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Lower bounds for Riesz-Fischer maps in rigged Hilbert spaces

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Abstract. This note concerns a further study about Riesz-Fischer maps, already introduced by the author in a recent work, that is a notion that extends to the spaces of distributions the sequences that are known as Riesz-Fischer sequences. In particular it is proved a characterizing inequality that has as consequence the existence of the continuous inverse of the synthesis operator.

Keywords: distributions, rigged Hilbert spaces, frames, Riesz-Fischer sequences

MSC 2022 classification: primary 42C15, secondary 47A70

Introduction

As it is known, given an element $f \in \mathcal{H}$ and a sequence of elements $\{f_n\}_{n\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ endowed of the inner product $\langle \cdot|\cdot \rangle$, the sequence $a_n : \mathbb{N} \to \mathbb{C}$, $a_n := \langle f|f_n\rangle$ is called *moment sequence* -briefly *moments- of* $f \in \mathcal{H}$. The problem to find a solution $f \in \mathcal{H}$ of the equations:

$$\langle f|f_n\rangle = a_n, \quad n \in \mathbb{N}$$

given $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ and $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$, is known as *moment problem.*

In particular, the sequence $\{f_n\}_{n\in\mathbb{N}}$ is called *Riesz-Fischer sequence* if, for every $\{a_n\}_{n\in\mathbb{N}} \in \ell^2$ (i.e. such that $\sum_1^\infty |a_n|^2 < \infty$), there exists a solution f of the moment problem.

On the other hand, the sequence $\{f_n\}_{n\in\mathbb{N}}$ is called *Bessel sequence* if, for all $f \in \mathcal{H}$, one has $\{\langle f|f_n\rangle\}_{n\in\mathbb{N}} \in \ell^2$.

We have the well-known characterization results [22, Ch. 4, Sec. 2,Th. 3]:

- $\{f_n\}_{n\in\mathbb{N}}$ is a Riesz-Fischer sequence if, and only if, there exists $A > 0$ such that:
$$A \sum |c_n|^2 \leq \| \sum c_n f_n\|^2 \tag{0.1}$$
for all finite scalar sequences $\{c_n\} \subset \mathbb{C}$;

- $\{f_n\}_{n\in\mathbb{N}}$ is a Bessel sequence if, and only if, there exists $B > 0$ such that:

$$\|\sum c_n f_n\|^2 \leq B \sum |c_n|^2 \tag{0.2}$$

for all finite scalar sequences $\{c_n\} \subset \mathbb{C}$.

Bessel and Riesz-Fischer sequences play an important role in the theory of frames and, in particular, in the study of Riesz bases [22, 8]. Roughly speaking, a frame is an extension of a basis in a Hilbert space, in the sense that every vector of $\mathcal{H}$ can be decomposed in terms of elements of a frame, but this decomposition is not unique. This "loss of constraints" or "more leeway" allows several applications in many branches of mathematical sciences and technology.

More precisely, a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ is a *frame* if there exist $A, B > 0$ such that:

$$A\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f|f_n\rangle|^2 \leq B\|f\|^2, \forall f \in \mathcal{H}.$$

A frame $\{f_n\}_{n\in\mathbb{N}}$ that is also a basis for $\mathcal{H}$, is called *Riesz basis*. One has that Bessel, Riesz-Fischer sequences and Riesz basis are related via linear operators to orthonormal basis (see [22, 8, 5]). Furthermore, if $\{f_n\}_{n\in\mathbb{N}}$ is complete or total (i.e. the set of its linear span is dense in $\mathcal{H}$), then it is a Riesz basis if, and only if, it is both Bessel and Riesz-Fischer sequence [22].

If it is not diversely specified, a frame is intended as discrete. However, a notion of *continuous frame* have been introduced by S. T. Ali, J. P. Antoine, J. P. Gazeau in [1, 2] in order to study coherent states, and, independently, by G. Kaiser in [17]. Instead of a sequence, it is considered a map F from a measure space (X, μ) (μ is a positive measure) to a Hilbert space $\mathcal{H}$, i.e.: $F : X \to \mathcal{H}$, $F : x \mapsto F_x$. The map F is called continuous frame with respect to (X, μ) if:

- F is weakly measurable, i.e. that is $x \to \langle f|F_x\rangle$ is μ-measurable for every $f \in \mathcal{H}$;

- there exist $A, B > 0$ such that:

$$A\|f\|^2 \leq \int_X |\langle f|F_x\rangle|^2 d\mu \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

In [20], C. Trapani, S. Triolo and the author have extended the notion of frame, and related topics as Bessel, bases, Riesz basis, etc., to the spaces of distributions; in [21] and [10] a further study has been done respectively for Riesz-Fischer sequences and multipliers.

An appropriate framework for the spaces of distributions is given by the *rigged Hilbert space* i.e. a triple $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$ where $\mathcal{D}$ is a locally convex space,

$\mathcal{D}^\times$ the conjugate dual to $\mathcal{D}$, and where the inclusions have to be intended as continuous and dense embedding. They have been introduced by Gel'fand in [13, 14] with the aim to define the *generalized eigenvectors* of an essential self-adjoint operator on $\mathcal{D}$ and to prove the theorem known as Gel'fand-Maurin theorem, on the existence of a complete system of generalized eigenvector (see also [15]). For that it is called also *Gel'fand triple*, also denoted by $(\mathcal{D}, \mathcal{H}, \mathcal{D})$.

However *Gel'fand triple* play a relevant role also in other branches of mathematics, such as Gabor Analysis: see for example [9, 11, 12]. More exhaustive references, papers and talks can be found on: www.nuhag.eu/talks .

Reconsidering Riesz-Fischer maps introduced in [21], the aim of this paper is to continue the study, proving characterizing conditions, analogously to (0.1) for the Riesz-Fischer sequences. The paper is organized as follows. In Section 2 are recalled some preliminaries, definitions and previous results. In Section 3 are proved some characterizations of Riesz-Fischer maps in term of lower bounds properties.

1 Preliminary definitions and facts

As usual, let us denote by $\mathcal{H}$ a Hilbert space, $\langle\cdot|\cdot\rangle$ its inner product and $\|\cdot\|$ the Hilbert norm. Let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$ endowed with a locally convex topology t stronger than the topology induced by the Hilbert norm. The embedding of $\mathcal{D}$ in $\mathcal{H}$ is continuous and dense, and it is denoted by $\mathcal{D} \hookrightarrow \mathcal{H}$. The space of conjugate linear continuous forms on $\mathcal{D}$ is called *conjugate dual of* $\mathcal{D}$ and it is denoted by $\mathcal{D}^\times$. Unless otherwise stated, the value of $F \in \mathcal{D}^\times$ on $f \in \mathcal{D}$ is denoted by $\langle f|F\rangle$. The space $\mathcal{D}^\times$ is endowed with the *strong dual topology* $t^\times = \beta(\mathcal{D}^\times, \mathcal{D})$ defined by the set of seminorms:

$$p_{\mathcal{M}}(F) = \sup_{g\in\mathcal{M}} |\langle g|F\rangle|, \quad F \in \mathcal{D}^\times, \tag{1.3}$$

where $\mathcal{M}$ is a bounded subset of $\mathcal{D}[t]$. In this way, the Hilbert space $\mathcal{H}$ can be continuously embedded as subspace of $\mathcal{D}^\times$ (see [16]). If $\mathcal{D}$ is reflexive, i.e. $\mathcal{D}^{\times\times} = \mathcal{D}$, the embedding is dense and the Gel'fand triple is obtained:

$$\mathcal{D}[t] \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times[t^\times], \tag{1.4}$$

where $\hookrightarrow$ denotes a continuous and dense embedding. The sesquilinear form $\langle\cdot|\cdot\rangle$ that puts $\mathcal{D}$ and $\mathcal{D}^\times$ in duality is an extension of the inner product of $\mathcal{H}$ and has the same notation. We put: $\langle F|f\rangle := \overline{\langle f|F\rangle}$.

Throughout the paper, (X, μ) will be measure space, where μ is a σ-finite positive measure. By $L^1(X, \mu), L^2(X, \mu), \cdots$ we mean the usual spaces of measurable functions; if $X = \mathbb{R}$, and μ is the Lebesgue measure, we denote them as

$L^p(\mathbb{R})$. Furthermore, $\mathcal{S}$ stands for the *Schwartz space*, i.e. the space of infinitely differentiable and rapidly decreasing functions on $\mathbb{R}$. The conjugate dual of $\mathcal{S}$, denoted by $\mathcal{S}^\times$, is known as the space of *tempered distributions* (see [18] for more accurate definitions). An usual example of rigged Hilbert space is given by:

$$\mathcal{S} \hookrightarrow L^2(\mathbb{R}) \hookrightarrow \mathcal{S}^\times.$$

The vector space of all continuous linear maps from $\mathcal{D}[t]$ into $\mathcal{D}^\times[t^\times]$ will be denoted by $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$. If $\mathcal{D}[t]$ is barreled (e.g. reflexive), an involution can be introduced in $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$, $X \mapsto X^\dagger$, by:

$$\left\langle X^\dagger \eta | \xi \right\rangle = \overline{\langle X\xi | \eta \rangle}, \quad \forall \xi, \eta \in \mathcal{D}. \tag{1.5}$$

Hence, in this case, $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$ is a †-invariant vector space. For a detailed study, see [3].

In this paper we consider maps with values in a distribution space, defined in [20] as *weakly measurable maps*, and here denoted by ω. The definition extends the notion, previously recalled in the introduction, of weakly measurable functions considered for continuous frames (see [2]).

Definition 1. The correspondence $\omega : X \to \mathcal{D}^\times$, $x \mapsto \omega_x$ is called *weakly measurable map* if the complex valued function $x \mapsto \langle f|\omega_x\rangle \in \mathbb{C}$ is μ-measurable for all $f \in \mathcal{D}$.

In particular, the notions of completeness and independence of sequences in the Hilbert space is extended to the spaces of distributions by the following:

Definition 2. Let $\omega : x \in X \to \omega_x \in \mathcal{D}^\times$ be a weakly measurable map, then:

i) ω is *total* or *complete* if, $f \in \mathcal{D}$ and $\langle f|\omega_x\rangle = 0$ μ-a.e. $x \in X$ implies $f = 0$;

ii) ω is *μ-independent* if the unique measurable function $\xi : X \to \mathbb{C}$ such that $\int_X \xi(x) \langle g|\omega_x\rangle \, d\mu = 0$ for every $g \in \mathcal{D}$, is $\xi(x) = 0$ μ-a.e.

Let us recall the notion of *Bessel map*:

Definition 3. [20, Definition 3.2] A weakly measurable map ω is a *Bessel distribution map* (briefly: Bessel map) if for every $f \in \mathcal{D}$, $\int_X |\langle f|\omega_x\rangle|^2 d\mu < \infty$.

As a consequence of the closed graph theorem, if $\mathcal{D}$ is a Frèchet space, for the Bessel maps one has the following characterization result:

Proposition 1. [20, Proposition 3.1] *Let $\mathcal{D}[t]$ be a Frèchet space, and $\omega : x \in X \to \omega_x \in \mathcal{D}^\times$ be a weakly measurable map. The following statements are equivalent.*

(i) ω is a Bessel map;

(ii) there exists a continuous seminorm p on $\mathcal{D}[t]$ such that:

$$\left(\int_X |\langle f|\omega_x\rangle|^2 d\mu\right)^{1/2} \leq p(f), \quad \forall f \in \mathcal{D}.$$

(iii) for every bounded subset $\mathcal{M}$ of $\mathcal{D}$ there exists $C_{\mathcal{M}} > 0$ such that:

$$\sup_{f\in\mathcal{M}} \left|\int_X \xi(x) \langle \omega_x|f\rangle \, d\mu\right| \leq C_{\mathcal{M}}\|\xi\|_2, \quad \forall \xi \in L^2(X,\mu).$$

The previous proposition has the following consequences [20]:

- If $\xi \in L^2(X,\mu)$, then the conjugate linear functional Λ_ω^ξ on $\mathcal{D}$ defined by:

$$\left\langle f|\Lambda_\omega^\xi\right\rangle := \int_X \xi(x) \langle f|\omega_x\rangle \, d\mu, \quad \forall f \in \mathcal{D} \tag{1.6}$$

  is defined and continuous, i.e. $\Lambda_\omega^\xi \in \mathcal{D}^\times[t^\times]$;

- the *synthesis operator* $D_\omega : L^2(X,\mu) \to \mathcal{D}^\times[t^\times]$ weakly defined by $D_\omega : \xi \mapsto \Lambda_\omega^\xi$, i.e. such that:

$$\langle f|D_\omega\xi\rangle := \int_X \xi(x) \langle f|\omega_x\rangle \, d\mu, \quad \forall f \in \mathcal{D}$$

  is continuous;

- the *analysis operator* $C_\omega : \mathcal{D}[t] \to L^2(X,\mu)$ defined by $(C_\omega f)(x) = \langle f|\omega_x\rangle$ is continuous;

- the *frame operator* $S_\omega : \mathcal{D} \to \mathcal{D}^\times$, $S_\omega := D_\omega C_\omega$ is continuous, i.e. $S_\omega \in \mathcal{L}(\mathcal{D},\mathcal{D}^\times)$.

Remark 1. If $\mathcal{D}$ is a Frèchet space, a Bessel map ω is bounded from above by a continuous seminorm on $\mathcal{D}$, but, in general, is not bounded from above by the Hilbert norm. An example, considered in [21], is the system of derivative of Dirac deltas on $\mathcal{S} \hookrightarrow L^2(\mathbb{R}) \hookrightarrow \mathcal{S}^\times$ denoted by $\{\delta'_x\}_{x\in\mathbb{R}}$, and defined by: $\langle f|\delta'_x\rangle := -f'(x)$. Then δ'_x is a Bessel map but is not bounded from above by the Hilbert norm. In [20] the notion of *bounded Bessel map* is defined, i.e. a Bessel map ω such that there exists $B > 0$ such that: $\int_X |\langle f|\omega_x\rangle|^2 d\mu \leq B\|f\|^2$ for all $f \in \mathcal{D}$. In particular, if ω is also total and if there exists $B > 0$ such that $0 < \int_X |\langle f|\omega_x\rangle|^2 d\mu \leq B\|f\|^2$ forall $f \neq 0$, then ω is called *distribution upper semiframe* [21], as extension to the space of distributions of the corresponding notion of *continuous upper semiframe* introduced in [4].

If a bounded Bessel map ω is also bounded from below by the Hilbert norm, we have the definition of *distribution frame*:

Definition 4. [20, Definition 3.6] Let $\mathcal{D}[t] \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times[t^\times]$ be a rigged Hilbert space, with $\mathcal{D}[t]$ a reflexive space and ω is a Bessel map. We say that ω is a *distribution frame* if there exist $A, B > 0$ such that:

$$A\|f\|^2 \leq \int_X |\langle f|\omega_x\rangle|^2 d\mu \leq B\|f\|^2, \quad \forall f \in \mathcal{D}.$$

We have that (see [20] for details): Λ_ω^ξ is bounded in $(\mathcal{D}, \|\cdot\|)$ and the bounded extension to $\mathcal{H}$ is denoted by $\widetilde{\Lambda}_\omega^\xi$; the synthesis operator D_ω has range in $\mathcal{H}$ and it is bounded; the Hilbert adjoint D_ω^* extends C_ω to $\mathcal{H}$; the operator $\widetilde{S}_\omega = D_\omega D_\omega^*$ is bounded and extends the frame operator S_ω.

If ω is a distribution frame, then the frame operator $\hat{S}_\omega$ satisfies the inequality:

$$A\|f\| \leq \|\hat{S}_\omega f\| \leq B\|f\|, \quad \forall f \in \mathcal{H},$$

with $A, B > 0$ frame bounds for ω. Since $\hat{S}_\omega$ is symmetric, this implies that $\hat{S}_\omega$ has a bounded inverse $\hat{S}_\omega^{-1}$ everywhere defined in $\mathcal{H}$.

In [21] are defined the Riesz-Fischer maps in the space of distributions. They are the analogous of that notion in Hilbert spaces whose extension to the continuous case is given in [19].

Definition 5. [21, Definition 3.4] Let $\mathcal{D}[t]$ be a locally convex space. A weakly measurable map $\omega : x \in X \mapsto \omega_x \in \mathcal{D}^\times$ is called a *Riesz-Fischer distribution map* (briefly: Riesz-Fischer map) if, for every $h \in L^2(X, \mu)$, there exists $f \in \mathcal{D}$ such that:

$$\langle f|\omega_x\rangle = h(x) \quad \mu\text{-a.e.} \tag{1.7}$$

In this case, we say that f is a solution of equation $\langle f|\omega_x\rangle = h(x)$.

Clearly, if f_1 and f_2 are solutions of (1.7), then $f_1 - f_2 \in \omega^\perp := \{g \in \mathcal{D} : \langle g|\omega_x\rangle = 0, \quad \mu - a.e.\}$. If ω is total, the solution is unique.

The analysis operator C_ω is defined on $dom(C_\omega) := \{f \in \mathcal{D} : \langle f|\omega_x\rangle \in L^2(X, \mu)\}$ as $C_\omega : f \in dom(C_\omega) \mapsto \langle f|\omega_x\rangle \in L^2(X, \mu)$. Clearly, ω is a Riesz-Fischer map if and only if $C_\omega : dom(C_\omega) \to L^2(X, \mu)$ is surjective. If ω is total, it is injective too, so, in this case, C_ω is invertible.

To define the synthesis operator D_ω we consider the following subset of $L^2(X, \mu)$:

$$dom(D_\omega) := \{\xi \in L^2(X, \mu), \text{s.t.} \int_X \xi(x)\omega_x d\mu \, \text{is convergent in } \mathcal{D}^\times\}.$$

Where *convergent in* $\mathcal{D}^\times$ means that: $\int_X \xi(x)\langle f|\omega_x\rangle d\mu$ is convergent for all $f \in \mathcal{D}$ and the conjugate functional on $\mathcal{D}$ defined in (1.6) by Λ_ω^ξ, is continuous, so $\Lambda_\omega^\xi \in \mathcal{D}^\times$. Then the synthesis operator $D_\omega : dom(D_\omega) \to \mathcal{D}^\times$ is weakly defined by:

$$D_\omega : \xi \mapsto \Lambda_\omega^\xi := \int_X \xi(x)\omega_x d\mu.$$

The range of D_ω is denoted by $Ran(D_\omega)$:

$$Ran(D_\omega) := \Big\{F \in \mathcal{D}^\times : \exists \xi \in dom(D_\omega) : \forall f \in \mathcal{D}, \langle f|F\rangle := \int_X \xi(x) \langle f|\omega_x\rangle \, d\mu\Big\}.$$

If $\mathcal{D}$ is a Frèchet space, as a consequence of the closed graph theorem one has, for a total Riesz-Fischer map ω, the following inequality:

Corollary 1. *[21, Corollary 3.7] Assume that $\mathcal{D}[t]$ is a Frèchet space. If the map $\omega : x \in X \to \omega_x \in \mathcal{D}^\times$ is a total Riesz-Fischer map, then for every continuous seminorm p on $\mathcal{D}$, there exists a constant $C > 0$ such that, for the solution f of (1.7):*

$$p(f) \leq C\| \langle f|\omega_x\rangle \|_2.$$

It follows that, if ω is a total Riesz-Fischer map, the inverse of the analysis operator $C_\omega^{-1} : L^2(X, \mu) \to dom(C_\omega)$ is continuous.

2 Main results

In this section are proved some characterization properties of Riesz-Fischer maps. We have the following:

Proposition 2. *Let (X, μ) be a measure space, $h(x) \in L^2(X, \mu)$ and $\omega : X \ni x \mapsto \omega_x \in \mathcal{D}^\times$ a weakly measurable map. Then ω is a Riesz-Fischer map if, and only if, there exists a bounded subset $\mathcal{M} \subset \mathcal{D}$ such that:*

$$\Big|\int_X \xi(x)\overline{h(x)}d\mu\Big| \leq \sup_{f\in\mathcal{M}} \Big|\int_X \xi(x) \langle \omega_x|f\rangle \, d\mu\Big| \tag{2.8}$$

for all $\xi(x) \in L^2(X, \mu)$ such that $\int_X \xi(x)\omega_x d\mu$ is convergent in $\mathcal{D}^\times$.

Proof. Necessity is obvious: let $\bar{f}$ be a solution of (1.7), then, for all $\xi(x)$, one has:

$$\Big|\int_X \xi(x)\overline{h(x)}d\mu\Big| = \Big|\int_X \xi(x) \langle \omega_x|\bar{f}\rangle \, d\mu\Big| \leq \sup_{f\in\mathcal{M}} \Big|\int_X \xi(x) \langle \omega_x|f\rangle \, d\mu\Big|.$$

Sufficiency: if the condition (2.8) holds, consider the subspace $\mathcal{E} \subset \mathcal{D}^\times$ defined by the set of elements $F \in \mathcal{D}^\times$ such that there exists $\xi \in L^2(X, \mu)$: $F = \int_X \xi(x)\omega_x d\mu$, and let us define the linear functional ν on $\mathcal{E}$ by:

$$\nu(F) = \nu\Big(\int_X \xi(x)\omega_x d\mu\Big) := \int_X \xi(x)\overline{h(x)}d\mu.$$

It follows immediately from hypothesis that ν is defined unambiguously. Furthermore, from (2.8) one has:

$$\big|\nu(F)\big| = \Big|\int_X \xi(x)\overline{h(x)}d\mu\Big| \leq \sup_{f\in\mathcal{M}}\Big|\int_X \xi(x)\left\langle \omega_x|f\right\rangle d\mu\Big| = \sup_{f\in\mathcal{M}} |\left\langle f|F\right\rangle|, \forall F\in\mathcal{E}$$

i.e. $\nu(F)$ is bounded by a seminorm of $\mathcal{D}^\times[t^\times]$. By Hahn-Banach theorem, there exists an extension $\widetilde{\nu}$ of ν to $\mathcal{D}^\times$. Since $\mathcal{D}$ is reflexive, there exists $\bar{f}\in\mathcal{D}$ such that: $\widetilde{\nu}(F) = \big\langle F|\bar{f}\big\rangle$. Since:

$$\int_X \xi(x)[\big\langle\omega_x|\bar{f}\big\rangle - \overline{h(x)}]d\mu = \nu(F) - \int_X \xi(x)\overline{h(x)}d\mu = 0, \quad \forall\,\xi\in L^2(X,\mu)$$

then $\big\langle\bar{f}|\omega_x\big\rangle = h(x)$ μ-a.e. QED

As consequence, we have the following:

Corollary 2. *ω is a Riesz-Fischer map if, and only if, there exists a bounded subset $\mathcal{M}\subset\mathcal{D}$ such that:*

$$\|\xi\|_2 \leq \sup_{f\in\mathcal{M}}\Big|\int_X \xi(x)\left\langle f|\omega_x\right\rangle d\mu\Big|, \tag{2.9}$$

for all $\xi(x)\in L^2(X,\mu)$ such that $\int_X \xi(x)\omega_x d\mu$ is convergent in $\mathcal{D}^\times$.

Proof. Sufficiency: if the condition (2.9) holds, consider $h(x)\in L^2(X,\mu)$ with $\|h(x)\|_2\leq 1$, then:

$$\Big|\int_X \xi(x)\overline{h(x)}d\mu\Big| \leq \|\xi\|_2 \leq \sup_{f\in\mathcal{M}}\Big|\int_X \xi(x)\left\langle f|\omega_x\right\rangle d\mu\Big|.$$

Then, for the previous proposition, ω is a Riesz-Fischer map.
Necessity: lete ω be a Riesz-Fischer map and put $\frac{\xi(x)}{\|\xi(x)\|_2} = h(x)$ (for $\xi\neq 0$), by Proposition 2 there exists a bounded subset $\mathcal{M}\subset\mathcal{D}$ such that:

$$\|\xi\|_2 = \int_X \xi(x)\frac{\overline{\xi(x)}}{\|\xi(x\|_2}d\mu \leq \sup_{f\in\mathcal{M}}\Big|\int_X \xi(x)\left\langle f|\omega_x\right\rangle d\mu\Big|.$$

QED

The previous Corollary can be rephrased as:

Corollary 3. *ω is a Riesz-Fischer map if, and only if, the synthesis operator D_ω is invertible and the inverse $D_\omega^{-1}: Ran(D_\omega)\to L^2(X,\mu)$ is continuous.*

3 Conclusions

The inequalities in Proposition 1(iii) and in Corollary 2 are an extension to the rigged Hilbert spaces respectively of the inequalities (0.2), and (0.1) for sequences in Hilbert spaces (see [22, Ch. 4, Sec. 2, Th. 2, Th. 3]). In the case of sequences, it follows immediately that: if $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$, then $\{f_n\}_{n\in\mathbb{N}}$ is a Bessel sequence if, and only if, there exists a bounded operator $T : \mathcal{H} \to \mathcal{H}$ such that $f_n = Te_n$; $\{f_n\}_{n\in\mathbb{N}}$ is a Riesz-Fischer sequence if, and only if, there exists a bounded operator $V : \mathcal{H} \to \mathcal{H}$ such that $Vf_n = e_n$ (for frames and Riesz-bases see also [5, Proposition 4.6]). Since in the spaces of distributions the orthonormality is not defined, a sort of "orthonormal basis" is played by the *Gel'fand basis*: see [20] and [21, Definition 5.3]. So, it would be appropriate to carry on a further study, started in [20], about the transformations between Gal'fand basis, Bessel, Riesz-Fischer maps, distribution frames, and Riesz distribution basis.

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