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On Jordan ideals with generalized left derivations in 3-prime Near-rings

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Abstract. In this paper, we will extend some results on the commutativity of Jordan ideals proved in [8] and [5]. However, instead of left generalized derivations, we will consider generalized left derivations, which are sufficient to obtain good results with respect to the structure of near-rings. We will also show that the conditions imposed in the paper cannot be removed.

Keywords: 3-prime near-rings, generalized left derivations, commutativity.

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Introduction

A right (resp. left) near-ring $\mathcal{N}$ is a triple $(\mathcal{N},+,.)$ with two binary operations " $+$ " and " $.$ " such that (i) $(\mathcal{N},+)$ is a group (not necessarily abelian), (ii) $(\mathcal{N},.)$ is a semigroup, (iii) $(r+s).t = r.t+s.t$ (resp. $r.(s+t) = r.s+r.t$) for all $r,s,t \in \mathcal{N}$. We denote by $Z(\mathcal{N})$ the multiplicative center of $\mathcal{N}$, and usually $\mathcal{N}$ will be 3-prime, that is, for $r,s \in \mathcal{N}$, $r\mathcal{N}s = \{0\}$ implies $r = 0$ or $s = 0$. A right (resp. left) near-ring $\mathcal{N}$ is a zero symmetric if $r.0 = 0$ (resp. $0.r = 0$) for all $r \in \mathcal{N}$, (recall that right distributive yields $0.r = 0$ and left distributive yields $r.0 = 0$). For any pair of elements $r,s \in \mathcal{N}$, $[r,s] = rs - sr$ and $r \circ s = rs + sr$ stand for Lie product and Jordan product respectively. Recall that $\mathcal{N}$ is called

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2-torsion free if $2r = 0$ implies $r = 0$ for all $r \in \mathcal{N}$. As noted in [5], an additive subgroup J of $\mathcal{N}$ is said to be a Jordan ideal of $\mathcal{N}$ if $r \circ i \in J$ and $i \circ r \in J$ for all $i \in J, r \in \mathcal{N}$. An additive mapping $H : \mathcal{N} \to \mathcal{N}$ is said to be a multiplier if $H(rs) = rH(s) = H(r)s$ for all $r, s \in \mathcal{N}$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is called left derivation (resp. Jordan left derivation) if $d(rs) = rd(s) + sd(r)$ (resp. $d(r^2) = 2rd(r)$) holds for all $r, s \in \mathcal{N}$. The concepts of left derivations and Jordan left derivations were introduced by Breşar et al. in [7], and it was shown that if a prime ring $\mathcal{R}$ of characteristic different from 2 and 3 admits a nonzero Jordan left derivation, then $\mathcal{R}$ must be commutative. Obviously, every left derivation is a Jordan left derivation, but the converse need not be true in general (see, [10, Example 1.1]). In [1], M. Ashraf et al. proved that the converse statement is true in the case if the underlying ring is prime and 2-torsion free. The left derivation study was introduced by S. M. A. Zaidi et al. in [10] and they showed that if J is a Jordan ideal and a subring of a 2 torsion-free prime ring R admits a non-zero Jordan left derivation and an automorphism T such that $d(r^2) = 2T(r)d(r)$ holds for all $r \in J$, then either $J \subseteq Z(\mathcal{R})$ or $d(J) = \{0\}$. Recently there has been a lot of work on Jordan ideals of near rings involving derivations; see for example [4], [5], [6], etc. For more details, in [6, Theorems 3.6 & 3.12] we only manage to show the commutativity of the Jordan ideal, but we don't manage to show the commutativity of our studied near-rings, so our goal is to extend these results to the generalized left derivations.

Definition 1. An additive mapping $\mathfrak{D} : \mathcal{N} \to \mathcal{N}$ is called a generalized left derivation if there exists a left derivation $d : \mathcal{N} \to \mathcal{N}$ such that $\mathfrak{D}(xy) = x\mathfrak{D}(y) + yd(x)$ for all $x, y \in \mathcal{N}$.

It is obvious to see that every left derivation on a near-ring $\mathcal{N}$ is a left gneralized derivation. But the opposite is not true in general. The following example illustrates this fact:

Example 1. Let $\mathcal{S}$ be a near-ring. Define the set $\mathcal{N}$ and the maps $d, \mathfrak{D} : \mathcal{N} \to \mathcal{N}$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} \mid x, y, z, 0 \in \mathcal{S} \right\},$$

$$\mathfrak{D} \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix}, \; d \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix}.$$

Then $\mathfrak{D}$ is a generalized left derivation of $\mathcal{N}$ associated with a left derivation d of $\mathcal{N}$, but $\mathfrak{D}$ is not a left derivation of $\mathcal{N}$.
Also for $d = 0$, a generalized derivation covers the notion of a right multiplier, i.e., an additive mapping H satisfying $H(xy) = xH(y)$ for all $x, y \in \mathcal{N}$.

1 Some preliminaries

To facilitate the proof of our main results, the following lemmas are essential.

Lemma 1. *Let $\mathcal{N}$ be a 3-prime near-ring.*

(i) [3, Lemma 1.2 (iii)] *If $z \in Z(\mathcal{N}) \setminus \{0\}$ and $xz \in Z(\mathcal{N})$ or $zx \in Z(\mathcal{N})$, then $x \in Z(\mathcal{N})$.*

(ii) [2, Lemma 3 (ii)] *If $Z(\mathcal{N})$ contains a nonzero element z of $\mathcal{N}$ which $z+z \in Z(\mathcal{N})$, then $(\mathcal{N},+)$ is abelian.*

(iii) [5, Lemma 3] *If $J \subseteq Z(\mathcal{N})$ and $\mathcal{N}$ is a 2 torsion free, then $\mathcal{N}$ is a commutative ring.*

Lemma 2. [9, Theorem 3.1] *Let $\mathcal{N}$ be a 3-prime right near-ring. If $\mathcal{N}$ admits a nonzero left derivation d, then the following properties hold true:*

(i) *If there exists a nonzero element a such that $d(a) = 0$, then $a \in Z(\mathcal{N})$,*

(ii) *$(\mathcal{N},+)$ is abelian, if and only if $\mathcal{N}$ is a commutative ring.*

Lemma 3. [4, Lemma 2.2] *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero Jordan ideal J, then $j^2 \neq 0$ for all $j \in J \setminus \{0\}$.*

Lemma 4. [4, Theorem 3.1] *Let $\mathcal{N}$ be a 2-torsion free 3-prime right near-ring and J a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero left multiplier H, then the following assertions are equivalent:*

(i) *$H(J) \subseteq Z(\mathcal{N})$,*

(ii) *$H(J^2) \subseteq Z(\mathcal{N})$,*

(iii) *$\mathcal{N}$ is a commutative ring.*

Lemma 5. *Let $\mathcal{N}$ be a left near-ring. If $\mathcal{N}$ admits a left derivation d, then we have the following identity:*

$$xyd(y^n) = yxd(y^n) \text{ for all } n \in \mathbb{N} \setminus \{0\},\ x, y \in \mathcal{N}.$$

Proof. Using the definition of d. On one hand, we have

$$\begin{aligned} d(xy^{n+1}) &= xd(y^{n+1}) + y^{n+1}d(x) \text{ for all } n \in \mathbb{N} \setminus \{0\},\ x, y \in \mathcal{N} \\ &= xy^n d(y) + xyd(y^n) + y^{n+1}d(x) \text{ for all } n \in \mathbb{N} \setminus \{0\},\ x, y \in \mathcal{N}. \end{aligned}$$

On the other hand,

$$\begin{aligned} d(xy^{n+1}) &= xy^n d(y) + yd(xy^n) \text{ for all } n \in \mathbb{N} \setminus \{0\},\ x, y \in \mathcal{N} \\ &= xy^n d(y) + yxd(y^n) + y^{n+1}d(x) \text{ for all } n \in \mathbb{N} \setminus \{0\},\ x, y \in \mathcal{N}. \end{aligned}$$

Comparing the above expressions of $d(xy^{n+1})$, we obtain the required result. QED

Lemma 6. *Let $\mathcal{N}$ be a 3-prime right near-ring. If $\mathcal{N}$ admits a nonzero generalized left derivation $\mathfrak{D}$ associated with a left derivation d such that $\mathfrak{D}(a) = 0$, then*

$$a(xd(y) + yd(x)) = xad(y) + yad(x) \text{ for all } x, y \in \mathcal{N}.$$

Proof. By defining $\mathfrak{D}$, we have

$$\begin{aligned} \mathfrak{D}(xya) &= x\mathfrak{D}(ya) + yad(x) \\ &= x(y\mathfrak{D}(a) + ad(x)) + yad(x) \\ &= xad(x) + yad(x) \text{ for all } x, y \in \mathcal{N}, \end{aligned}$$

and

$$\begin{aligned} \mathfrak{D}(xya) &= xy\mathfrak{D}(a) + ad(xy) \\ &= a(xd(y) + yd(x)) \text{ for all } x, y \in \mathcal{N}. \end{aligned}$$

Combining the last two expressions, we find that

$$a(xd(y) + yd(x)) = xad(y) + yad(x) \text{ for all } x, y \in \mathcal{N}.$$

QED

2 Results characterize generalized left derivations in 3-prime near-rings

In [2] the author proved that if $\mathcal{N}$ is a 3-prime 2-torsion-free near-ring admitting a non-zero derivation d for which $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring. In this section we investigate possible analogues of these results, where d is replaced by a generalized left derivation $\mathfrak{D}$ associated with a left derivation d, and by involving the concept of Jordan ideals.

2.1 Results concerning 3-prime right near-rings

Theorem 1. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a generalized left derivation $\mathfrak{D}$ associated with a left derivation d, then the following assertions are equivalent:*

(i) $\mathfrak{D}(J) \subseteq Z(\mathcal{N})$,

(ii) $\mathfrak{D}(J^2) \subseteq Z(\mathcal{N})$,

(iii) *$\mathcal{N}$ is a commutative ring or $\mathfrak{D} = 0$.*

Proof. It is obvious that (iii) implies (i) and (ii). Therefore, we only need to prove $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (iii)$.
$(i) \Rightarrow (iii)$ Suppose that $Z(\mathcal{N}) = \{0\}$, then $\mathfrak{D}(J) = \{0\}$, which implies that $\mathfrak{D}(j \circ j) = 0$ for all $j \in J$. Using the definition of $\mathfrak{D}$, we get

$$jd(j) = 0 \text{ for all } j \in J. \tag{2.1}$$

On the other hand, we have $\mathfrak{D}(j \circ nj) = 0$ for all $j \in J,\ n \in \mathcal{N}$. Using the fact that $j \circ nj = (j \circ n)j$ together with the definition of $\mathfrak{D}$, we obtain $(j \circ n)\mathfrak{D}(j) + jd(j \circ n) = 0$ for all $j \in J,\ n \in \mathcal{N}$. It follows that

$$jd(j \circ n) = 0 \text{ for all } j \in J,\ n \in \mathcal{N}. \tag{2.2}$$

Replacing n by nj in (2.2) and using (2.2), we get $j^2nd(j) = \{0\}$ for all $j \in J, n \in \mathcal{N}$. In view of $\mathcal{N}$ is 3-prime together with Lemma 3, the last result reduces to $d(J) = \{0\}$. By Lemma 2 (i), we conclude that $J \subseteq Z(\mathcal{N})$ and therefore $\mathcal{N}$ must be a commutative ring by Lemma 2 (iii).
Now, if $\mathcal{N}$ is a commutative ring, using our hypothesis, then $\mathfrak{D}(j \circ n) = 0$ for all $j \in J,\ n \in \mathcal{N}$, by the 2-torssion freeness of $\mathcal{N}$, we get $\mathfrak{D}(nj) = 0$ for all $j \in J, n \in \mathcal{N}$. Using the definition of $\mathfrak{D}$ together with the fact that $\mathfrak{D}(J) = \{0\}$, we obtain

$$jd(n) = 0 \text{ for all } j \in J. \tag{2.3}$$

Taking $j \circ m$ of j, where $m \in \mathcal{N}$ in (2.3) and using it, we get $J\mathcal{N}d(n) = \{0\}$ for all $n \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime and $J \neq \{0\}$, we obtain $d = 0$. In this case, we obtain

$$j\mathfrak{D}(n) = 0 \text{ for all } j \in J \ n \in \mathcal{N}. \tag{2.4}$$

Substituting $j \circ m$ of j, where $m \in \mathcal{N}$ in (2.4) and using it, we get $J\mathcal{N}\mathcal{D}(n) = \{0\}$ for all $n \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime and $J \neq \{0\}$, we obtain $\mathfrak{D} = 0$.
Now, suppose that $\mathfrak{D}(J) \subseteq Z(\mathcal{N})$ and $Z(\mathcal{N}) \neq \{0\}$, then there exists $z \in J \setminus \{0\}$ such that $\mathfrak{D}(z) \in Z(\mathcal{N})$ and $\mathfrak{D}(z) + \mathfrak{D}(z) = \mathfrak{D}(2z) \in Z(\mathcal{N})$, which implies that

$(\mathcal{N},+)$ is abelian by Lemma 1 (ii). From Lemma 2 (ii), we conclude that $\mathcal{N}$ is a commutative ring.
$(ii) \Rightarrow (iii)$ Suppose that $Z(\mathcal{N}) = \{0\}$, then $\mathfrak{D}(J^2) = \{0\}$, which implies that

$$\mathfrak{D}(j(i \circ ni)) = 0 \text{ for all } i, j \in J,\ n \in \mathcal{N}.$$

By the definition of $\mathfrak{D}$, we get

$$(i \circ ni)d(j) = 0 \text{ for all } i, j \in J,\ n \in \mathcal{N}.$$

Which implies that

$$inid(j) = -ni^2 d(j) \text{ for all } i, j \in J,\ n \in \mathcal{N}.$$

Substituting nm instead of n in the above equation and using it, we find that

$$\begin{aligned} inmid(j) &= -nmi^2 d(j) \\ &= -n(-i)md(j) \text{ for all } i, j \in J,\ n, m \in \mathcal{N}. \end{aligned}$$

Then the above equation becomes

$$n(-i)mid(j) + inid(j) = 0 \text{ for all } i, j \in J,\ n, m \in \mathcal{N}.$$

Replacing i by $-i$ in the last equation, we infer that

$$nim(-i)d(j) - inm(-i)d(j) = 0 \text{ for all } i, j \in J,\ n, m \in \mathcal{N}.$$

It folows that

$$[n, i]m(-i)d(j) = 0 \text{ for all } i, j \in J,\ n, m \in \mathcal{N}.$$

And therefore $[n, i]\mathcal{N}(-i)d(j) = \{0\}$ for all $i, j \in J,\ n \in \mathcal{N}$. Using the 3-primeness of $\mathcal{N}$, we obtain

$$i \in Z(\mathcal{N}) \text{ or } id(j) = 0 \text{ for all } i, j \in J. \tag{2.5}$$

If there exists $i_0 \in Z(\mathcal{N})$, using our hypothesis, then $\mathfrak{D}(j(i_0 \circ ni_0)) = 0$ for all $j \in J,\ n \in \mathcal{N}$. By the definition of $\mathfrak{D}$, we find that $(i_0 \circ ni_0)d(j) = 0$ for all $j \in J,\ n \in \mathcal{N}$ and by the 2-torsion freeness of $\mathcal{N}$, we obtain $i_0 n i_0 d(j) = 0$ for all $j \in J,\ n \in \mathcal{N}$, which implies that $i_0 \mathcal{N} i_0 d(j) = \{0\}$. Since $\mathcal{N}$ is 3-prime, the last equation implies that $i_0 d(j) = 0$ and therefore (2.5) becomes

$$id(j) = 0 \text{ for all } i, j \in J. \tag{2.6}$$

Taking $i \circ m$ instead of i in (2.6), where $m \in \mathcal{N}$, we arrive at $J\mathcal{N}d(J) = \{0\}$. Since $\mathcal{N}$ is 3-prime and $J \neq \{0\}$, we obtain $d(J) = \{0\}$. Using Lemma 2 (i) we obtain $J \subseteq Z(\mathcal{N})$, and Lemma 2 (iii) forces that $\mathcal{N}$ is a commutative ring.
Now, assume that $\mathfrak{D}(J^2) \subseteq Z(\mathcal{N})$ and $Z(\mathcal{N}) \neq \{0\}$. Thus there exists $t \in J \setminus \{0\}$ such that $\mathfrak{D}(t^2) \in Z(\mathcal{N})$ and $\mathfrak{D}(t^2) + \mathfrak{D}(t^2) = \mathfrak{D}(2t^2) = \mathfrak{D}((2t)t) \in Z(\mathcal{N})$, it follows that $(\mathcal{N}, +)$ is abelian by Lemma 1 (ii) and therefore $\mathcal{N}$ is a commutative ring by Lemma 2 (ii). QED

Corollary 1. Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a generalized left derivation $\mathfrak{D}$ associated with a left derivation d, then the following assertions are equivalent:

(i) $\mathfrak{D}(\mathcal{N}) \subseteq Z(\mathcal{N})$,

(ii) $\mathfrak{D}(\mathcal{N}^2) \subseteq Z(\mathcal{N})$,

(iii) $\mathcal{N}$ is a commutative ring or $\mathfrak{D} = 0$.

Corollary 2. Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a left derivation d, then the following assertions are equivalent:

(i) $d(J) \subseteq Z(\mathcal{N})$,

(ii) $d(J^2) \subseteq Z(\mathcal{N})$,

(iii) $\mathcal{N}$ is a commutative ring or $d = 0$.

Corollary 3. Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a left derivation d, then the following assertions are equivalent:

(i) $d(\mathcal{N}) \subseteq Z(\mathcal{N})$,

(ii) $d(\mathcal{N}^2) \subseteq Z(\mathcal{N})$,

(iii) $\mathcal{N}$ is a commutative ring or $d = 0$.

Theorem 2. Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admit a nonzero multiplier H and a generalized left derivation $\mathfrak{D}$ associated with a left derivation d which satisfy any one of the following identities:

(i) $\mathfrak{D}(H(J)) = \{0\}$,

(ii) $\mathfrak{D}(H(n \circ j)) = \mathfrak{D}(H([n, j]))$ for all $j \in J,\ n \in \mathcal{N}$, then $\mathfrak{D} = 0$.

Proof. **(i)** Assume that $\mathfrak{D}\left(H(J)\right) = \{0\}$, then $\mathfrak{D}\left(j \circ H^2(j)\right) = 0$ for all $j \in J$. Since $j \circ H^2(j) = 2H(j)H(j)$, by our hypotheses we arrive at

$$H(j)d(H(j)) = 0 \text{ for all } j \in J. \tag{2.7}$$

On the other hand, we have $\mathfrak{D}\left(H((j \circ n)j)\right) = 0$ for all $j \in J,\ n \in \mathcal{N}$ which is equivalently to

$$H(j)d(j \circ n) = 0 \text{ for all } j \in J,\ n \in \mathcal{N}. \tag{2.8}$$

Replacing n by $nH(j)$ in (2.8) and using the fact that $j \circ nH(j) = (j \circ n)H(j)$, we can arrive at

$$H(j)\Big((j \circ n)d(H(j)) + H(j)d(j \circ n)\Big) = 0 \text{ for all } j \in J,\ n \in \mathcal{N}.$$

Using (2.7) and (2.8), then the above equation becomes

$$j^2 H(n)d(H(j)) = 0 \text{ for all } j \in J,\ n \in \mathcal{N}.$$

Substituting yzt for n, where $y, z, t \in \mathcal{N}$, in the latter equation, we prove that

$$j^2 yH(z)td(H(j)) = 0 \text{ for all } j \in J,\ y, z, t \in \mathcal{N}.$$

From Lemma 3 with the 3-primeness of $\mathcal{N}$ we obtain $d(H(J)) = \{0\}$, which implies that $H(J) \subseteq Z(\mathcal{N})$ by Lemma 2 (i), and Lemma 4 (i) forces that $\mathcal{N}$ is a commutative ring. In this case, returning to our hypotheses, we can easily arrive at

$$\mathfrak{D}(nH(j)) = 0 \text{ for all } j \in J,\ n \in \mathcal{N}$$

equivalently,

$$H(j)d(n) = 0 \text{ for all } j \in J,\ n \in \mathcal{N}. \tag{2.9}$$

Replacing j by $j \circ m$, where $m \in \mathcal{N}$ in (2.9), and using it again, we find that $H(J)\mathcal{N}d(n) = \{0\}$ for all $n \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, we conclude that $d = 0$ or $H(J) = \{0\}$. If $H(J) = \{0\}$, then $H((j \circ m) \circ n)) = 0$ for all $j \in J,\ n, m \in \mathcal{N}$. In view of the 2-torsion freeness of $\mathcal{N}$, we get $J\mathcal{N}H(n) = \{0\}$ which assures that $J = \{0\}$ or $H = 0$ by the 3-primeness of $\mathcal{N}$; a contradiction. Hence, $d = 0$. Using the same techniques as used in the proof of $(i) \Rightarrow (iii)$ in Theorem 1, we conclude that $\mathfrak{D} = 0$.
(ii) Suppose that $\mathfrak{D}(H(n \circ j)) = \mathfrak{D}(H([n, j]))$ for all $j \in J,\ n \in \mathcal{N}$. It follows that

$$\mathfrak{D}(H(jn)) = 0 \text{ for all } j \in J,\ n \in \mathcal{N}. \tag{2.10}$$

Taking in instead of n in (2.10) and using it again, we can easily arrive at

$$iH(n)d(j) = 0 \text{ for all } i, j \in J, \; n \in \mathcal{N}.$$

Substituting yzt for n in the last equation, we can see that

$$iyH(z)td(j) = 0 \text{ for all } i, j \in J, \; y, z, t \in \mathcal{N},$$

which can be rewritten as $J\mathcal{N}H(z)\mathcal{N}d(J) = \{0\}$ for all $z \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime, $J \neq \{0\}$ and $H \neq 0$, we obtain $d(J) = \{0\}$, then $d = 0$ by using the proof of the case **(i)**. According to (2.10), we can show that

$$H(j)\mathfrak{D}(n) = 0 \text{ for all } j \in J, \; n \in \mathcal{N}. \tag{2.11}$$

Placing $j \circ m$ for j, where $m \in \mathcal{N}$, in (2.11) we infer that $H(J)\mathcal{N}\mathfrak{D}(n) = \{0\}$ for all $n \in \mathcal{N}$. As $\mathcal{N}$ is 3-prime and $J \neq \{0\}$, we conclude that $\mathfrak{D} = 0$. QED

The next result is a consequence immediate of Theorem 3, just to take $H = id_{\mathcal{N}}$.

Corollary 4. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admit a left derivation d and a nonzero multiplier H that satisfy any one of the following identities:*

(i) $\mathfrak{D}(J) = \{0\}$,

(ii) $\mathfrak{D}(n \circ j) = \mathfrak{D}([n, j])$ *for all* $j \in J, \; n \in \mathcal{N}$,

then $\mathfrak{D} = 0$.

2.2 Results about 3-prime left near-rings

Theorem 3. Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a generalized left derivation $\mathfrak{D}$ associated with a left derivation d satisfying any one of the following identities:

(i) $\mathfrak{D}(J) = \{0\}$,

(ii) $\mathfrak{D}(J^2) = \{0\}$,

then $\mathfrak{D} = 0$.

Proof. **(i)** By our hypothesis, we have $\mathfrak{D}(j \circ j) = 0$ for all $j \in J$. It follows that

$$jd(j) = 0 \text{ for all } j \in J. \tag{2.12}$$

From Lemma 5, we get $jnd(j) = njd(j) = 0$ for all $j \in J,\ n \in \mathcal{N}$, so $J\mathcal{N}d(J) = \{0\}$, by 3-primeness of $\mathcal{N}$, we arrive at

$$d(J) = \{0\}. \tag{2.13}$$

Using (2.13), it follows that $d(j \circ n) = 0$ for all $j \in J,\ n \in \mathcal{N}$, so we have

$$jd(n) = 0 \text{ for all } n \in \mathcal{N}. \tag{2.14}$$

Replacing n by jnm in (2.14) and using it with (2.13), we can see that $j^2\mathcal{N}d(m) = \{0\}$ for all $j \in J,\ m \in \mathcal{N}$. In view of Lemma 3 and the 3-primeness of $\mathcal{N}$, we conclude that $d = 0$. Since $\mathfrak{D}(j \circ n) = 0$ for all $j \in J,\ n \in \mathcal{N}$, by defining $\mathfrak{D}$ with the same techniques as used previously, we find that $\mathfrak{D} = 0$.
(**ii**) Suppose that $\mathfrak{D}(J^2) = \{0\}$, then $\mathfrak{D}(j \circ j^2) = \mathfrak{D}(j(j \circ j)) = 0$ for all $j \in J$, by the 2-torsion freeness of $\mathcal{N}$, we get $\mathfrak{D}(j^3) = 0$ for all $j \in J$ and hence

$$j^2d(j) = 0 \text{ for all } j \in J. \tag{2.15}$$

From Lemma 5, we can write $j(nj)d(j) = (nj)jd(j)$ for all $j \in J, n \in \mathcal{N}$, and by (2.15) we get $jnjd(j) = 0$ for all $j \in J,\ n \in \mathcal{N}$ which implies $j\mathcal{N}jd(j) = \{0\}$ for all $j \in J$. The 3-primeness of $\mathcal{N}$ gives $jd(j) = 0$ for all $j \in J$, arguing as the same techniques as used previously, we conclude that $\mathfrak{D} = 0$. QED

Corollary 5. Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a generalized left derivation $\mathfrak{D}$ associated with a left derivation d satisfying any one of the following identities:

(i) $\mathfrak{D}(\mathcal{N}) = \{0\}$,

(ii) $\mathfrak{D}(\mathcal{N}^2) = \{0\}$,

then $\mathfrak{D} = 0$.

Corollary 6. Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a left derivation d satisfying any one of the following identities:

(i) $d(J) = \{0\}$,

(ii) $d(J^2) = \{0\}$,

then $d = 0$.

Corollary 7. Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a left derivation d satisfying any one of the following identities:

(i) $d(\mathcal{N}) = \{0\}$,

(ii) $d(\mathcal{N}^2) = \{0\}$,

then $d = 0$.

The following example proves that the 3-primeness of $\mathcal{N}$ cannot be omitted in our Theorems.

Example 2. Let $\mathcal{R}$ be a right (or left) near-ring which is not abelian. Define the sets $\mathcal{N}$, J and the maps d, $\mathfrak{D}$ of $\mathcal{N}$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} \mid x, y, z, 0 \in \mathcal{R} \right\}, J = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix} \mid u, 0 \in \mathcal{R} \right\},$$

$$d\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \text{ and } \mathfrak{D}\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix}.$$

Then $\mathcal{N}$ is a right (or left) near-ring which is not 3-prime, J is a nonzero Jordan ideal of $\mathcal{N}$, d is a nonzero left derivation of $\mathcal{N}$ which is not a derivation. Also, $\mathfrak{D}$ is a nonzero generalized left derivation associated with a left derivation d of $\mathcal{N}$ which is not a left derivation, which satisfies all identities mentioned in this paper. However, $\mathcal{N}$ is not a commutative ring.

3 Conclusion

In this paper, we study the 3-prime near-rings with generalized left derivations. We prove that 3-prime near-rings admitting generalized left derivations satisfying certain types of differential identities on Jordan ideals become commutative rings. Compared to some recent studies using some additive maps, our results are considered more advanced. In future research, we will try to generalize some existing results using a new concept, which we will define later for quotient near-rings, called generalized P-left derivation, where P is a prime ideal of the studied near-ring.

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