to prove that all base point free g^1_{d-1} on X are induced by a pencil of lines through a smooth point of Y. This observation gives us no simplification for the proof.

Of course, one can give additional conditions on Z and get, perhaps, results for other linear series. A standard way to improve cohomological conditions (or to have them for free) is to assume something about the singularities, e.g. only ordinary nodes or ordinary cusps, so that the conduction scheme Z is just a finite set, or to further require that the finite set is general and for a fixed Z the plane curve Y is general ([3, 4, 5]). If we prescribe that Y has only ordinary multiple points with, say, multiplicity $m_1 \geq \cdots \geq m_s \geq 2$ and that the s singular points of Y are general in $\mathbb{P}^2$, the computation of the lowest integer ρ such that $h^1(\mathcal{I}_Z(\rho)) = 0$ is related to the famous Segre - Harbourne - Gimigliano - Hirschowitz conjecture ([2]).

We work over an arbitrary algebraically closed field. Some steps could be done in characteristic zero using a monodromy argument for g^r_t, $r \geq 2$, not composed with an involution ([1, p. 111]) and we expect that a characteristic zero assumption may help to generalize Theorem 1.

1 Proof of Theorem 1:

Let ρ be the minimal integer ≥ 0 such that $h^1(\mathcal{I}_Z(\rho)) = 0$. Adjunction gives $\rho \leq d-3$. Note that $\rho = 0$ if and only if $\deg(Z) \leq 1$, i.e. if and only if Y is either smooth or it has a unique singular point which is either an ordinary node or an ordinary cusp, i.e. if and only $g \geq (d-1)(d-2)/2 - 1$.

Lemma 1. *The integer ρ is the maximal non-negative integer $x \leq d-3$ such that $h^0(X, v^*(\mathcal{O}_Y(x))) = \binom{x+2}{2}$.*

Proof. Fix a non-negative integer $x \leq d-3$. Since $\deg(Y) = d > x$, we have $h^0(X, v^*(\mathcal{O}_Y(x))) \geq \binom{x+2}{2}$. Fix a general degree x curve $J \subset \mathbb{P}^2$ and set $E :=$

$Y \cap J$. Since J is general, $Z \cap J = \emptyset$. Consider the residual exact sequence of $Z \cup A$ with respect to J:

$$0 \to \mathcal{I}_Z(d-3-x) \to \mathcal{I}_{Z\cup E}(d-3) \to \mathcal{I}_{E,J}(d-3) \to 0 \qquad (1.1)$$

By the definition of ρ we have $h^1(\mathbb{P}^2, \mathcal{I}_Z(d-3-x)) = 0$ if and only if $x \le d-3-\rho$. $\boxed{QED}$

Remark 1. In the set-up of Theorem 1 we have $\rho \ge 2$.

Lemma 2. *Fix integers $d \ge 4$, $t \in \{1,2\}$ and a degree d integral plane curve $Y \subset \mathbb{P}^2$. Let $v : X \to Y$ denote the normalization map and let $Z \subset \mathbb{P}^2$ be the conductor of v. If $h^1(\mathbb{P}^2, \mathcal{I}_Z(d-3-t)) = 0$, then $h^0(X, v^*(\mathcal{O}_Y(t)) = \binom{t+2}{2}$.*

Proof. Since Y is neither a line nor a conic, $h^0(v^*(\mathcal{O}_Y(1))) \ge \binom{t+2}{2}$. Fix a general degree t curve $L \subset \mathbb{P}^2$ and set $A := Y \cap L$. Since L is general, $Z \cap L = \emptyset$. Consider the residual exact sequence of $Z \cup A$ with respect to A:

$$0 \to \mathcal{I}_Z(d-3-t) \to \mathcal{I}_{Z\cup A}(d-3) \to \mathcal{I}_{A,L}(d-3) \to 0 \qquad (1.2)$$

By assumption $h^1(\mathbb{P}^2, \mathcal{I}_Z(d-3-t)) = 0$. Thus the long cohomology exact sequence of (1.2) gives $h^1(\mathbb{P}^2, \mathcal{I}_{Z\cup A}(d-3-t)) \le h^1(L, \mathcal{I}_{A,L}(d-3))$. Since $\deg(A) = td$, $\deg(\mathcal{O}_L(d-3)) = td - 3t$, $t \in \{1,2\}$, and $L \cong \mathbb{P}^1$, we have $h^1(L, \mathcal{I}_{A,L}(d-3)) = \binom{t+2}{2} - 1$. Thus $h^0(X, v^*(\mathcal{O}_Y(t)) \le \binom{t+2}{2}$. $\boxed{QED}$

Remark 2. ([6, Rem. at p. 116]) Let $A \subset \mathbb{P}^2$ be a zero-dimensional scheme, $A \ne \emptyset$. Set $z := \deg(A)$. Let τ be the maximal integer such that $h^1(\mathcal{I}_A(\tau)) > 0$. If $z \ge 9$ and $z < 3\tau$, then either there is a line L such that $\deg(A \cap L) \ge \tau + 2$ or there is conic D (possibly singular) such that $\deg(D \cap A) \ge 2\tau + 2$. If $z \ge 4$ and $z \le 2\tau + 1$, there is a line L such that $\deg(A \cap L) \ge \tau + 2$.

Lemma 3. *Fix an integer $d \ge 6$ and a degree d integral plane curve $Y \subset \mathbb{P}^2$. Let $v : X \to Y$ denote the normalization map and let $Z \subset \mathbb{P}^2$ be the conductor of v. Assume $h^1(\mathbb{P}^2, \mathcal{I}_Z(d-5)) = 0$. Take a base point free g^r_d which is a subseries of $v^*(\mathcal{O}_Y(2))$. Then $r = 1$ and there is a line $L \subset \mathbb{P}^2$ such that $g^1_d = v^*(\mathcal{O}_Y(2))(-Y \cap L)$.*

Proof. Lemma 2 gives $h^0(X, v^*(\mathcal{O}_Y(2))) = 6$. The line bundle $v^*(\mathcal{O}_Y(2))$ has degree $2d$ and it is base point free. For any zero-dimensional scheme $W \subset \mathbb{P}^2$ such that $\deg(W) \ge 4$ we have $h^0(\mathbb{P}^2, \mathcal{I}_W(2)) \ge 3$ if and only if W is contained in a line. Apply this observation to the degree d zero-dimensional subscheme B of X such that $v^*(\mathcal{O}_Y(2))(-B)$ is the line bundle associated to the g^r_d. $\boxed{QED}$

Proof of Theorem 1: Let t be the minimal degree of a g^2_t on X. Since t is minimal, the g^2_t is base point free and complete, say $g^2_t = |R|$ for some degree t line bundle R on X. To conclude the proof it is sufficient to prove that $R = v^*(\mathcal{O}_Y(1))$ and $h^0(X, v^*(\mathcal{O}_Y(1))) = 3$. The latter equality is true by Lemma 2. Take a general $B \in |R|$ and set $A := v(B)$. Since B is general and $|R|$ is base point free, $A \cap Z = \emptyset$. Adjunction gives $h^1(\mathbb{P}^2, \mathcal{I}_{A\cup Z}(d-3)) = 2$ and $h^1(\mathbb{P}^2, \mathcal{I}_{Z\cup A'}(d-3)) < 2$ for all $A' \subsetneq A$. For each $p \in \mathbb{P}^2$ the set of all lines $L \subset \mathbb{P}^2$ containing p is one-dimensional. Since B is a general element of a g^2_t and Z_{red} is a finite set, if a line L satisfies $\#(A \cap L) \geq 2$, then $L \cap Z = \emptyset$. Moreover, only finitely many lines contain at least 2 points of Z_{red}, while for each $p \in Z_{\mathrm{red}}$ all lines through p, except finitely many, intersect the connected component, Z_p, of Z with p as its reduction in a scheme of degree μ_p, where μ_p is the multiplicity of Z_p. Thus if $J \subset \mathbb{P}^2$ is a line such that $A\cap J \neq \emptyset$ and $Z\cap J \neq \emptyset$, then $\#(A \cap J) = 1$, $R \cap Z_{\mathrm{red}}$ is a single point, p, and $\deg(Z \cap J) = \mu_p$. Note that if Z_p has multiplicity μ_p, then $\deg(Z_p) \geq \binom{\mu_p+1}{2}$ and $h^1(\mathcal{I}_{Z_p}(\mu_p - 2)) > 0$.

Since $h^1(\mathbb{P}^2, \mathcal{I}_Z(\lceil d/2\rceil - 3)) = 0$, the Castelnuovo Mumford Lemma gives that the sheaf $\mathcal{I}_Z(\lceil d/2\rceil - 2)$ is globally generated. Thus $A \cap W = \emptyset$ for a general $W \in |\mathcal{I}_Z(\lceil d/2\rceil - 2)|$. Consider the residual exact sequence of $Z \cup A$ with respect to W:

$$0 \to \mathcal{I}_A(\lfloor d/2\rfloor - 1) \to \mathcal{I}_{Z\cup A}(d-3) \to \mathcal{I}_{Z,W}(d-3) \to 0 \qquad (1.3)$$

Since $h^1(\mathbb{P}^2, \mathcal{I}_Z(d-3)) = 0$, we have $h^1(W, \mathcal{I}_{Z,W}(d-3)) = 0$. Thus the long cohomology exact sequence of (1.3) gives $h^1(\mathcal{I}_A(\lfloor d/2\rfloor - 1) \geq 2$. Let τ be the largest integer such that $h^1(\mathcal{I}_A(\tau)) > 0$. Since $\tau \geq \lfloor d/2\rfloor - 1$, we have $\#A \leq 2\tau + 3$. Thus either there is a line L such that $\#(A \cap L) \geq \tau + 2$ or there is a conic C such that $\#(C \cap A) \geq 2\tau + 2$ (Remark 2).

(a) Assume the existence of a line L such that $\#(A\cap L) \geq \tau+2$. If $A \subset L$, then the g^2_t is a subseries of $v^*(\mathcal{O}_Y(1))$. Since $h^0(v^*(\mathcal{O}_Y(1))) = 3$ (Remark 1), we get a contradiction. Now assume $A \nsubseteq L$. Consider the residual exact sequence of $Z \cup A$ with respect to $W \cup L$:

$$0 \to \mathcal{I}_{A\setminus A\cap L}(\lfloor d/2\rfloor - 2) \to \mathcal{I}_{Z\cup A}(d-3) \to \mathcal{I}_{Z\cup(A\cap L),W\cup L}(d-3) \to 0 \qquad (1.4)$$

Since $A\cap L \neq A$, we have $h^1(\mathbb{P}^2, \mathcal{I}_{Z\cup(A\cap L)}(d-3)) < 2$. Thus the long cohomology exact sequence of (1.4) gives $h^1(\mathcal{I}_{A\setminus A\cap L}(\lfloor d/2\rfloor - 2)) > 0$. Since $\#(A \cap L) \geq \tau + 2 \geq \lfloor d/2\rfloor$, we have $\#(A \setminus A \cap L) \leq \lceil d/2\rceil$. Thus there is a line J such that $\#(J \cap (A \setminus A \cap L)) \geq \lfloor d/2\rfloor$. If d is even, we get $A \subset L \cup J$. If d is odd, we get $\#(A \setminus A \cap (L \cup J)) \leq 1$. Using the residual exact sequence of $W \cup L \cup J$ and that $h^1(\mathbb{P}^2, \mathcal{I}_{Z\cup A'}(d-3)) < 2$ for all $A' \subsetneq A$, we get $A \subset L \cup J$ for all integers d. Set $C := L \cup J$. Consider the residual exact sequence of C:

$$0 \to \mathcal{I}_{\mathrm{Res}_C(Z)}(d-5) \to \mathcal{I}_{Z\cup A}(d-3) \to\in \mathcal{I}_{(C\cap Z)\cup A,C}(d-3) \to 0 \qquad (1.5)$$

Since $\mathrm{Res}_C(Z) \subseteq Z$, we have $h^1(\mathcal{I}_{\mathrm{Res}_C(Z)}(d-5)) = 0$. Thus the long cohomology exact sequence of (1.5) gives $h^1(C, \mathcal{I}_{A\cup(Z\cap C),C}(d-3)) \geq 2$. Since $\#(A \cap L) \geq 2$, we have $Z \cap L = \emptyset$. First assume $\#(A \cap J) \geq 2$ and hence $Z \cap J = \emptyset$. Each line contains at most $t-1 \leq d-1$ points of A and $\#A = t \leq d$. Since $h^1(\mathcal{I}_A(d-3)) = h^1(C, \mathcal{I}_{A,C}(d-3)) \geq 2$, we get $\#A \geq 2d-3$ (e.g., by Remark 2 or a residual exact sequence of L), a contradiction. Now assume $\#(A \cap J) = 1$. Since $A \nsubseteq L$, $J \cap L \notin A$. Remark 2 (or the residual exact sequence of $A \cup (C \cap Z)$ with respect to L) gives $\deg(J \cap (A \cup Z)) \geq d-2$ and hence $\deg(Z \cap J) \geq d-3$. We saw that the unique connected component, Z_p, of Z intersecting J has multiplicity $\geq d-3$. Thus $h^1(\mathcal{I}_{Z_p}(d-5)) > 0$, a contradiction.

(b) By step (a) we may assume that there is no line L such that $\#(A\cap L) \geq \tau+2$. Thus there is a conic C such that $\#(C \cap A) \geq 2\tau+2$. Assume for the moment $A \nsubseteq C$. Consider the residual exact sequence of $Z \cup A$ with respect to $W \cup C$:

$$0 \to \mathcal{I}_{A\setminus A\cap C}(\lfloor d/2 \rfloor - 3) \to \mathcal{I}_{Z\cup A}(d-3) \to \mathcal{I}_{Z\cup(A\cap C),W}(d-3) \to 0 \qquad (1.6)$$

Since $\#(A \setminus A \cap C) \leq 1$ and $d \geq 6$, we have $h^1(\mathbb{P}^2, \mathcal{I}_{A\setminus A\cap C}(\lfloor d/2\rfloor - 3)) = 0$. Thus the long cohomology exact sequence of (1.6) gives $h^1(\mathbb{P}^2, \mathcal{I}_{Z\cup(A\cap C)}(d-3)) = 2$. Since $h^1(\mathcal{I}_{Z\cup A'}(d-3)) < 2$ for all $A' \subsetneq A$, we get $A \subset C$. Since $h^1(\mathcal{I}_{\mathrm{Res}_C(Z)}(d-5)) = 0$, (1.5) gives $h^1(C, \mathcal{I}_{(Z\cap C)\cup A,C}(d-3)) \geq 2$. If C is reducible, then we conclude as in step (a). Now assume C irreducible. Set $\alpha := \deg(Z \cap C)$. Since C is a smooth conic, $h^1(C, \mathcal{I}_{(Z\cap C)\cup A,C}(d-3)) \geq 2$ if and only if $t+\alpha \geq 2d-3$. Since $t \leq d$, we get $\alpha \geq d-3$. Thus $h^1(\mathcal{I}_{Z\cap C}(x)) > 0$ if $2x+2 \leq d-3$, i.e. if $x \leq \lfloor (d-5)/2 \rfloor$. If d is even we get $h^1(\mathcal{I}_{Z\cap C}(d/2-3)) > 0$, a contradiction. If $d = 2k+1$ is odd, we get $h^1(\mathcal{I}_{Z\cap C}(k-2)) > 0$, a contradiction. $\boxed{QED}$

References

[1] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris: Geometry of algebraic curves, I. Springer, Berlin, 1985.

[2] S. M. Cooper, B. Harbourne: *Regina lectures on fat points.* In: Connections between algebra, combinatorics, and geometry, 147–187. Springer Proc. Math. Stat., 76, Springer, New York, 2014.

[3] M. Coppens: *The gonality of general smooth curves with a prescribed plane nodal model.* Math. Ann. **289** (1991), 89–93.

[4] M. Coppens, T. Kato: *Correction to: "The gonality of smooth curves with plane models".* Manuscripta Math. **70** (1990), no. 1, 5–25.

[5] M. Coppens, T. Kato: *The gonality of smooth curves with plane models.* Manuscripta Math. **71** (1991), no. 3, 337–338.

[6] Ph. Ellia, Ch. Peskine: *Groupes de points de* $\mathbf{P}^2$*: caractère et position uniforme.* In: Algebraic geometry (L' Aquila, 1988), 111–116, Lecture Notes in Math., 1417, Springer, Berlin, 1990.