An interesting table related to Fibonacci numbers

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Abstract. In this paper, we show that sum of the row elements on the table formed by a given recurrence relation, and each element on this table can be obtained using two different Fibonacci numbers.

Keywords: Matrices, Fibonacci numbers, recurrence relation.

MSC 2000 classification: 11B37, 11B39, 11B20

1 Introduction

There are interesting recurrence relations in [2] and then, we take an infinite-dimensional matrix $A = (a_{i,j})$ formed by the following recurrence relation.

\[
\begin{align*}
a_{1,j} &= F_j + (-1)^j F_1 \\
a_{2,j} &= F_{j+1} + (-1)^j F_2 \\
a_{i,j} + a_{i+1,j} &= a_{i+2,j}, \quad i > 1
\end{align*}
\]

2 Results on the table

Using the recurrence relation (1), we have the following results.

1 Theorem.

\[
a_{i,j} = F_{i+j-1} + (-1)^j F_i, \quad i \geq 1.
\]

Proof. By the recurrence relation (1), the result is true for $i = 1, 2$.

We now assume that it is true for all integer less than $n$, where $n > 2$:

\[
a_{n-1,j} = F_{n+j-2} + (-1)^j F_{n-1}.
\]
We then get

\[ a_{n,j} = a_{n-2,j} + a_{n-1,j} = F_{n+j-3} + (-1)^j F_{n-2} + F_{n+j-2} + (-1)^j F_{n-1} \]

\[ = F_{n+j-1} + (-1)^j F_n. \]

Thus, the result is true for all \( n \geq 2 \).

\[ \Box \]

**TABLE**

<table>
<thead>
<tr>
<th>i/j</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>9</td>
<td>12</td>
<td>22</td>
<td>33</td>
<td>56</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>7</td>
<td>14</td>
<td>20</td>
<td>35</td>
<td>54</td>
<td>90</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>5</td>
<td>3</td>
<td>10</td>
<td>11</td>
<td>23</td>
<td>32</td>
<td>57</td>
<td>87</td>
<td>146</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>8</td>
<td>5</td>
<td>16</td>
<td>18</td>
<td>37</td>
<td>52</td>
<td>92</td>
<td>141</td>
<td>236</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>13</td>
<td>8</td>
<td>26</td>
<td>29</td>
<td>60</td>
<td>84</td>
<td>149</td>
<td>228</td>
<td>382</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>21</td>
<td>13</td>
<td>42</td>
<td>47</td>
<td>97</td>
<td>136</td>
<td>241</td>
<td>369</td>
<td>618</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>34</td>
<td>21</td>
<td>68</td>
<td>76</td>
<td>157</td>
<td>220</td>
<td>390</td>
<td>597</td>
<td>1000</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>55</td>
<td>34</td>
<td>110</td>
<td>123</td>
<td>254</td>
<td>356</td>
<td>631</td>
<td>966</td>
<td>1618</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>89</td>
<td>55</td>
<td>178</td>
<td>199</td>
<td>411</td>
<td>576</td>
<td>1021</td>
<td>1563</td>
<td>2618</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>144</td>
<td>89</td>
<td>288</td>
<td>322</td>
<td>665</td>
<td>932</td>
<td>1652</td>
<td>2529</td>
<td>4236</td>
<td></td>
</tr>
</tbody>
</table>

\[ : \]

**2 Corollary.**

\[ a_{i,j} + a_{i,j+1} = F_{i+j+1}, \quad i, j \geq 1. \]

**Proof.** Since \( a_{i,j} = F_{i+j-1} + (-1)^j F_i \), by Theorem 1,

\[ a_{i,j} + a_{i,j+1} = F_{i+j-1} + (-1)^j F_i + F_{i+j} + (-1)^{j+1} F_i \]

\[ = F_{i+j-1} + F_{i+j} = F_{i+j+1}. \]

\[ \Box \]

We now examine the sum of the first \( n \) elements on any row:

\[ a_{i,1} = F_i + (-1)^1 F_i = F_{i-1}, \]

\[ a_{i,2} = F_{i+1} + (-1)^2 F_i = F_{i+1} + F_i, \]

\[ a_{i,3} = F_{i+2} + (-1)^3 F_i = F_{i+2} - F_i, \]

\[ : \]

\[ a_{i,n} = F_{i+n-1} + (-1)^n F_i. \]
Here, it is important that $n$ is even or odd in these equations. Therefore, we first assume that $n$ is even. Adding these equations, we get

$$
\sum_{j=1}^{n} a_{i,j} = F_{i+1} + F_{i+2} + \cdots + F_{i+n-1} + F_{i}
$$

$$
= \sum_{k=0}^{n-1} F_{i+k} = \sum_{r=1}^{i+n-1} F_{r} - \sum_{r=1}^{i-1} F_{r} = F_{i+n+1} - F_{i+1}.
$$

Now let $n$ be odd. Then, we get

$$
\sum_{j=1}^{n} a_{i,j} = F_{i+1} + F_{i+2} + \cdots + F_{i+n-1}
$$

$$
= \sum_{k=1}^{n-1} F_{i+k} = \sum_{r=1}^{i+n-1} F_{r} - \sum_{r=1}^{i} F_{r} = F_{i+n+1} - F_{i+2}.
$$

Thus,

**3 Theorem.**

$$
\sum_{j=1}^{n} a_{i,j} = \begin{cases} 
F_{i+n+1} - F_{i+2} & \text{if } n \text{ is odd}, \\
F_{i+n+1} - F_{i+1} & \text{if } n \text{ is even}.
\end{cases}
$$

We now examine a determinant of a special matrix $A = (a_{i,j})_{2 \times 2}$ taken from the above table. We now take the matrix

$$
A = \begin{bmatrix} a_{i,j} & a_{i,j+1} \\
                  a_{i+1,j} & a_{i+1,j+1} \end{bmatrix}
$$

and calculate the determinant of this matrix as follows.

**4 Theorem.**

$$
\begin{vmatrix} a_{i,j} & a_{i,j+1} \\
                  a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} = (-1)^{i+j+1}[F_{j+1} - 1], \quad i, j \geq 1.
$$

**Proof.** Using the Theorem 1, we get

$$
\begin{vmatrix} a_{i,j} & a_{i,j+1} \\
                  a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} = F_{i+j-1}F_{i+j+1} + (-1)^{j+1}F_{i+j-1}F_{i+1} + 
$$

$$
(-1)^{j}F_{i+j+1}F_{i} - F_{i+j}^2 - (-1)^{j}F_{i+j}F_{i+1} - (-1)^{j+1}F_{i+j}F_{i} = F_{i+j-1}F_{i+j+1} - F_{i+j}^2 + (-1)^{j+1}(F_{i+j-1}F_{i+1} - F_{i+j}F_{i}) + 
$$

$$
(-1)^{j}(F_{i+j+1}F_{i} - F_{i+j}F_{i+1}).
$$
By the well-known identities about the Fibonacci numbers [2, p.74, 87]
\[ F_{r-1}F_{r+1} - F_r^2 = (-1)^r, \quad F_mF_n - F_{m+k}F_{n-k} = (-1)^iF_{m+k-n}F_k, \]
we obtain
\[
\begin{vmatrix}
  a_{i,j} & a_{i,j+1} \\
  a_{i+1,j} & a_{i+1,j+1}
\end{vmatrix} = (-1)^{i+j} + (-1)^{i+j+1}F_{j-1} + (-1)^{i+j+2}F_{i-j}
\]
\[
= (-1)^{i+j} + (-1)^{i+j+1}F_{j-1} + (-1)^{i+j+3}F_j
\]
\[
= (-1)^{i+j+1}[F_{j-1} + F_j - 1] = (-1)^{i+j+1}[F_{j+1} - 1].
\]
We can generalize Theorem 4, as the next theorem shows.

5 Theorem.
\[
\begin{vmatrix}
  a_{i,j} & a_{i,k} \\
  a_{i+1,j+1} & a_{i+1,k}
\end{vmatrix} = (-1)^{i+j+1}F_{k-j} + (-1)^{i+j+k}[F_{1-j} - F_{1-k}], \quad i, j \geq 1, k > j.
\]

Proof. From the definition of determinant of a matrix, we obtain
\[
\begin{vmatrix}
  a_{i,j} & a_{i,k} \\
  a_{i+1,j+1} & a_{i+1,k}
\end{vmatrix} = a_{i,j}a_{i+1,k} - a_{i,k}a_{i+1,j+1}
\]
\[
= (F_{i+j-1} - (-1)^jF_i)(F_{i+k} + (-1)^kF_{i+1}) - (F_{i+k-1} - (-1)^kF_i)(F_{i+j} + (-1)^jF_{i+1})
\]
\[
= F_{i+j-1}F_{i+k} + (-1)^kF_{i+j-1}F_{i+1} + (-1)^jF_{i+k} + (-1)^jF_{i+i+k}F_{i+1}
\]
\[
- F_{i+j}F_{i+k-1} - (-1)^kF_{i+i}F_{i+k-1} - (-1)^{i+j+k}F_{i+1}F_i
\]
\[
= F_{i+j-1}F_{i+k} + (-1)^k(F_{i+j-1}F_{i+1} + F_{i+j}F_i) + (-1)^j(F_{i+k}F_{i+k-1} - F_{i+i+k}F_{i+1})
\]
Since by [1]
\[ F_{n+k}F_{n+k} - F_nF_{n+h+k} = (-1)^nF_{h}F_{k}, \]
we get
\[
\begin{vmatrix}
  a_{i,j} & a_{i,k} \\
  a_{i+1,j+1} & a_{i+1,k}
\end{vmatrix} = (-1)^{i+j}F_{k-j} + (-1)^k(-1)^{i+j}F_{1-j} + (-1)^j(-1)^{i+k}F_{1-k}
\]
\[
= (-1)^{i+j}F_{k-j} + (-1)^{i+j+k}(F_{1-j} + F_{1-k}),
\]
as desired.

References
