Note di Matematica
Note Mat. **43** (2023) no. 2, 67–81.
ISSN 1123-2536, e-ISSN 1590-0932
doi:10.1285/i15900932v43n2p67

Genus One Almost Simple Groups of Lie Rank Two

Haval M. Mohammed Salih
Department of Mathematics, Faculty of Science, Soran University, Kawa St. Soran, Iraq.
havalmahmood07@gmail.com

Received: 1.6.2023; accepted: 22.9.2023.

Abstract. In this paper, we assume that G is a finite group with socle $PSL(3, q)$ and G acts on the projective points of 2-dimensional projective geometry $PG(2, q)$, q is a prime power. By using a new method, we show that G possesses no genus one group if $q > 13$. Furthermore, we study the connectedness of the Hurwitz space $\mathcal{H}_r^{in}(G)$ for a given group G, genus one and $q \leq 13$.

Keywords: Projective special linear group; Fixed point ratio; Genus one group

MSC 2022 classification: primary 20B05, secondary 20C30

1 Introduction

Let X be a compact connected Riemann surface of genus g and that $f\colon X \to \mathbb{P}^1$ is a meromorphic function where $\mathbb{P}^1$ is the Riemann sphere. For every meromorphic function, there is a number N such that the fiber $f^{-1}(q)$ is of size N for all but finitely many points $q \in \mathbb{P}^1$. The number N is called the degree of f. So every compact Riemann surface can be made into the branched covering of $\mathbb{P}^1$. The points p are called the branch points of f if $|f^{-1}(p)| < N$. It is well known that the set of branch points is finite and it will be denoted by $B = \{p_1, ..., p_r\}$. For $q \in \mathbb{P}^1 \setminus B$, the fundamental group $\pi_1(\mathbb{P}^1 \setminus B, q)$ is a group which is generated by all homotopy classes of loops γ_i winding once around the point p_i. γ_i are subject to the relation that $\gamma_1 \dots \gamma_r = 1$ in $\pi_1(\mathbb{P}^1 \setminus B, q)$. The explicit and well known construction of Hurwitz shows that a Riemann surface X with N branching coverings of $\mathbb{P}^1$ is defined in the following way: consider the preimage $f^{-1}(q) = \{x_1, ..., x_N\}$, every loop in γ in $\mathbb{P}^1 \setminus B$ can be lifted to N paths $\tilde{\gamma_1}, \cdots, \tilde{\gamma_N}$ where $\tilde{\gamma_i}$ is the unique path lift of γ and $\tilde{\gamma_i}(0) = x_i$ for every i. The endpoints $\tilde{\gamma_i}(1)$ also lie over q. That is

$$G = \langle x_1, x_2, ..., x_r \rangle \tag{1.1}$$

$$\prod_{i=1}^{r} x_i = 1,\ x_i \in G^{\#} = G \setminus \{1\},\ i = 1, ..., r \tag{1.2}$$

$$\sum_{i=1}^{r} ind\, x_i = 2(N + g - 1) \tag{1.3}$$

where $ind\, x_i$ is the minimal number of 2-cycles needed to express x_i as a product. Equation (1.2) is called the Riemann Hurwitz formula. A transitive subgroup $G \leq S_N$ is called a genus g group if there exist $x_1, ..., x_r \in G$ satisfying (1.1), (1.1) and (1.2) and then we call $(x_1, ..., x_r)$ the genus g tuple of G.

A subset Y of Ω is a block for G if for all $g \in G$ either $Y^g = Y$ or $Y^g \cup Y = \emptyset$. The action of G is primitive if it is transitive and all blocks are trivial. If the action of G on Ω is primitive, we call G a primitive genus g group.

A group G is said to be almost simple if it contains a non-abelian simple group S such that $S \leq G \leq Aut(S)$. In [7], Kong worked on almost simple groups whose socle is a projective special linear group. Moreover, she gave a complete list for some almost simple groups of Lie rank 2 up to ramification type in her PhD thesis for genus 0,1 and 2 system. Furthermore, she showed that the almost simple groups with socle $PSL(3, q)$ do not posses genus low tuples if $q \geq 16$. In [8], Mohammed Salih gave the classification of some almost simple groups with socle $PSL(3, q)$ for $g = 0$ up to braid action and diagonal conjugation. Our calculations are done with the aid of GAP; see [11].

We will now describe the work carried out in this paper. In the second section we review some basic concepts and results will be used later. In the third section, we provide some basic facts for computing fixed points and generating tuples. Moreover, we show that the almost simple groups with socle $PSL(3, q)$ do not posses genus one tuples if $q \geq 13$. In the fourth section, we show that the connectedness of the almost simple groups with socle $PSL(3, q)$ if $q \leq 13$.

2 Preliminary results

Assume that G is a finite permutation group of degree N and $x_i \in G \setminus \{1\}$. The signature of the r-tuple $x = (x_1, ..., x_r)$ is the r-tuple $d = (d_1, ..., d_r)$ where $d_i = o(x_i)$. We assume that $d_i \leq d_j$ if $i \leq j$, because of braid action on x. The center of $GL(n, q)$ is the set of all scalar matrices and denoted by $Z(GL(n, q))$. The projective general linear group and the projective special linear group are defined by $PGL(n, q) = \frac{GL(n,q)}{Z(GL(n,q))}$ and $PSL(n, q) = \frac{SL(n,q)}{Z(SL(n,q))}$ respectively where $Z(SL(n, q)) = SL(n, q) \cap Z(GL(n, q))$.

Let Ω be the set of the projective points of projective geometry $PG(n - 1, q)$ that is the set of 1-dimensional subspaces of vector space over a finite field $GF(q)$. So we have $|\Omega| = \frac{q^n-1}{q-1}$. Moreover, $PGL(n,q)$ and $PSL(n,q)$ act primitively on Ω. In this paper, we take $n = 3$, so we have $|\Omega| = q^2 + q + 1$.

The following result will tell us the tuple x can not generate G, where $G = PGL(3,q)$ or $PSL(3,q)$ if 2., 3. and 4. below hold. So, setting $A(d) = \sum_{i=1}^{r} \frac{d_i-1}{d_i}$, we have $A(d) \geq \frac{85}{42}$.

Proposition 1. *[2] Assume that a group G acts transitively and faithfully on Ω and $|\Omega| = N$. Let $r \geq 2$, $G = \langle x_1, ..., x_r \rangle$, $\prod_{i=1}^{r} x_i = 1$ and $o(x_i) = d_i > 1$, $i = 1, ..., r$. Then one of the following holds:*

(1) $\sum_{i=1}^{r} \frac{d_i-1}{d_i} \geq \frac{85}{42}$.

(2) $r = 4$, $d_i = 2$ *for each* $i = 1$ *and* $G'' = 1$.

(3) $r = 3$ *and (up to permutation)*$(d_1, d_2, d_3) =$

- **(a)** $(3,3,3), (2,3,6)$ *or* $(2,4,4)$ *and* $G'' = 1$.
- **(b)** $(2,2,d)$ *and G is dihedral.*
- **(c)** $(2,3,3)$ *and* $G \cong A_4$.
- **(d)** $(2,3,4)$ *and* $G \cong S_4$.
- **(e)** $(2,3,5)$ *and* $G \cong A_5$.

(4) $r = 2$ *and G is cyclic.*

For a permutation x of the finite set Ω, let $Fix(x)$ denote the fixed points of x on Ω and $f(x) = |Fix(x)|$ is the number of fixed point of x. Note that the conjugate elements have the same number of fixed points.

The following result provides a useful connection between fixed points and indices.

Lemma 1. *[2] If x is a permutation of order d on a set of size N, then* $ind\, x = N - \frac{1}{d}\sum_{y \in \langle x \rangle} f(y)$ *where $\langle x \rangle$ is the cyclic group generated by x.*

The fixed point ratio of x is defined by $fpr(x) = \frac{f(x)}{N}$. The codimension of the largest eigenspace of a linear transformation $\bar{g}$ in $GL(n,q)$ is denoted by $v(\bar{g})$.

The classification of non identity elements in $PGL(3,q)$ by their fixed points can be found in Table 1 in [7], we have two cases. If $v(\bar{g}) = 1$, then g fixes $q+1$ or $q+2$ points. Otherwise, it fixes at most 3 points. From this fact, we will show that there are no genus g systems for $PSL(3,q)$ and $PGL(3,q)$ when $q > 25$ and $g \leq 2$.

Table 1: Number of Fixed points

$v(\bar{g})$	Type of eigenspaces of $\bar{g} \in GL(n,q)$	Number of fixed points of $g \in PGL(3,q)$
3	no eigenspace	0
2	one 1-dimensional eigenspace	1
1	one 1-dimensional eigenspaces	$q+1$
2	two 1-dimensional eigenspaces	2
1	one 1-dimensional and one 2-dimensional eigenspaces	$q+2$
2	three one dimensional eigenspaces	3

3 Existence of genus g system

The following lemma is well known in [14, 6]

Lemma 2. *(1) If $\frac{1}{N}\sum_{i=1}^{r}\sum_{j=1}^{d_i-1}\frac{f(x^j)}{d_i} < A(d)-2$, then d is not genus zero system.*

(2) If $\frac{1}{N}\sum_{i=1}^{r}\sum_{j=1}^{d_i-1}\frac{f(x^j)}{d_i} \neq A(d)-2$, then d is not genus one system.

(3) If $\frac{1}{N}\sum_{i=1}^{r}\sum_{j=1}^{d_i-1}\frac{f(x^j)}{d_i} > A(d)-2$, then d is not genus two system.

Now, we are going to apply Lemma 2, to exclude all signatures which do not have a genus $g \leq 2$ system. As a result, we will obtain Theorem 1.

Let F be the set of elements with $q+1$ or $q+2$ fixed points in $PGL(3,q)$. So we have $fpr(x) \leq \begin{cases} \frac{q+3}{N} & if\ x \in F \\ \frac{3}{N} & if\ x \notin F \end{cases}$

Assume that $\alpha = \frac{q^2+q+2}{N}$ and $\gamma = \frac{3}{N}$. Combining the Riemann Hurwitz formula as done in [7], we obtain the following inequality

$$A(d) \leq \frac{2+\epsilon+\beta(\alpha-\gamma)}{1-\gamma} \tag{3.4}$$

where $\epsilon \leq \frac{2}{N}$ and $\beta = \sum_{i=1}^{r}\frac{|\langle x_i\rangle^{\#}\cap F|}{d_i}$ and $\langle x_i\rangle^{\#}$ is the set of all elements in the group generated by x_i except the identity [7].

If $\alpha = \gamma$ in inequality (2.4), then we obtain the following

$$A(d) \leq \frac{2+\epsilon}{1-\alpha} \tag{3.5}$$

and we have $fpr(x) \leq \alpha$.

The following result is an interesting tools to compute β.

Lemma 3. *(Scott Bound)[15] Let $G \leq GL(n,q)$. If a triple $x = (x_1, x_2, x_3)$ satisfies $G = \langle x_1, x_2, x_3 \rangle$ and $x_1x_2x_3 = 1$, then $v(x_i) + v(x_j) \geq n$ where $i \neq j$ and $1 \leq i, j \leq 3$. In particular if $n \geq 3$ and $i \neq j$, then $v(x_i) \geq 2$ or $v(x_j) \geq 2$.*

Now bound β for the tuple $x = (x_1, ..., x_r)$ can be computed by the following formula which exists in [7].

$$\beta \leqslant \sum_{x_i \in F} \frac{d_i - 1}{d_i} + \sum_{x_i \notin F} \frac{d_i - \phi(d_i) - 1}{d_i}.$$

Notice that Lemma 3 will tell us for the tuple of length 3, that at most one element lies in F.

Now consider $PGL(3,q)$ and $PSL(3,q)$. Let $q \geq 16$, then using inequality (3.5), we have $A(d) \leq \frac{548}{255}$. If $r \geq 4$, then $A(d) \geq A((2,2,2,3)) = \frac{13}{6}$. But $\frac{548}{255} < \frac{13}{6}$. So the number of branch points r must be 3.

Now we are looking for signatures which satisfy the inequality $\frac{85}{42} \leq A(d) \leq \frac{32}{15}$. This leads d only can be $(2,3,d_3)$ with $7 \leq d_3 \leq 57$, $(2,4,d_3)$ with $5 \leq d_3 \leq 10$, $(2,5,d)$ with $5 \leq d \leq 7$, $(3,3,d)$ with $4 \leq d \leq 6$.

Now, we compute β for all signatures which satisfy $\frac{85}{42} \leq A(d) \leq \frac{32}{15}$.

$d =$	$\beta \leqslant$
$(2,3,n)$ with $7 \leqslant n \leqslant 57$	$\frac{41}{30}$
$(2,4,n)$ with $5 \leqslant n \leqslant 10$	$\frac{5}{4}$
$(2,5,n)$ with $5 \leqslant n \leqslant 7$	$\frac{13}{10}$
$(3,3,n)$ with $4 \leqslant n \leqslant 6$	$\frac{7}{6}$

In the above table the maximum β is $\frac{41}{30}$. Now set $\beta \leq \frac{41}{30}$ and $q \geq 16$. We substitute both of them in inequality (2.4) and we obtain that $A(d) \leq \frac{3572243}{1702740}$. From this, we find all signatures d, which are the following:

$d =$	$\beta \leqslant$
$(2,3,n)$ with $7 \leqslant n \leqslant 15$	$\frac{5}{4}$
$(2,4,n)$ with $4 \leqslant n \leqslant 5$	$\frac{5}{4}$
$(3,3,4)$	$\frac{11}{12}$

Again, we choose the maximum β in the above table which is $\beta \leq \frac{5}{4}$. So we put $\beta \leq \frac{5}{4}$ and $q \geq 16$ in inequality (2.4) and hence $A(d) \leq \frac{3012}{1445}$. Therefore, all signatures are $(2,3,d_3)$ with $7 \leq d_3 \leq 14$, $(2,4,5), (2,4,6), (3,3,4)$.

Finally, for each signature d we can compute β and $A(d)$ and put in inequality (2.4). So we can solve it and obtaining the values of q.

Here we reduce the prime powers and give the new version of Theorem 7 in [7].

Theorem 1. *If $PGL(3,q)$ or $PSL(3,q)$ possesses genus $g \leq 2$ system, then one of the following is hold:*

(1) $q \leq 13$;

(2) d *and* q *as shown in the following table*

d	β	$A(d)$	q
(2,3,7)	1/2	85/42	16,17,19,23,25
(2,3,8)	7/8	49/24	16,17,19,23
(2,3,10)	7/6	31/15	16
(2,4,5)	3/4	41/20	16,17
(2,4,6)	5/4	25/12	16

The following results are devoted to compute indices of elements of order 2, 3, 4, 5, 7 and 8 in $PSL(3,q)$ which will be used in Equation (1.2). This tells us the given signature possess genus g group or not. Let e_d be an element of order d in G.

Lemma 4. *In* $G = PSL(3,q)$.

(1) *If* $2 \nmid q$, *then* $f(e_2) = q+2$, *ind* $e_2 = \frac{q^2-1}{2}$.

(2) *If* $2 \mid q$, *then* $f(e_2) = q+1$ *and ind* $e_2 = \frac{q^2}{2}$.

Proof. The proof can be found in [7]. QED

Lemma 5. *In* $G = PSL(3,q)$.

(1) *If* $q \equiv 2 \mod 3$, *then* $f(e_3) = 1$, *ind* $e_3 = \frac{2}{3}(q^2+q)$.

(2) *If* $q \equiv 0 \mod 3$, *then* $f(e_3) \in \{1, q+1\}$, *ind* $e_3 \in \{\frac{2}{3}(q^2+q), \frac{2}{3}(q^2)\}$.

(3) *If* $q \equiv 1 \mod 3$, *then* $f(e_3) \in \{0, 3, q+2\}$, *ind* $e_3 \in \{\frac{2}{3}(q^2+q+1), \frac{2}{3}(q^2+q-2), \frac{2}{3}(q^2-1)\}$.

Proof. The proof can be found in [7]. QED

Lemma 6. *In* $PSL(3,q)$.

(1) *If* $q \equiv 1 \mod 4$, *then* $f(e_4) \in \{3, q+2\}, f(e_4^2) = q+2$ *and ind* $e_4 \in \{\frac{3q^2+2q-5}{4}, \frac{3}{4}(q^2-1)\}$.

(2) *If* $q \equiv 3 \mod 4$, *then* $f(e_4) = 1$, $f(e_4^2) = q+2$ *and ind* $e_4 = \frac{3q^2+2q-1}{4}$.

(3) *If* $q \equiv 0 \mod 4$, *then* $f(e_4) = 1, f(e_4^2) = q+1$ *and ind* $e_4 = \frac{3q^2+2q}{4}$.

PROOF. Suppose element e_4 has order 4 in G. Now $ind\ e_4 = \frac{3N-(2f(e_4)+f(e_4^2))}{4}$.

(1) Since $4|q-1$, then $\bar{e_4}$ is conjugate to one of the following: $\begin{pmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & 1\end{pmatrix}$, where $\alpha = \beta^{-1}$ where α is a fixed element of order 4 or $\begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \beta\end{pmatrix}$ where $\alpha^2 = \beta^{-1}$ where α is a fixed element of order 4. The associated eigen spaces of α, β and 1 have dimensions 1,1 and 1. Thus $f(e_4) = 3$ or the associated eigen spaces of α and β have dimensions 1 and 2. So $f(e_4) = q+2$. Also $\bar{e_4}^2$ is conjugate to $\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}$. From Lemma 5, we have $f(e_4^2) = q+2$ fixed points. Therefore, $ind\ e_4 \in \{\frac{3q^2+2q-5}{4}, \frac{3}{4}(q^2-1)\}$.

(2) Since $\bar{e_4}$ is conjugate to one of the following: $\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & p-1\\ 0 & 1 & 0\end{pmatrix}$, where $q = p^k$. The associated eigen space of 1 has dimension 1. Thus $f(e_4) = 1$. Also $\bar{e_4}^2$ is conjugate to $\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}$. From Lemma 5, we have $f(e_4^2) = q+2$ fixed points. Therefore, $ind\ e_4 = \frac{3q^2+2q-1}{4}$.

(3) The proof is similar to part 2.

QED

Lemma 7. *In $PSL(3,q)$.*

(1) If $q \equiv 1 \mod 5$, then $f(e_5) \in \{3, q+2\}$ and $ind\ e_5 \in \{\frac{4}{5}(q^2+q-2), \frac{4}{5}(q^2-1)\}$.

(2) If $q \equiv 4 \mod 5$, then $f(e_5) = 1$ and $ind\ e_5 = \frac{4}{5}(q^2+q)$.

(3) If $q \equiv 0 \mod 5$, than $f(e_5) \in \{1, q+1\}$ and $ind\ e_5 \in \{\frac{4}{5}(q^2+q), \frac{4}{5}q^2\}$.

PROOF. The proof is similar as Lemma 6. QED

Lemma 8. *In $PSL(3,q)$.*

(1) If $q \equiv 2$ or $4 \mod 7$, then $f(e_7) = 0$ and $ind\ e_7 = \frac{6}{7}(q^2+q+1)$.

(2) *If* $q \equiv 6 \mod 7$, *then* $f(e_7) = 1$ *and ind* $e_7 = \frac{6}{7}(q^2 + q)$.

(3) *If* $q \equiv 0 \mod 7$, *than* $f(e_7) \in \{1, q+1\}$ *and ind* $e_7 \in \{\frac{6}{7}(q^2+q), \frac{6}{7}q^2\}$.

(4) *If* $q \equiv 0 \mod 7$, *than* $f(e_7) \in \{3, q+2\}$ *and ind* $e_7 \in \{\frac{6}{7}(q^2+q-2), \frac{6}{7}(q^2-1)\}$.

Proof. The proof is similar as Lemma 6. QED

Lemma 9. *In* $PSL(3,q)$.

(1) *If* $q \equiv 7$ *or* $3 \mod 8$, *then* $f(e_8) = 1 = f(e_8^2), f(e_8^4) = q+2$ *and ind* $e_8 = \frac{7q^2+6q-1}{8}$.

(2) *If* $q \equiv 5 \mod 8$, *then* $f(e_8) = 1, f(e_8^2) = f(e_8^4) = q+2$ *andind* $e_8 = \frac{7q^2+4q-3}{8}$.

(3) *If* $q \equiv 1 \mod 8$, *than* $f(e_8) \in \{3, q+2\}, f(e_8^4) = q+2$ *and ind* $e_8 \in \{\frac{7}{8}(q^2-1), \frac{7q^2+6q-13}{8}\}$.

Proof. The proof is similar as Lemma 6. QED

Lemma 10. *In* $PSL(3,q)$.

(1) *If* $q \equiv 5 \mod 6$, *then* $f(e_6) = 1 = f(e_6^2), f(e_6^3) = q+2$ *and ind* $e_8 = \frac{5q^2+4q+3}{6}$.

(2) *If* $q \equiv 8 \mod 9$, *then* $f(e_9) = 1 = f(e_9^3)$ *and ind* $e_9 = \frac{4}{3}(q^2+q)$.

(3) *If* $q \equiv 6 \mod 10$, *than* $f(e_{10}) = 2, f(e_{10}^2) = q+2, f(e_{10}) = q+1$ *and ind* $e_{10} = \frac{9q^2+4q-8}{10}$.

Proof. The proof is similar as Lemma 6. QED

Proposition 2. *In* $PSL(3,q)$, *there is no generating tuple of genus* $g \leq 2$ *if* $16 \leq q \leq 25$.

Proof. From Theorem 1, we have to deal with some possible signatures in the different groups $PSL(3,q)$. Since $PSL(3,q)$ does not contain an element of order 7 where $q = 17, 19$, there is no signature (2,3,7) in $PSL(3,q)$. For $q = 23, 25$, the signature $(2,3,7)$ does not satisfy equation (1.1). If $q = 16$, then there is no signature (2,3,8) in $PSL(3,16)$. For $q = 17, 19, 23$, the signature $(2,3,8)$ does not satisfy equation (1.1). We can compute the indices of elements of order 2,3,4,5,6,9 and 10 by Lemma 4,..., Lemma 10. The sum of the indices of the signatures (2,3,9),(2,3,10), (2,4,5) and (2,4,6) do not fit the Riemann Hurwtiz formula (1.2). This completes the proof. QED

Now we are considering $q \leq 13$ and $g = 1$, we will find all tuples which satisfy equations (1.1),(1.1) and (1.2) and then classify them up to braid action and diagonal conjugation in the next section.

Let C_i be the conjugacy class of x_i. Then the multi set of non trivial conjugacy classes $\bar{C} = \{C_1, ..., C_r\}$ in G is called the ramification type of the branched cover f.

The following example show how to achieve the generating tuples and braid orbits for the group $PSL(3,4)$.

Example 1. We use the GAP and package MAPClASS to illustrate some of these computations more concretely. Let $G = PSL(3,4)$ be the projective special linear group, acting on 21 points. First, we find the tuples satisfy the Riemann Hurwitz formula (1.2).

```
gap> LoadPackage( "mapclass", false );
true
gap> rts:=[];
[  ]
gap> k:=AllPrimitiveGroups(DegreeOperation,21);
[ PGL(2, 7), A(7), S(7), PSL(3, 4)=M(21), PSigmaL(3, 4), PGL(3, 4),
PGammaL(3, 4), A(21), S(21) ]
gap> g:=k[4];
PSL(3, 4)=M(21)
gap> m:=MaximalSubgroupClassReps(a);;
gap> d:=List(m,x->Index(a,x));
[ 21, 21, 56, 56, 56, 120, 120, 120, 280 ]
gap> ccl:= ConjugacyClasses( g );;
gap> reps:= List( ccl, Representative );;
gap> orders:= List( reps, Order );;
gap> Ind:= pi -> NrMovedPoints( pi ) - Sum( CycleStructurePerm( pi ), 0 );;
gap> ind:= List( reps, Ind );
[ 0, 16, 16, 12, 18, 18, 8, 14, 14, 14 ]
gap> N:=21;;
gap> cand:= RestrictedPartitions( 2*N, Set( ind{ [ 2 .. Length( ind ) ] } ) );;
gap> for l in cand do
> UniteSet( rts, Set( Cartesian( List( l, x -> Positions( ind, x ) ) )
,SortedList));> od;
gap> rts;
[ [ 2, 4, 8 ], [ 2, 4, 9 ], [ 2, 4, 10 ], [ 2, 5, 7 ], [ 2, 6, 7 ],
[ 3, 4, 8 ],[ 3, 4, 9 ], [ 3, 4, 10 ], [ 3, 5, 7 ], [ 3, 6, 7 ],
[ 4, 4, 5 ], [ 4, 4, 6 ], [ 4, 7, 7, 8 ], [ 4, 7, 7, 9 ], [ 4, 7, 7, 10 ],
[ 5, 7, 7, 7 ], [ 6, 7, 7, 7 ], [ 8, 8, 8 ], [ 8, 8, 9 ], [ 8, 8, 10 ],
```

```
[ 8, 9, 9 ], [ 8, 9, 10 ], [ 8, 10, 10 ], [ 9, 9, 9 ], [ 9, 9, 10 ],
[ 9, 10, 10 ], [ 10, 10, 10 ] ]
```

Pick one example of the ramification type $\bar{C} = (2A, 2A, 2A, 7A)$. Find the braid orbits if its exist.

```
gap> cand:=rts[16];
[ 5, 7, 7, 7 ]
gap> orbs:= GeneratingMCOrbits( g, 0, reps{ cand } : OutputStyle:=
"silent" );;
gap> Length(orbs);
7
gap> tup:= orbs[1].TupleTable[1].tuple;
[ (1,9,3,14,8,15,11)(2,12,16,10,17,4,18)(5,7,19,20,13,6,21),
(1,9)(2,18)(3,12)(4,14)(7,19)(8,10)(11,16)(20,21), (2,3)(5,8)
(6,20)(7,18)(10,17)(11,12)(15,21)(16,19),
(3,4)(5,16)(7,18)(8,21)(9,11)(13,20)(14,17)(15,19) ]
gap> g = Group( tup );
true
gap> Product( tup );
()
gap> Sum( List( tup, Ind ) );
42
```

The tuple of type $\bar{C}$ passes Equations (1.1),(1.1) and (1.2). So it is a genus one system. It has 3 braid orbits of length 1.

4 On the Hurwitz space $\mathcal{H}_r^{in}(G, \bar{C})$

Let C_i be the conjugacy class of x_i. Then the multi set of non trivial conjugacy classes $\bar{C} = \{C_1, ..., C_r\}$ in G is called the ramification type of the branched cover f [9]. Now we define the subset $\mathcal{N}(\bar{C}) = \{(x_1, ..., x_r) : G = \langle x_1, ..., x_r\rangle, \prod_{i=1}^{r} x_i = 1, \exists \sigma \in S_n \text{ such that } x_i \in C_{i\sigma} \text{ for all } i\}$ which is called the Nielsen class of $\bar{C}$ [5]. From now on we assume that $g = 1$ and we give some results which tells us whether or not the Hurwitz space $\mathcal{H}_r^{in}(G, \bar{C})$ is connected. This will be done by both the calculations of GAP (Groups, Algorithms, Programming) software and Proposition 2.4 in [8]. Proposition 2.4 tells us there is a one to one correspondence between components of $\mathcal{H}_r^{A}(\bar{C})$ and the braid orbits on $\mathcal{N}^A(\bar{C})$. In particular, $\mathcal{H}_r^{in}(G, \bar{C})$ is connected if and only if there is only one braid orbit. The algorithm which will be used here, can be found in [10].

Proposition 3. *If G is isomorphic to one of the following groups:* $PGL(3,4)$, $PSL(3,4), PSL(3,4).2, PSL(3,4).2^2, PSL(3,4).2_1, PSL(3,4).2_3, PSL(3,4).D_{12}$, $PSL(3,4).6, PSL(3,4).S_3$, $PSL(3,5).2, PSL(3,7), P\Sigma L(3,4)$ *and* $g = 1$, *then* $\mathcal{H}_r^{in}(G,\bar{C})$ *is disconnected.*

Proof. Since we have at least two braid orbits for some ramification type $\bar{C}$ and the Nielsen classes $\mathcal{N}(\bar{C})$ are the disjoint union of braid orbits. From Proposition 2.4 in [8], we obtain that the Hurwitz space $\mathcal{H}_r^{in}(G,\bar{C})$ is disconnected. QED

Proposition 4. *If G is isomorphic to one of the following groups:* $PSL(3,5)$, $PSL(3,7), P\Gamma(3,8), PSL(3,9), P\Gamma L(3,9), P\Gamma L(3,13)$ *and* $g = 1$ *then* $\mathcal{H}_r^{in}(G,\bar{C})$ *is connected.*

Proof. Since we have just one braid orbit for all ramification types $\bar{C}$ and the Nielsen classes $\mathcal{N}(\bar{C})$ are the disjoint union of braid orbits. From Proposition 2.4 in [8], we obtain that the Hurwitz space $\mathcal{H}_r^{in}(G,\bar{C})$ is connected. QED

The proof of the following results are similar to the previous results.

Proposition 5. *If* $r \geq 4, g = 1$ *and G is isomorphic to one of the following groups:* $PSL(3,2), PSL(3,3)$, $PSL(3,3).2$, *then* $\mathcal{H}_r^{in}(G,\bar{C})$ *is connected.*

Proposition 6. *If* $r \geq 5, g = 1$ *and* $G = P\Gamma L(3,4)$, *then* $\mathcal{H}_r^{in}(G,\bar{C})$ *is connected where* $\bar{C} \neq (2B,2B,6A,8A)$.

5 Appendix

Throughout this section, N.O means number of orbits, L.O means length of the orbis and GOS means Genus one System. The number p^b means b copies of p. Note that $PSL(3,2) \cong PSL(2,7)$, which is almost simple group of lie rank one.

Table 2: GOSs for Almost Simple Groups of Lie Rank Two

	(4A,4A,8B)	10	1	(4A,4A,8A)	10	1
	(4A,6B,8B)	6	1	(4A,6A,8A)	9	1
	(6A,6A,8B)	8	1	(6A,6A,8B)	8	1
	(2A,8B,13D)	1	1	(2A,8B,13C)	1	1
	(2A,8B,13B)	1	1	(2A,8B,13A)	1	1
	(2A,8A,13D)	1	1	(2A,8A,13C)	1	1
	(2A,8A,13B)	1	1	(2A,8A,13A)	1	1
	(3B,4A,13D)	1	1	(3B,4A,13C)	1	1
	(3B,4A,13B)	1	1	(2A,4A,13A)	1	1
	(3B,6A,13D)	1	1	(3B,6A,13C)	1	1
	(3B,6A,13B)	1	1	(2A,6A,13A)	1	1
	(3B,3A,13D)	1	1	(3B,3A,13C)	1	1
	(3B,3A,13B)	1	1	(2A,3A,13A)	1	1
	(3A,4A,8B)	1	1	(3A,4A,8A)	1	1
	(3A,6A,8B)	1	1	(3A,6A,8A)	1	1
	(3A,3A,8B)	1	1	(3A,3A,8A)	1	1
	(2A,3B,6A,6A)	1	128	(2A,3A,3B,6A)	1	144
	(2A,3B,3A,3A)	1	128	(2A,3B,4A,6A)	1	140
$PSL(3,3)$	(2A,3A,3B,4A)	1	140	(2A,3B,4A,4A)	1	156
$n = 13$	(2A,2A,6A,8A)	1	136	(2A,2A,6A,8B)	1	136
	(2A,2A,3B,13A)	1	13	(2A,2A,3B,13B)	1	13
	(2A,2A,3B,13C)	1	13	(2A,2A,3B,13D)	1	13
	(2A,2A,3A,8A)	1	144	(2A,2A,3A,8B)	1	144
	(2A,2A,4A,8A)	1	152	(2A,2A,4A,8B)	1	152
	(2A,2A,2A,3B,6A)	1	2124	(2A,2A,2A,3B,3B)	1	2232
	(2A,2A,2A,3B,4A)	1	2352	(2A,2A,2A,2A,8A)	1	2304
	(2A,2A,2A,2A,2A,3B)	1	36720	(2A,2A,2A,2A,8B)	1	2304
	(4B,4B,4A)	1	1	(2B,6A,6B)	6	1
	(2B,4A,12B)	3	1	(2B,4A,12A)	3	1
$PSL(3,3).2$	(2A,6A,6B)	4	1	(2B,4A,8A)	3	1
$n = 52$	(2A,2A,2B,4A)	1	62			
$PSL(3,3).2$, $n = 117$	(2B,4A,6B)	4	1			
$PSL(3,3).2$, $n = 234$				(2B,3B,8B)	3	1
	(3A,3A,7B)	24	1	(3A,3A,7A)	24	1
	(3A,4C,5B)	16	1	(3A,4C,5A)	16	1
	(3A,4B,5B)	16	1	(3A,4B,5A)	16	1
	(3A,4A,5B)	16	1	(3A,4A,5A)	16	1
	(4B,4C,4C)	4	1	(4B,4B,4C)	4	1
	(4A,4C,4C)	4	1	(4A,4B,4C)	2	1
	(4A,4B,4B)	4	1	(4A,4A,4C)	4	1
	(2A,5B,7B)	9	1	(4A,4A,4B)	4	1
$PSL(3,4)$	(2A,5B,7A)	9	1	(2A,5A,7B)	9	1
$n = 21$	(2A,4C,7B)	2	1	(2A,5A,7A)	9	1
	(2A,4C,7A)	2	1	(2A,5A,5B)	6	1
	(2A,4B,7B)	2	1	(2A,4B,7A)	2	1
	(2A,2A,3A,4B)	2	192,168	(2A,2A,3A,4A)	2	192,168
	(2A,2A,3A,4C)	2	192,168	(2A,2A,2A,7A)	7	21^6,42
				(2A,2A,2A,7B)	7	21^6,42
$PSL(3,4).2$	(2A,3A,10B)	2	1	(2A,3A,10B)	2	1
$n = 105$	(2A,4A,6A)	8	1			
$PSL(3,4).2$				(2B,4B,6A)	8	1
$n = 120$	(2B,3A,10A)	2	1	(2B,3A,10B)	2	1

Table 3: GOSs for Almost Simple Groups of Lie Rank Two

group and degree	ramification type	N.O	L.O	ramification type	N.O	L.O
$PSL(3,4).S_3$, $n=280$	(2A,3A,8A)	2	1	(2A,3C,8B)	2	1
$PSL(3,4).6$,$n=120$	(2B,3D,12B)	1	1	(2B,3C,12A)	1	1
$PSL(3,4).D_{12}$, $n=120$	(2D,4C,6A)	7	1	(2B,2A,2D,3A)	1	11
$PSL(3,4).2_3$,$n=56$	(2B,5A,8A)	2	1	(2B,6A,7B)	2	1
	(2B,4B,14A)	1	1	(2B,6A,7A)	2	1
	(2B,4B,14B)	1	1			
$PSL(3,4).2_1$, $n=56$	(2A,4B,10A)	1	1	(2B,4B,10B)	1	1
	(2A,4A,8A)	2	1	(2B,2A,2A,4B)	1	28
	(2A,4A,10A)	1	1	(2B,4A,10B)	1	1
	(2A,4B,8B)	2	1			
$PSL(3,4).2^2$, $n=56$	(2B,4B,8EA)	4	1			
	(3C,3D,4A)	2	1	(3D,3D,6B)	2	1
	(3C,4A,6B)	12	1	(3D,6B,6B)	18	1
	(6B,6B,6B)	54	1	(3C,3C,6A)	2	1
	(3D,4A,6A)	12	1	(4A,6A,6B)	54	1
	(3C,6A,6A)	18	1	(6A,6A,6A)	54	1
	(3B,3E,21D)	1	1	(3B,3E,21B)	1	1
	(3B,3D,7B)	1	1	(3B,3D,7A)	1	1
	(3B,3C,15C)	1	1	(3B,3C,15A)	1	1
$PGL(3,4)$,$n=21$	(3B,4A,15D)	1	1	(3B,4A,15B)	1	1
	(3B,6B,7B)	3	1	(3B,6B,7A)	3	1
	(3B,6A,15C)	1	1	(3B,6A,15A)	1	1
	(3A,3E,21C)	1	1	(3A,3E,21A)	1	1
	(3A,3D,15D)	1	1	(3A,3D,15B)	1	1
	(3A,3C,7B)	1	1	(3A,3C,7A)	1	1
	(3A,4A,15C)	1	1	(3A,4A,15A)	1	1
	(3A,6B,15B)	1	1	(3A,6B,15D)	1	1
	(3A,6A,7B)	1	1	(3A,6A,7A)	1	1
	(2A,3D,21C)	1	1	(2A,3D,21A)	1	1
	(2A,3C,21D)	1	1	(2A,3C,21B)	1	1
	(2A,6B,21C)	1	1	(2A,6B,21A)	1	1
	(2A,6A,21D)	1	1	(2A,6A,21B)	1	1
	(2A,3B,3B,6A)	1	18	(2A,3B,3B,3D)	1	18
	(2A,3A,3B,4A)	2	18	(2A,3A,3A,6B)	1	18
	(2A,3A,3A,3C)	1	18			
	(4B,6A,6B)	102	1	(4B,3B,6B)	40	1
	(3A,6B,8A)	10	1	(3A,4B,14B)	3	1
	(2A,6B,21A)	5	1	(3A,4B,14A)	3	1
	(2A,6B,21B)	5	1	(2A,8A,15A)	2	1
				(2A,8A,15B)	2	1
	(2A,2A,4A,6A)	1	480	(2A,2A,2B,21A)	1	21
	(2A,2A,2AB,21B)	1	21	(2A,2A,4A,3B)	1	96
$P\Gamma L(3,4)$, $n=21$	(2A,3A,3C,4B)	1	144	(2A,3A,3A,6B)	1	96
	(2A,2A,3B,3B)	1	44	(2A,2A,3B,6A)	1	264
	(2A,2A,6A,8A)	2	192,672	(2A,2A,3A,15A)	1	20
	(2A,2A,3A,7A)	1	21	(2A,2A,3A,15B)	1	20
	(2A,2A,3A,7B)	1	21	(2A,2A,4B,6B)	1	1152
	(2B,2A,3A,8A)	1	40	(2B,2A,3B,4B)	1	204
	(2B,2A,4B,6A)	1	480	(2A,2A,2B,3B,3B)	1	456
	(2A,2A,2A,2A,3B)	1	2952	(2A,2A,2A,2A,6A)	1	10656
	(2A,2A,2A,2A,4A)	1	4224	(2A,2A,2A,2B,4B)	1	5760
	(2A,2A,2A,2A,2A,2A)	1	131200			

Table 4: GOSs for Almost Simple Groups of Lie Rank Two

group and degree	ramification type	N.O	L.O	ramification type	N.O	L.O
	(4A, 4C, 6A)	8	1	(4A,4B,6A)	40	1
	(4A,4A,5A)	18	1	(2B,8A,8A)	2	1
	(2B,6A,14B)	7	1	(2B, 6A, 14A)	7	1
	(3A,4A,8A)	16	1	(3A,6A,6A)	70	1
	(2A, 5A,14B)	3	1	(2A,5A,14A)	3	1
	(2A, 8A,7B)	2	1	(2A,8A,7A)	2	1
	(2B,2B,4B,4B)	2	128,104	(2A,2B,4B,4C)	1	192
$P\Sigma L(3,4)$,$n=21$	(2A,2B,4B,4A)	1	40	(2A,2B,3A,6A)	2	168^2
	(2A,2B,2B,14A)	2	14^2	(2A,2B,2B,14B)	2	14^2
	(2A,2A,4A,6A)	1	72	(2A,2A,4C,4C)	1	64
	(2A,2A,3A,5A)	3	40,40,50	(2A,2A,4A,4C)	1	40
	(2A,2A,2B,2B,3A)	2	864,756	(2A,2A,2A,2B,4B)	1	336
				(2A,2A,2A,2A,4C)	1	384
	(4B,4C,8B)	1	1	(3A,4B,5B)	1	1
	(3A,4B,10A)	1	1	(3A,4B,8B)	1	1
	(3A,4B,6A)	1	1	(3A,4B,5B)	1	1
	(3A,4A,10A)	1	1	(3A,4A,8A)	1	1
	(3A,4A,6A)	1	1	(4A,4C,8A)	1	1
	(2A,4C,31J)	1	1	(2A,4C,31I)	1	1
	(2A,4C,31H)	1	1	(2A,4C,31G)	1	1
	(2A,4C,31F)	1	1	(2A,4C,31E)	1	1
	(2A,4C,31D)	1	1	(2A,4C,31C)	1	1
	(2A,4C,31A)	1	1	(2A,4C,31B)	1	1
$PSL(3,5)$, $n=31$	(2A,8B,20B)	1	1	(2A,8A,20A)	1	1
	(2A,8B,12B)	1	1	(2A,8B,12A)	1	1
	(2A,8A,12B)	1	1	(2A,8A,12A)	1	1
	(2A,3A,31A)	1	1	(2A,3A,31B)	1	1
	(2A,3A,31C)	1	1	(2A,3A,31D)	1	1
	(2A,3A,31E)	1	1	(2A,3A,31F)	1	1
	(2A,3A,31G)	1	1	(2A,3A,31H)	1	1
	(2A,3A,31I)	1	1	(2A,3A,31J)	1	1
	(2A,2A,4B,5A)	1	1	(2A,2A,3A,4B)	1	33
	(2A,2A,4A,5A)	1	1	(2A,2A,3A,4A)	1	33
$PSL(3,5).2$,$n=186$	(2B,3A,8A)	14	1			
$PSL(3,7)$, $n=57$	(2A,2A,2A,14A)	2	3,3			
	(2A,3A,19A)	3	1	(2A,3A,19B)	3	1
	(2A,3A,19C)	3	1	(2A,3A,19D)	3	1
	(2A,3A,19E)	3	1	(2A,3A,19F)	3	1
	(3B,3D,6E)	1	1	(3B,4A,6C)	1	1
	(3A,3C,6E)	1	1	(3A,4A,6D)	1	1
$PGL(3,7)$, $n=57$	(2A,6D,12A)	1	1	(2A,6C,12B)	1	1
	(2A,3D,42B)	1	1	(2A,3D,24B)	1	1
	(2A,3D,24C)	1	1	(2A,3C,42A)	1	1
	(2A,3C,24D)	1	1	(2A,3C,24C)	1	1
	(2A,4A,14F)	1	1	(2A,4A,14E)	1	1
	(2A,4A,14D)	1	1	(2A,4A,14C)	1	1
$P\Gamma L(3,8)$,$n=73$	(2A,4A,14A)	1	1	(2A,4A,14B)	1	1
	(2A,3A,21A)	1	1	(2A,3A,21B)	1	1
	(2A,3A,21C)	1	1	(2A,3A,21D)	1	1
	(2A,3A,21E)	1	1	(2A,3A,21F)	1	1
	(2A,3B,24D)	1	1	(2A,3B,24C)	1	1
$PSL(3,9)$, $n=91$	(2A,3B,24A)	1	1	(2A,3B,24B)	1	1
$P\Gamma L(3,9)$,$n=57$	(2A,3B,16D)	1	1	(2A,3B,16C)	1	1
$P\Gamma L(3,13)$,$n=183$	(2A,3B,12N)	1	1	(2A,3B,12L)	1	1
	(2A,3A,12M)	1	1	(2A,3A,12K)	1	1

Acknowledgements. I would like thank to the referee for careful reading of the article and detailed report including corrections and comments; and I appreciate his/her effort on reviewing the article.

References

[1] Neubauer, M. G: *On solvable monodromy groups of fixed genus*, Thesis (Ph.D.)–University of Southern California, 1989.

[2] Guralnick, R. and Thompson, J.: *Finite groups of genus zero*, Journal of Algebra,**131**, n.1, 303–341. 1990

[3] Aschbacher, M.: *On conjectures of Guralnick and Thompson*, Journal of Algebra,**135**, n.2, 277–343. 1990

[4] Neubauer, Michael G: *On monodromy groups of fixed genus*, Journal of Algebra,**153**, n.1, 215–261. 1992.

[5] Völklein, H.: Groups as Galois groups, Cambridge University Press, Cambridge, 1996.

[6] Khudhur, Peshawa Mohammed: *Sporadic simple groups of low genus*, Thesis (Ph.D.)–University of Birmingham, 2016.

[7] Kong, X.: *Genus 0, 1, 2 actions of some almost simple groups of lie rank 2*, Thesis (Ph.D.)–Wayne State University, 2011.

[8] Salih, H. M. Mohammed: *Hurwitz components of groups with socle PSL (3; q)*,EXTRACTA MATHEMATICAE,**36**, n.1, 51–62. 2021.

[9] Salih, H. M. Mohammed: *Finite Groups of Small Genus*, Thesis (Ph.D.)–University of Birmingham, 2014.

[10] Salih, H. M. Mohammed: *Connected components of affine primitive permutation groups*, Journal of Algebra,**561**, n.1, 355–373. 2020.

[11] The GAP Group: *GAP – Groups, Algorithms, and Programming,Version 4.9.3*, http://www.gap-system.org, 2018.

[12] Frohardt, D. and Magaard, K.: *Composition factors of monodromy groups*, Ann. of Math. (2) ,**154**, n.2, 327–345. 2001.

[13] Aschbacher, M. and Scott, L.: *Maximal subgroups of finite groups*, Journal of Algebra ,**92**, n.1, 44–80. 1985.

[14] Magaard, Kay.: *Monodromy and sporadic groups*, Communications in Algebra ,**21**, n.12, 4271–4297. 1985.

[15] Scott, Leonard L.: *Matrices and cohomology*, Annals of Mathematics ,**105**, n.3, 473–492. 1977.