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An application of a method of summability to Fourier series

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Abstract. Applying r-repeated de la Vallée Poussin sums, we have proved four theorems which show the upper bound of the r-repeated de la Vallée Poussin kernel, their convergence at a point, the deviation between a continuous function and the r-repeated de la Vallée Poussin sums of partial sums of its Fourier series, and finally we determine the degree of approximation of functions belonging to ordinary Lipschitz class.

Keywords: Fourier series, de la Vallée Poussin sums, Lipschitz class, modulus of continuity, the best approximation.

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1 Introduction

It is a well-known that the de la Vallée-Poussin sums have several properties which are of interest in the theory of summation of Fourier series. These sums as well as their generalization are used extensively by many researchers in the recent and in past, having a great impact in mathematical research since they has been introduced by Ch. J. de la Vallée-Poussin [27]. He was the first to study the method of approximating periodic functions by polynomials bearing his name. Similar works of this type appeared for these sums in articles [2], [3], [4], [7], [8], [9], [10], [11], [18] or in another ones [1], [6], [19], [20], [22], [23], [25], [5], and in references in that matter.

To begin with bringing to light of our intention we write some notions and notations. Indeed, let $f \in L[0, 2\pi]$ be a 2π-periodic function,

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx), \tag{1.1}$$

its Fourier series at the point x, where a_k, b_k are defined by

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx,\ b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx,\ (k = 0, 1, \dots),$$

and

$$s_m(f;x) = \frac{a_0}{2} + \sum_{k=1}^{m}(a_k \cos kx + b_k \sin kx), \tag{1.2}$$

m-th partial sums of the series (1.1).

We write $u = \mathcal{O}(v)$ whenever there exists a positive constant K, not necessarily the same at each occurrence, such that $u \leq Kv$. Also, we write

$$\varphi_x(t) := f(x+2t) - 2f(x) + f(x-2t),$$

and

$$\Phi_x(h) := \int_0^h |\varphi_x(t)|dt, \quad h > 0.$$

Let $\lambda := (\lambda_n)$ be a monotone non-decreasing sequence of integers such that $\lambda_1 = 1$ and $\lambda_{n+1} - \lambda_n \leq 1$. The means

$$V_n(\lambda, f; x) = \frac{1}{\lambda_n} \sum_{m=n-\lambda_n}^{n-1} s_m(f;x), \quad (n \geq 1), \tag{1.3}$$

are called the generalized de la Vallée Poussin mean of the sequence $(s_m(f;x))$.

Regarding to summability of Fourier series Leindler (see [17]) proved the following four theorems.

Theorem 1 ([17])**.** If the function $f(x)$ is bounded, i.e. $|f(x)| \leq K$, then the means $V_n(\lambda, f; x)$ satisfy the inequality

$$|V_n(\lambda, f; x)| \leq K\left(3 + \log\frac{2n - \lambda_n}{\lambda_n}\right).$$

Theorem 2 ([17])**.** If the sequence (λ_n) tends to infinity and conditions

$$\int_{1/n}^{1/\lambda_n} \frac{|\varphi_x(t)|}{t} dt = o(1), \quad n\Phi_x\left(\frac{1}{n}\right) = o(1), \quad (n \to \infty)$$

are satisfied, then $V_n(\lambda, f; x)$ converge to $f(x)$.

Denote $E_n := E_n(f)$ the degree of the best approximation of $f(x)$ in the space $C(0, 2\pi)$ of all continuous functions in $(0, 2\pi)$ by trigonometric polynomials $T_n(x)$ of order not exceeding n, i.e.,

$$E_n := E_n(f) = \inf_{T_n}\left\{\max_{0<x<2\pi} |f(x) - T_n(x)|\right\}.$$

Theorem 3 ([17])**.** If the function $f(x)$ is continuous, then the estimate

$$|V_n(\lambda, f; x) - f(x)| \le \left(4 + \log \frac{2n - \lambda_n}{\lambda_n}\right) E_{n-\lambda_n}$$

holds true uniformly in all x.

The modulus of continuity of $f(x)$, for a given real number $\delta > 0$, is defined as follows

$$\omega(f; \delta) := \sup_{|x-y|\le\delta} |f(x) - f(y)|,$$

where $x, y \in [0, 2\pi]$.

If $f(x) \in C(0, 2\pi)$ and $\omega(f; \delta) = \mathcal{O}(\delta^\alpha)$, $(0 < \alpha \le 1)$, then it is said that $f \in \mathrm{Lip}\alpha$.

Theorem 4 ([17])**.** If $f \in \mathrm{Lip}\alpha$, then

$$|V_n(\lambda, f; x) - f(x)| = \begin{cases} \mathcal{O}\left(\frac{1}{\lambda_n^\alpha}\right), & \alpha < 1; \\ \mathcal{O}\left(\frac{1+\log\lambda_n}{\lambda_n}\right), & \alpha = 1. \end{cases}$$

holds true uniformly in all x.

The homologous of these theorems are reported in [12] using the so-called the repeated de la Vallée Poussin sums. Some other results on the topic, treated in the present paper, can be found in [13]–[15]. For example, in [12] are employed the following repeated sums. Let p_1 and p_2 be two positive integers, and $n \in \mathbf{N}$ with condition

$$p_1 + p_2 < n.$$

The repeated de la Vallée Poussin sums of $s_m(f; x)$, i.e.

$$\mathbb{V}_n^{p_1,p_2}(f; x) := \frac{1}{p_1 p_2} \sum_{k=n-p_1}^{n-1} \sum_{m=k-p_2+1}^{k} s_m(f; x).$$

have been introduced in [23]. More details on these repeated de la Vallée Poussin sums the interested reader could consults the article [25]. For $p_1 = 1$ or $p_2 = 1$ the sums $\mathbb{V}_n^{1,p_2}(f; x)$ and $\mathbb{V}_n^{p_1,1}(f; x)$ are indeed the ordinary de la Vallée Poussin sums. Also, for $p_1 = p_2 = 1$ these sums become the Fourier sums $\mathbb{V}_n^{1,1}(f; x) \equiv s_n(f; x)$. In addition, for $p_1 = n$ and $p_2 = 1$ we obtain the Fejér sums $\mathbb{V}_n^{n,1}(f; x) \equiv \sigma_n(f; x)$.

In the sequel let $m \in \{1, 2, \dots\}$. As we know the ordinary (first) de la Vallée Poussin sums of $s_m(f; x)$ are

$${}_1\mathbb{V}_{n,m}(f; x) = \frac{1}{n} \sum_{k=m}^{m+n-1} s_k(f; x),$$

while the ordinary second repeated de la Vallée Poussin sums are

$$ {}_2\mathbb{V}_{n,m}(f;x) = \frac{1}{n}\sum_{k=m}^{m+n-1} {}_1\mathbb{V}_{n,k}(f;x). $$

In general, for $r \geq 2$, the ordinary r-repeated de la Vallée Poussin sums are

$$ {}_r\mathbb{V}_{n,m}(f;x) = \frac{1}{n}\sum_{k=m}^{m+n-1} {}_{r-1}\mathbb{V}_{n,k}(f;x) $$

see [26].

It worth to emphasize here that the particular cases of these sums are the following: for $r = 1$ and $n = 1$ we obtain ${}_1\mathbb{V}_{1,m}(f;x) = s_m(f;x)$, while for $r = 1$ and $m = 1$ we have ${}_1\mathbb{V}_{n,1}(f;x) = \sigma_n(f;x)$.

Provoked by these sums, we are going to use them to prove the homologous of Theorems 1-4 which is the main objective of this paper. In order to do this, we need two Lemmas given in next section.

2 Helpful Lemmas

The following integral representation (see also [26]) of ${}_r\mathbb{V}_{n,m}(f;x)$,

$$ {}_r\mathbb{V}_{n,m}(f;x) = \frac{1}{\pi}\int_0^{\pi} [f(x+t) + f(x-t)]_r W_{n,m}(t)dt, \tag{2.4} $$

holds true, where

$$ {}_rW_{n,m}(t) := \frac{\sin^r(nt/2)\sin((2m + r(n-1) + 1)t/2)}{n^r \sin^{r+1}(t/2)}. $$

Lemma 1. Let $r \in \{1, 2, \dots\}$. Then,

$$ \frac{2}{\pi}\int_0^{\pi} {}_rW_{n,m}(t)dt = 1. $$

Proof. First of all we have

$$ {}_1W_{n,m}(t) = \frac{1}{n}\sum_{k=m}^{m+n-1} D_k(t) $$

where

$$ D_\ell(t) = \frac{\sin{(2\ell+1)}\frac{t}{2}}{2\sin\frac{t}{2}}. $$

Taking into account that (see [5], page 135)

$$\int_0^\pi D_\ell(t)dt = \frac{\pi}{2}, \quad \ell \in \{0,1,2,\dots\},$$

we obtain

$$\int_0^\pi {}_1W_{n,m}(t)dt = \frac{\pi}{2}. \tag{2.5}$$

Now, from

$${}_2W_{n,m}(t) = \frac{1}{n}\sum_{k=m}^{m+n-1} {}_1W_{n,k}(t)$$

and (2.5) we also get

$$\int_0^\pi {}_2W_{n,m}(t)dt = \frac{\pi}{2}.$$

Repeating this process r-times, we arrive at

$$\int_0^\pi {}_rW_{n,m}(t)dt = \frac{\pi}{2}.$$

With this we have finished the proof. QED

The next Lemma gives the upper estimates of absolute value of the "r-repeated kernel" ${}_rW_{n,m}(t)$.

Lemma 2. Let $r, m, n \in \{1,2,3,\dots\}$ and $0 < t \le \pi$. Then,

(i)

$$|{}_rW_{n,m}(t)| = \mathcal{O}((2m + r(n-1) + 1))$$

(ii)

$$|{}_rW_{n,m}(t)| = \mathcal{O}\left(\frac{1}{t}\right)$$

(iii)

$$|{}_rW_{n,m}(t)| = \mathcal{O}\left(\frac{1}{n^r t^{r+1}}\right).$$

Proof. The use of elementary inequality $|\sin(t)| \le |t|$ and the Jordan's inequality $\pi|\sin(t)| \ge 2t$, for $t \in [0, \pi/2]$, imply:

(i)

$$\begin{aligned}|{}_rW_{n,m}(t)| &= \frac{|\sin^r(nt/2)\sin((2m+r(n-1)+1)t/2)|}{n^r\left|\sin^{r+1}(t/2)\right|}\\ &\le \frac{(nt/2)^r\,((2m+r(n-1)+1)t/2)}{n^r\left(\frac{t}{\pi}\right)^{r+1}}\\ &= \left(\frac{\pi}{2}\right)^{r+1}(2m+r(n-1)+1),\end{aligned}$$

(ii)

$$|{}_rW_{n,m}(t)| \le \frac{(nt/2)^r}{n^r\left(\frac{t}{\pi}\right)^{r+1}} = \left(\frac{\pi}{2}\right)^r\frac{\pi}{t},$$

and

(iii)

$$\begin{aligned}|{}_rW_{n,m}(t)| =& \frac{|\sin^r(nt/2)||\sin((2m+r(n-1)+1)t/2)|}{n^r\left|\sin^{r+1}(t/2)\right|}\\ \le& \frac{1}{n^r\left(\frac{t}{\pi}\right)^{r+1}} = \frac{1}{n^r}\left(\frac{\pi}{t}\right)^{r+1}.\end{aligned}$$

The proof is completed. QED

3 Main Results

First main result is the following.

Theorem 1. Let r, n, m be any natural numbers. If the function $f(x)$ is bounded, i.e. $|f(x)| \le K$ for $0 < K < \infty$, then the means ${}_r\mathbb{V}_{n,m}(f;x)$ satisfy

$$|{}_r\mathbb{V}_{n,m}(f;x)| = \mathcal{O}\left(1+\ln\left(\frac{r(n-1)+2m+1}{n}\right)\right). \tag{3.6}$$

Proof. Using the ordinary r-repeated de la Vallée Poussin sums of $s_m(f;x)$, i.e.

$${}_r\mathbb{V}_{n,m}(f;x) = \frac{1}{n}\sum_{k=m}^{m+n-1} {}_{r-1}\mathbb{V}_{n,k}(f;x)$$

with agreement ${}_0\mathbb{V}_{n,k}(f;x) := s_k(f;x)$, we get

$${}_r\mathbb{V}_{n,m}(f;x) = \frac{1}{\pi}\int_0^{\pi}\phi_x(t)_rW_{n,m}(t)dt,$$

where $\phi_x(t) := f(x+t) + f(x-t)$.

Let

$$\eta_1(n) := \min\left(\frac{1}{\pi\left(r(n-1)+2m+1\right)}, \mu\right), \quad \eta_2(n) := \min\left(\frac{1}{\pi n}, \mu\right),$$

and $\mu \in (0, \pi]$. Then

$$|{}_r\mathbb{V}_{n,m}(f;x)| \leq \frac{1}{\pi}\Bigg(\underbrace{\int_0^{\eta_1(n)} |\phi_x(t)||{}_rW_{n,m}(t)|dt}_{:=\mathbb{I}_1} + \underbrace{\int_{\eta_1(n)}^{\eta_2(n)} |\phi_x(t)||{}_rW_{n,m}(t)|dt}_{:=\mathbb{I}_2}$$
$$+ \underbrace{\int_{\eta_2(n)}^{\mu} |\phi_x(t)||{}_rW_{n,m}(t)|dt}_{:=\mathbb{I}_3} + \underbrace{\int_{\mu}^{\pi} |\phi_x(t)||{}_rW_{n,m}(t)|dt}_{:=\mathbb{I}_4}\Bigg). \tag{3.7}$$

The use of Lemma 2 (i), implies

$$\mathbb{I}_1 = \mathcal{O}(1)\int_0^{\eta_1(n)} ((2m + r(n-1)+1)dt = \mathcal{O}(1). \tag{3.8}$$

Similarly, but this time using Lemma 2 (ii), we get

$$\mathbb{I}_2 = \mathcal{O}(1)\int_{\eta_1(n)}^{\eta_2(n)} \frac{dt}{t} = \mathcal{O}(1)\ln\left(\frac{r(n-1)+2m+1}{n}\right). \tag{3.9}$$

Applying Lemma 2 (iii), we have that

$$\mathbb{I}_3 = \mathcal{O}(n^{-r})\int_{\eta_2(n)}^{\mu} \frac{dt}{t^{r+1}} = \mathcal{O}(1). \tag{3.10}$$

Finally, we also can write

$$\mathbb{I}_4 = \mathcal{O}(1)\int_{\mu}^{\pi} \frac{|\phi_x(t)|}{n^r t^{r+1}}dt. \tag{3.11}$$

Combining (3.7), (3.8), (3.9), (3.10), and (3.11) with $\mu = \pi$ we obtain (3.6). The proof is completed.

QED

Theorem 2. Let r, n, m be any natural numbers and $m = \mathcal{O}(n)$. If the function $f(x)$ is continuous, then the estimate

$$|{}_r\mathbb{V}_{n,m}(f;x) - f(x)| = \mathcal{O}\left(2 + \ln\left(\frac{r(n-1)+2m+1}{n}\right)\right) E_{n+r-2m+1}$$

holds true uniformly in all x.

Proof. Let $\tau_m^*(x)$ denote the trigonometric polynomial of best approximation of $f(x)$ whose degree is not higher then m. It is clear, by definition of $s_n(f;x)$ and ${}_r\mathbb{V}_{n,m}(f;x)$, that

$${}_r\mathbb{V}_{n,m}(f-\tau_m^*;x) = {}_r\mathbb{V}_{n,m}(f;x) - \tau_m^*(x),$$

whenever $n+r-2m+1 \geq m$.

Whence, it holds

$$\begin{aligned}|{}_r\mathbb{V}_{n,m}(f;x) - f(x)| \leq& |{}_r\mathbb{V}_{n,m}(f;x) - t^*_{n+r-2m+1}(x)| + |t^*_{n+r-2m+1}(x) - f(x)| \\ \leq& |{}_r\mathbb{V}_{n,m}(f;x) - t^*_{n+r-2m+1}(x)| + E_{n+r-2m+1}. \end{aligned} \tag{3.12}$$

Also, by the definition of r-repeated de la Vallée Poussin means ${}_r\mathbb{V}_{n,m}(f;x)$, we have

$$|{}_r\mathbb{V}_{n,m}(f;x) - t^*_{n+r-2m+1}(x)| \leq \frac{2E_{n+r-2m+1}}{\pi}\int_0^\pi |{}_rW_{n,m}(t)|dt.$$

The remaining estimate which has been obtained, with same reasoning as in the proof of Theorem 1 for $\mu = \pi$, is

$$\begin{aligned}&|{}_r\mathbb{V}_{n,m}(f;x) - t^*_{n+r-2m+1}(x)| \\ &\qquad= \mathcal{O}\left(1+\ln\left(\frac{r(n-1)+2m+1}{n}\right)\right)E_{n+r-2m+1},\end{aligned} \tag{3.13}$$

in which we have used the constant function $h(t) := E_{n+r-2m+1}$ instead of $f(x)$. Subsequently, relations (3.12) and (3.13) imply the required inequality.

The proof is completed. $\boxed{QED}$

Theorem 3. Let $r,n,m \in \mathbb{N}$ be natural numbers. If $f \in \text{Lip}\alpha$ and $m = \mathcal{O}(n)$, then

$$|{}_r\mathbb{V}_{n,m}(f;x) - f(x)| = \begin{cases}\mathcal{O}\left(\frac{1}{(r(n-1)+2m+1)^\alpha}\right), & \text{for} \quad 0<\alpha<1 \\ \mathcal{O}\left(\frac{\ln(r(n-1)+2m+1)}{n}\right), & \text{for} \quad \alpha=r=1,\end{cases}$$

holds true uniformly in all x.

Proof. Since $f \in \text{Lip}\alpha$, then using the equality (2.4), we get

$$\begin{aligned}&|{}_r\mathbb{V}_{n,m}(f;x) - f(x)| = \mathcal{O}(1)\int_0^\pi t^\alpha|{}_rW_{n,m}(t)|dt \\ &= \mathcal{O}(1)\Bigg(\underbrace{\int_0^{\frac{1}{r(n-1)+2m+2}}(\cdot)dt}_{:=\mathbb{P}_1} + \underbrace{\int_{\frac{1}{r(n-1)+2m+2}}^{\frac{\pi}{r(n-1)+2m+1}}(\cdot)dt}_{:=\mathbb{P}_2} + \underbrace{\int_{\frac{\pi}{r(n-1)+2m+1}}^{\pi}(\cdot)dt}_{:=\mathbb{P}_3}\Bigg).\end{aligned} \tag{3.14}$$

For $0 < \alpha \leq 1$, we apply Lemma 2 (i) to obtain

$$\begin{aligned}\mathbb{P}_1 =&\mathcal{O}((2m + r(n-1) + 1)) \int_0^{\frac{1}{r(n-1)+2m+2}} t^\alpha dt \\ =&\frac{\mathcal{O}(1)}{(2m + r(n-1) + 1)^\alpha}.\end{aligned} \tag{3.15}$$

Likewise, we use Lemma 2 (ii) to get

$$\mathbb{P}_2 = \mathcal{O}(1) \int_{\frac{1}{r(n-1)+2m+2}}^{\frac{\pi}{r(n-1)+2m+1}} t^{\alpha-1} dt = \frac{\mathcal{O}(1)}{(r(n-1) + 2m + 1)^\alpha}. \tag{3.16}$$

Finally, we need to estimate $\mathbb{P}_3$ in the case when $0 < \alpha < 1$ and after that in the case when $\alpha = r = 1$. Indeed, taking into account Lemma 2 (iii) and $0 < \alpha < 1$, we get

$$\begin{aligned}\mathbb{P}_3 =&\mathcal{O}\left(\frac{1}{n^r}\right) \int_{\frac{\pi}{r(n-1)+2m+1}}^{\pi} t^{\alpha-r-1} dt \\ =&\mathcal{O}\left(\frac{1}{n^r}\right)\left(\frac{\pi^{\alpha-r}}{r-\alpha} \frac{1}{(r(n-1)+2m+1)^{\alpha-r}} - \frac{\pi^{\alpha-r}}{r-\alpha}\right) \\ =&\frac{\mathcal{O}(1)}{(r(n-1)+2m+1)^\alpha}.\end{aligned} \tag{3.17}$$

For $\alpha = 1$ and $r = 1$, we have

$$\mathbb{P}_3 = \mathcal{O}\left(\frac{1}{n}\right) \int_{\frac{\pi}{r(n-1)+2m+1}}^{\pi} t^{-1} dt = \mathcal{O}\left(\frac{\ln(r(n-1)+2m+1)}{n}\right). \tag{3.18}$$

Therefore, our conclusion follows from (3.14), (3.15), (3.16), (3.17), and (3.18).

The proof is completed. QED

Remark 1. The same results, as Theorems 1–3, hold true for conjugate functions as well.

4 Conclusions

In this paper we have increased the utilization rate of the r-repeated de la Vallée-Poussin sums. Using these sums, several results on their upper bound, their convergence, and the degree of approximation of continuous and 2π-periodic functions and those belonging to Lipschitz class, has been proved. The results has opened the path for defining and applying the other repeated sums in problems treated here and in other papers.

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