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Spectral Analysis of A Difference Equation With Interface Conditions And Hyperbolic Eigenparameter on The Whole Axis

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Abstract. In this article, we present some spectral properties of a difference equation with interface (discontinuity) conditions and hyperbolic parameter on the whole axis. The purpose of this paper is to introduce the solutions and to investigate the qualitative properties of this equation such as finiteness of eigenvalues and spectral singularities. The analysis based on finding resolvent operator, Green function, continuous spectrum and some asymptotic equations.

Keywords: Difference equations; Interface condition; Eigenvalues; Hyperbolic eigenparameter; Spectral analysis; Spectral singularities

MSC 2022 classification: primary 34B37, secondary 34L05, 47A75, 58C40

1 Introduction

Difference equations with interface conditions involve discontinuties at one or more than one point in an interval and are a tool for mathematically explaining processes that are subject to sudden changes. These sudden changes depend on external factors and are negligibly short compared to the whole time. The conditions at discontinuity points are called interface conditions, impulsive conditions, jump conditions or transmission conditions in literature. Interface actions have important consequences for mathematical theory. Firstly, Myshkis and Mil'man studied that kinds of problems for systems of differential equations with interface conditions [22]. Then, these equations were examined in detail by Samoilenko and Perestyuk [26], Perestyuk et al [25] and Lakshmikantham et al [19]. Recently, such problems arise in many areas of mathematical modeling including population dynamics, infectious diseases, control problems, economic problems, biotechnology, industrial robotics, ecology, optimal control, industrial robotics, medicine, control theory and so forth [15, 17, 18, 20, 24]. Although the theory of difference equations with interface conditions has many applications, there are insufficient studies examining the spectral analysis of these problems. One can find many books and studies consisting the examination of the spectral analysis of Sturm-Liouville, Dirac, Klein Gordon and other types of operators and equations in the literature [1, 2, 5, 6, 7, 8, 4], there are few studies about interface cases of such equations [3, 12, 23, 27]. Differently from these works, we present spectral properties of a difference equation with interface conditions and hyperbolic parameter on the whole axis in this study. The purpose of this paper is to introduce the solutions and to investigate the qualitative properties of this equation such as finiteness of eigenvalues and spectral singularities. The analysis bases on finding resolvent operator, Green function, continuous spectrum and some asymptotic equations. Let us consider the following second-order difference equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{Z} \setminus \{-1, 0, 1\} \tag{1.1}$$

with the interface conditions

$$\begin{cases} y_1 = \zeta_1 y_{-1} \\ y_2 = \zeta_2 y_{-2}, \end{cases} \tag{1.2}$$

here $\lambda = 2\cosh z$ is a spectral parameter, ζ_1 and ζ_2 are complex numbers such that $\zeta_1\zeta_2 \neq 0$, $\{a_n\}_{n\in\mathbb{Z}}$ and $\{b_n\}_{n\in\mathbb{Z}}$ are complex sequences satisfying the following condition

$$\sum_{n\in\mathbb{Z}} |\, n \,| \left(|1 - a_n| + |b_n|\right) < \infty. \tag{1.3}$$

Throughout the remainder of the paper, we assume that $a_n \neq 0$ for all $n \in \mathbb{Z}$. In this work, we investigate the spectral analysis of (1.1)-(1.2). Differently from [10, 11], this paper includes hyperbolic parameter. Hence, the analytical region of the Jost solution changes and the regions of the problem are renewed. This gives a different perspective to researchers working on these topics.

This study is organized as follows:

- Firstly, we give some basic definitions and Jost solutions of difference equation without the interface conditions given by (1.2) for use in other chapters.
- Later, we obtain the solutions of (1.1)-(1.2).
- Next, we find resolvent operator and Green function of the problem (1.1)-(1.2). Furthermore, by using the poles of the resolvent operator's kernel, we define the sets of eigenvalues and spectral singularities of this problem.

- At last, we present a condition that guarantees finiteness of the multiplicities of the eigenvalues and spectral singularities of (1.1)-(1.2).

2 Statement of the problem

In this part, we give some auxiliary definition and lemmas. We introduce the solutions of (1.1)-(1.2) with the help of the solutions of (1.1) and we give an important asymptotic equation which is necessary to get the main results. Related the (1.1)-(1.2), let us introduce a difference operator L in the Hilbert space $\ell_2(\mathbb{Z})$ such that

$$\ell_2(\mathbb{Z}) := \left\{ y = \{y_n\}_{n\in\mathbb{Z}},\ y_n \in \mathbb{C},\ \|y\|^2 := \sum_{n\in\mathbb{Z}} |y_n|^2 < \infty \right\},$$

created by the following difference expression

$$l(y) := a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1}, \quad n \in \mathbb{Z} \setminus \{-1, 0, 1\}$$

and the interface conditions (1.2). Equation (1.1) has the bounded solutions $f_n^+(z)$ and $f_n^-(z)$ which are represented by

$$f_n^+(z) = \rho_n^+ e^{nz} \left(1 + \sum_{m=1}^{\infty} A_{n,m}^+ e^{mz} \right), \quad n \in \mathbb{Z} \tag{2.4}$$

and

$$f_n^-(z) = \rho_n^- e^{-nz} \left(1 + \sum_{-\infty}^{m=-1} A_{n,m}^- e^{-mz} \right), \quad n \in \mathbb{Z}, \tag{2.5}$$

where

$$\rho_n^+ = \left\{ \prod_{k=n}^{\infty} a_k \right\}^{-1}, \quad \rho_n^- = \left\{ \prod_{-\infty}^{k=n-1} a_k \right\}^{-1}$$

for $z \in \overline{\mathbb{C}}_- := \{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$ [16]. Furthermore, $A_{n,m}^{\pm}$ are expressed in terms of the sequences $\{a_n\}_{n\in\mathbb{Z}}$, $\{b_n\}_{n\in\mathbb{Z}}$ and satisfy

$$\left| A_{n,m}^+ \right| \leq v_1 \sum_{k=n+\left[\left|\frac{m}{2}\right|\right]}^{\infty} \left(|1 - a_k| + |b_k| \right), \tag{2.6}$$

$$\left| A_{n,m}^- \right| \leq v_2 \sum_{-\infty}^{k=n+\left[\left|\frac{m}{2}\right|\right]+1} \left(|1 - a_k| + |b_k| \right), \tag{2.7}$$

here $\left[\left|\dfrac{m}{2}\right|\right]$ denotes the integer part of $\dfrac{m}{2}$ and v_1, v_2 are positive constants. The solutions $f^+(z) := \{f_n^+(z)\}_{n\in\mathbb{Z}}$ and $f^-(z) := \{f_n^-(z)\}_{n\in\mathbb{Z}}$ which are called the Jost solutions of (1.1) are analytic with respect to z in $\mathbb{C}_- := \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$, continuous in $\overline{\mathbb{C}}_-$ and provide the following asymptotic equations

$$f_n^\pm(z) = e^{\pm nz}\left[1 + o(1)\right], \quad z \in \overline{\mathbb{C}}_-, \quad n \to \pm\infty, \tag{2.8}$$

$$f_n^\pm(z) = p_n^\pm e^{\pm nz}\left[1 + o(1)\right], \quad n \in \mathbb{Z}, \quad z = x + iy, \quad x \to -\infty. \tag{2.9}$$

Definition 1. The Wronskian of two solutions $y = \{y_n\}_{n\in\mathbb{Z}}$ and $u = \{u_n\}_{n\in\mathbb{Z}}$ of (1.1) is defined by

$$W\left[y, u\right] := a_n\left[y_n(z)u_{n+1}(z) - y_{n+1}(z)u_n(z)\right].$$

It follows from the Definition 1 that

$$W\left[f_n^\pm(z), f_n^\pm(-z)\right] = \mp 2\sinh z, \quad z \in \mathbb{C}^* := \mathbb{C} \setminus \{z : z = k\pi i,\ k \in \mathbb{Z}\}.$$

Now, we'll consider the equation (1.1) with the interface conditions (1.2). Firstly, we define two solutions of (1.1)-(1.2) as follows

$$E_n^+(z) = \begin{cases} \beta_1(z)f_n^-(z) + \beta_2(z)f_n^-(-z); & n \in \mathbb{Z}^- \\ f_n^+(z); & n \in \mathbb{Z}^+ \end{cases} \tag{2.10}$$

and

$$E_n^-(z) = \begin{cases} f_n^-(z); & n \in \mathbb{Z}^- \\ \beta_3(z)f_n^+(z) + \beta_4(z)f_n^+(-z); & n \in \mathbb{Z}^+ \end{cases} \tag{2.11}$$

for $\lambda = 2\cosh z$, $z \in \mathbb{C}^*$, where β_i are arbitrary coefficients for $i = 1, 2, 3, 4$ depending on z. By the help of interface conditions (1.2) and Definition 1, we find uniquely

$$\beta_1(z) = -\frac{a_{-2}}{2\sinh z\zeta_1\zeta_2}\left[\zeta_2 f_1^+(z)f_{-2}^-(-z) - \zeta_1 f_2^+(z)f_{-1}^-(-z)\right] \tag{2.12}$$

$$\beta_2(z) = \frac{a_{-2}}{2\sinh z\zeta_1\zeta_2}\left[\zeta_2 f_1^+(z)f_{-2}^-(z) - \zeta_1 f_2^+(z)f_{-1}^-(z)\right] \tag{2.13}$$

$$\beta_3(z) = -\frac{a_1}{2\sinh z}\left[\zeta_1 f_{-1}^-(z)f_2^+(-z) - \zeta_2 f_{-2}^-(z)f_1^+(-z)\right] \tag{2.14}$$

$$\beta_4(z) = \frac{a_1}{2\sinh z}\left[\zeta_1 f_{-1}^-(z)f_2^+(z) - \zeta_2 f_{-2}^-(z)f_1^+(z)\right] \tag{2.15}$$

for all $z \in \mathbb{C}^*$.

Corollary 1. *There is a following relation between the coefficients $\beta_2(z)$ and $\beta_4(z)$*

$$\beta_4(z) = -\frac{a_1}{a_{-2}}\zeta_1\zeta_2\beta_2(z), \quad z \in \mathbb{C}^*,$$

where ζ_1, ζ_2 are complex numbers and a_1, a_{-2} are also complex numbers obtaining from the terms of a_n.

Lemma 1. *For all $z \in \mathbb{C}^*$, the Wronskian of the solutions $E_n^+(z)$ and $E_n^-(z)$ is given by*

$$W[E_n^+(z), E_n^-(z)] = \begin{cases} -2\sinh z\beta_2(z); & n \in \mathbb{Z}^- \\ 2\sinh z\dfrac{a_1}{a_{-2}}\zeta_1\zeta_2\beta_2(z); & n \in \mathbb{Z}^+. \end{cases}$$

Theorem 1. *Assume* (1.3)*. Then the function β_2 has the following asymptotic equation for $n \in \mathbb{Z}$*

$$\beta_2(z) = a_{-2}e^{4z}\left(\frac{p_{-1}^- p_2^+}{\zeta_2} - \frac{p_1^+ p_{-2}^-}{\zeta_1}\right)[1 + o(1)], \quad \operatorname{Re} z \to -\infty.$$

Proof. By using (2.9), if we write equation (2.13) in limit form, we find

$$\beta_2(z) = -\frac{e^{4z}a_{-2}}{\zeta_1\zeta_2}\left(\zeta_2 p_1^+ p_{-2}^- - \zeta_1 p_2^+ p_{-1}^-\right)[1 + o(1)]$$

for $\operatorname{Re} z \to -\infty$, where

$$\rho_1^+ = \left\{\prod_{k=1}^{\infty} a_k\right\}^{-1}, \quad \rho_{-1}^- = \left\{\prod_{-\infty}^{k=-2} a_k\right\}^{-1},$$

$$\rho_{-2}^- = \left\{\prod_{-\infty}^{k=-3} a_k\right\}^{-1}, \quad \rho_2^+ = \left\{\prod_{k=2}^{\infty} a_k\right\}^{-1}.$$

It completes the proof of Theorem 1. $\boxed{QED}$

3 Resolvent operator and continuous spectrum of L

In this section, we give resolvent operator and continuous spectrum of L. Now, we will define two semi-strips

$$T_0 := \left\{z \in \mathbb{C} : \operatorname{Re} z < 0, -\frac{\pi}{2} \le \operatorname{Im} z \le \frac{3\pi}{2}\right\}$$

and

$$T := T_0 \cup \left\{ z \in \mathbb{C} : \ \operatorname{Re} z = 0, \ \operatorname{Im} z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2} \right] \right\}.$$

Throughout this work, we will show the set $\left\{ z \in \mathbb{C} : \ \operatorname{Re} z = 0, \ \operatorname{Im} z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2} \right] \right\}$ by $\left[-\frac{\pi}{2}i, \frac{3\pi}{2}i \right]$ shortly. To find the resolvent operator of L, we consider other solutions of (1.1)-(1.2) as

$$U_n^+(z) = \begin{cases} \hat{\beta}_1(z) f_n^-(z) + \beta_2(z) \hat{f}_n^-(z); & n \in \mathbb{Z}^- \\ f_n^+(z); & n \in \mathbb{Z}^+ \end{cases} \tag{3.16}$$

and

$$V_n^-(z) = \begin{cases} f_n^-(z); & n \in \mathbb{Z}^- \\ \hat{\beta}_3(z) f_n^+(z) + \beta_4(z) \hat{f}_n^+(z); & n \in \mathbb{Z}^+ \end{cases} \tag{3.17}$$

for $\lambda = 2\cosh z$ and $z \in T \setminus \{0, \pi i\}$, where $\hat{f}^\pm(z) := \left\{ \hat{f}_n^\pm(z) \right\}_{n \in \mathbb{Z}}$ are unbounded solutions of equation (1.1) fulfilling the asymptotic equations

$$\hat{f}_n^\pm(z) = e^{\mp nz} \left[1 + o(1) \right], \quad z \in \overline{\mathbb{C}}_-, \quad n \to \pm\infty.$$

To get the coefficients $\hat{\beta}_1(z)$ and $\hat{\beta}_3(z)$, we will use the same way as finding $\beta_1(z)$ and $\beta_3(z)$. We obtain

$$\hat{\beta}_1(z) = -\frac{a_{-2}}{2\sinh z \zeta_1 \zeta_2} \left[\zeta_2 f_1^+(z) \hat{f}_{-2}^-(z) - \zeta_1 f_2^+(z) \hat{f}_{-1}^-(z) \right]$$

$$\hat{\beta}_3(z) = -\frac{a_1}{2\sinh z} \left[\zeta_1 f_{-1}^-(z) \hat{f}_2^+(z) - \zeta_2 f_{-2}^-(z) \hat{f}_1^+(z) \right],$$

respectively. Similar to Lemma 1, for all $z \in T \setminus \{0, \pi i\}$, we conclude that

$$W[U_n^+(z), V_n^-(z)] = \begin{cases} -2\sinh z \beta_2(z); & n \in \mathbb{Z}^- \\ \\ 2\sinh z \dfrac{a_1}{a_{-2}} \zeta_1 \zeta_2 \beta_2(z); & n \in \mathbb{Z}^+. \end{cases}$$

Theorem 2. *For all $z \in T \setminus \{0, \pi i\}$, $\beta_2(z) \neq 0$ and $k, n \neq 0$, the resolvent operator of L is defined by*

$$\left(R_\lambda (L) \, g \right)_n := \sum_{k \in \mathbb{Z}} G_{n,k}(z) g(k), \quad g := \{g_k\} \in \ell_2(\mathbb{Z}),$$

where

$$G_{n,k}(z)=\begin{cases}-\dfrac{U_n^+(z)V_k^-(z)}{W[U^+,V^-](z)}; & k=n-1,n-2,...\\ \\ -\dfrac{V_n^-(z)U_k^+(z)}{W[U^+,V^-](z)}; & k=n,n+1,...\end{cases} \tag{3.18}$$

is the Green function of (1.1)-(1.2).

Proof. It is necessary to solve the equation in order to find resolvent operator and Green function of (1.1)-(1.2)

$$a_{n-1}y_{n-1}+b_ny_n+a_ny_{n+1}-\lambda y_n=g_n, \quad n\in\mathbb{Z}\setminus\{-1,0,1\}, \tag{3.19}$$

where $g_n\in\ell_2(\mathbb{Z})$. Because of the fact that $U_n^+(z)$ and $V_n^-(z)$ are the fundamental solutions of (1.1)-(1.2), we write the general solution of (3.19) as

$$y_n(z)=h_nU_n^+(z)+t_nV_n^-(z), \tag{3.20}$$

where h_n, t_n are coefficients and different from zero. By the help of variation of parameters method, h_n and t_n are obtained as follows

$$h_n-h_{n-1}=-\frac{g_nV_n^-(z)}{W[U^+,V^-](z)}, \quad k,n\neq 0 \tag{3.21}$$

$$t_n-t_{n-1}=\frac{g_nU_n^+(z)}{W[U^+,V^-](z)}, \quad k,n\neq 0, \tag{3.22}$$

respectively. In accordance with (3.20)-(3.22), we easily find Green function and resolvent operator of L given in Theorem (2). $\boxed{QED}$

Theorem 3. *Under the condition* (1.3)*, the continuous spectrum of the operator* L *is* $[-2,2]$*, i.e.,* $\sigma_c(L)=[-2,2]$.

Proof. Let L_1 and L_2 denote difference operators in $\ell_2(\mathbb{Z})$ by the following difference expressions

$$(l_0y)_n=y_{n-1}+y_{n+1}, \quad n\in\mathbb{Z}\setminus\{-1,1\},$$

$$(l_1y)_n=(a_{n-1}-1)\,y_{n-1}+b_ny_n+(a_n-1)\,y_{n+1}, \quad n\in\mathbb{Z}\setminus\{-1,0,1\},$$

respectively. It is obvious that $L=L_0+L_1$ and L_1 is a compact operator in $\ell_2(\mathbb{Z})$ under the condition (1.3) [21]. We also say that L_0 is a selfadjoint operator with $\sigma_c(L_0)=[-2,2]$. From the Weyl theorem of a compact perturbation [14], it is easy to write $\sigma_c(L_0)=\sigma_c(L)=[-2,2]$. $\boxed{QED}$

4 Main results

In this section, we will investigate the finiteness of eigenvalues, spectral singularities and their multiplicities under some special cases. Theorem 2 and equation (3.18) point us that in order to examine the quantitative properties of impulsive boundary value problem (1.1)-(1.2), it is necessary to obtain the quantitative properties of zeros of the function β_2. So the sets of eigenvalues and spectral singularities of the operator L are defined by

$$\sigma_d(L) = \{\lambda = 2\cosh z : \ z \in T_0, \ \beta_2(z) = 0\}$$

and

$$\sigma_{ss}(L) = \left\{\lambda = 2\cosh z : \ z = ix, \ x \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right], \ \beta_2(z) = 0\right\} \setminus \{0, \pi i\}.$$

Let D_1 and D_2 denote the set of all zeros of the function β_2 in T_0 and $\left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right]$, respectively. It is easily seen that

$$D_1 := \{z : z \in T_0, \ \beta_2(z) = 0\}, \tag{4.23}$$

$$D_2 := \left\{z : z \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right], \ \beta_2(z) = 0\right\}. \tag{4.24}$$

Lemma 2. *Assume the condition* (1.3)*. Then*
i) The set D_1 is bounded, has at most countable many elements and its limit points can lie only in $\left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right]$.
ii) The set D_2 is compact and its linear Lebesgue measure is zero.

Proof. i) Since $\zeta_1\zeta_2 \neq 0$, by using Theorem 1, we can say that the sets D_1 and D_2 are bounded. In addition, it follows from (2.13) that the function β_2 is analytic in T_0. So the limit points of zeros of β_2 in T_0 can only lie in $\left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right]$.
ii) Because of the fact that we have shown that the set D_2 is bounded, in order to prove the compactness of D_2, we need to show its closeness. Using the uniqueness theorem of analytic functions and Privalov Theorem [13], we get that D_2 is a closed set and its linear Lebesgue measure is zero. $\boxed{QED}$

From (4.23) and (4.24), the sets of eigenvalues and spectral singularities of L can be rewritten as

$$\sigma_d(L) := \{\lambda : \lambda = 2\cosh z, \ z \in D_1\} \tag{4.25}$$

and

$$\sigma_{ss}(L) := \{\lambda : \lambda = 2\cosh z,\ z \in D_2\} \setminus \{0, \pi i\}, \tag{4.26}$$

respectively. Now, we give the following theorem as a result of (4.25), (4.26) and Lemma 2.

Theorem 4. *Assume* (1.3). *Then we have the following results.*
i) The set of eigenvalues of L *is bounded and countable, its limit points can lie only in* $[-2, 2]$.
ii) The set of spectral singularities of L *is compact and its linear Lebesgue measure is zero.*

Definition 2. The multiplicity of the corresponding eigenvalue or spectral singularity of the operator L is called the multiplicity of a zero of the function β_2 in T.

We give the following definition and lemma to get the next results.

Definition 3. The convolution of the sequences $\{c_n\}$ and $\{d_n\}$ is defined by

$$c_n * d_n := \sum_{n\in\mathbb{Z}} c_n d_{n-m}, \tag{4.27}$$

here "$*$" denotes the convolution operation.

Lemma 3. *The following equation is satisfied for all* $\lambda \in \mathbb{C}$

$$\sum_{n\in\mathbb{Z}} (c_n * d_n)\, e^{\lambda n} = \sum_{n\in\mathbb{Z}} c_n e^{\lambda n} \sum_{n\in\mathbb{Z}} d_n e^{\lambda n}. \tag{4.28}$$

Now, we suppose that the complex sequences $\{a_n\}_{n\in\mathbb{Z}}$ and $\{b_n\}_{n\in\mathbb{Z}}$ satisfy the following inequality

$$\sup_{n\in\mathbb{Z}} \left\{ e^{\epsilon|n|} (|1 - a_n| + |b_n|) \right\} < \infty, \quad \varepsilon > 0. \tag{4.29}$$

Theorem 5. *If the condition* (4.29) *holds, then the operator* L *has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

Proof. By the help of (2.6), (2.7) and (4.29), it can be easily shown that

$$\left|A^+_{n,m}\right| \le \hat{v}_1 e^{-\frac{\varepsilon}{2}\left|\frac{m}{2}\right|}, \quad n = 1, 2; \quad m = 1, 2, ..., \tag{4.30}$$

$$\left|A^-_{n,m}\right| \le \hat{v}_2 e^{-\frac{\varepsilon}{2}\left|\frac{m}{2}\right|}, \quad n = -1, -2; \quad m = -1, -2, ..., \tag{4.31}$$

where $\hat{v}_1$ and $\hat{v}_2$ are arbitrary constants. In addition, by the help of (4.27)-(4.31), we calculate

$$\left|A^{-}_{n,-m} * A^{+}_{2,m}\right|, \left|A^{-}_{n,-m} * A^{+}_{1,m}\right| \leq \hat{v}_3 e^{-\frac{\varepsilon}{2}\left|\frac{m}{2}\right|} \quad n = -1, -2; \quad m \in \mathbb{N}. \tag{4.32}$$

From (2.4), (2.5) and (2.13), the function β_2 can be rewritten as follows

$$\beta_2(z) = \frac{a_{-2}}{2\zeta_1\zeta_2 \sinh z}\left\{\zeta_2 p_1^{+} p_{-1}^{-} e^{3z}\left(1 + \sum_{m=1}^{\infty} A^{+}_{1,m} e^{mz}\right)\left(1 + \sum_{-\infty}^{m=-1} A^{-}_{-2,m} e^{-mz}\right)\right.$$
$$\left. -\zeta_1 p_2^{+} p_{-1}^{-} e^{3z}\left(1 + \sum_{m=1}^{\infty} A^{+}_{2,m} e^{mz}\right)\left(1 + \sum_{-\infty}^{m=-1} A^{-}_{-1,m} e^{-mz}\right)\right\}.$$

By means of (4.30)-(4.32), the last equation shows that the function β_2 has analytical continuation for $\frac{\varepsilon}{4} > Rez$. So, the limit points of all zeros of the function β_2 in T_0 can not lie in $\left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right]$. Thus, we say that the bounded sets $\sigma_d(L)$ and $\sigma_{ss}(L)$ have no limit points using Theorem 4, in other words, these sets have a finite number of elements. Analyticity of β_2 in $\frac{\varepsilon}{4} > Rez$ proves that all zeros of β_2 in T have a finite multiplicity. Consequently, we obtain the finiteness of eigenvalues and spectral singularities of (1.1)-(1.2). $\boxed{QED}$

Let us assume that the following condition, which is weaker than (4.29), is satisfied

$$\sup_{n\in\mathbb{Z}}\left\{e^{\varepsilon|n|^{\gamma}}\left(|1-a_n| + |b_n|\right)\right\} < \infty \tag{4.33}$$

for $\varepsilon > 0$ and $\frac{1}{2} \leq \gamma < 1$.
Under the condition (4.33), the function β_2 is still analytic in $\mathbb{C}_-$ and has infinite derivatives by (2.6), (2.7) and (4.33). To examine the finiteness of eigenvalues and spectral singularities under condition (4.33), we need some notations.
We denote the sets of all limit points of D_1 and D_2 by D_3 and D_4, respectively and the set of all zeros of β_2 with infinite multiplicity in T by D_5.

Lemma 4. *Under the condition* (1.3)*, we have*
i) $D_3 \subset D_2$, $D_4 \subset D_2$, $D_5 \subset D_2$, $D_3 \subset D_5$, $D_4 \subset D_5$,
ii) $\mu(D_3) = \mu(D_4) = \mu(D_5) = 0$.

Proof. Using the boundary uniqueness theorems of analytic functions [13], the proof of Lemma 4 is easily completed. $\boxed{QED}$

For the sake of simplicity, let us consider the following function

$$H(z) := \beta_2(z)e^{-3z}2\sinh z. \tag{4.34}$$

It is evident that the function H is analytic in $\mathbb{C}_-$ and infinitely differentiable on imaginary axis.
In order to give our main result, we need two lemmas.

Lemma 5. *Assume* (4.33)*. Then the following inequality holds*

$$\left|H^{(k)}(z)\right| \le \eta_k, \quad z \in T, \quad k = 0,1,...,$$

where

$$\eta_k \le B\hat{D}d^k k! k^{k\left(\frac{1-\gamma}{\gamma}\right)},$$

B, $\hat{D}$ and d are positive constants depending on ϵ and γ.

Proof. From (2.13), we can regulate $H(z)$ as

$$\begin{aligned}&\left|H^{(k)}(z)\right| \le \\ &\widetilde{K}\left\{\widetilde{L}\left(\sum_{-\infty}^{m=-1}\left|A^-_{-2,m}\right||m|^k + \sum_{m=1}^{\infty}\left|A^+_{1,m}\right||m|^k + \sum_{m=1}^{\infty}\left|A^+_{1,m} * A^-_{-2,-m}\right||m|^k\right)\right. \\ &\left.+\widetilde{M}\left(\sum_{m=1}^{\infty}\left|A^+_{2,m}\right||m|^k + \sum_{-\infty}^{m=-1}\left|A^-_{-1,m}\right||m|^k + \sum_{m=1}^{\infty}\left|A^+_{2,m} * A^-_{-1,-m}\right||m|^k\right)\right\},\end{aligned} \tag{4.35}$$

where $\widetilde{K} = \left|\frac{a_{-2}}{\zeta_1\zeta_2}\right|$, $\widetilde{L} = \left|\zeta_2 p_1^+ p_{-2}^-\right|$ and $\widetilde{M} = \left|\zeta_1 p_2^+ p_{-1}^-\right|$.
By means of (2.6), (2.7) and (4.33), the following inequalities can be easily found

$$\left|A^+_{n,m}\right| \le \hat{v}_4 e^{-\frac{\varepsilon}{2}\left|\frac{m}{2}\right|^{\gamma}}, \quad n = 1,2; \quad m = 1,2,..., \tag{4.36}$$

$$\left|A^-_{n,m}\right| \le \hat{v}_5 e^{-\frac{\varepsilon}{2}\left|\frac{m}{2}\right|^{\gamma}}, \quad n = -1,-2; \quad m = -1-,2,..., \tag{4.37}$$

here $\hat{v}_4$ and $\hat{v}_5$ are arbitrary constants. Using Lemma 3, it is evident that

$$\left|A^-_{n,-m} * A^+_{2,m}\right|, \left|A^-_{n,-m} * A^+_{1,m}\right| \le \hat{v}_6 e^{-\frac{\varepsilon}{2}\left|\frac{m}{2}\right|^{\gamma}}, \quad n = -1,-2; \quad m \in \mathbb{N}. \tag{4.38}$$

If we use the inequalities (4.36)-(4.38) in (4.35), we find the following inequality

$$\left|H^{(k)}(z)\right| \leq B\hat{V} \sum_{m=1}^{\infty} m^k e^{-\frac{\varepsilon}{2}\left|\frac{m}{2}\right|^{\gamma}}, \tag{4.39}$$

where

$$B := \left|\frac{a_{-2}}{\zeta_1\zeta_2}\right| \left\{\left|\zeta_2 p_1^+ p_{-2}^-\right| + \left|\zeta_1 p_2^+ p_{-1}^-\right|\right\}.$$

In addition, we define

$$\mathcal{D}_k := \hat{V} \sum_{m=1}^{\infty} m^k e^{-\frac{\varepsilon}{2}\left(\frac{m}{2}\right)^{\gamma}}.$$

Using the last equation, we get the following inequality

$$\mathcal{D}_k \leq \hat{V}\left(\frac{4}{\varepsilon}\right)^{\frac{k+1}{\gamma}} \frac{1}{\gamma}\int_0^{\infty} y^{\frac{k+1}{\gamma}-1} e^{-y} dy.$$

Then, using the gamma function, $(1+k)^{\frac{1}{\gamma}-1} < e^{\frac{k}{\gamma}}$ and $k^k < k!e^k$, we find

$$\mathcal{D}_k \leq \hat{D} d^k k! k^{k\left(\frac{1-\gamma}{\gamma}\right)}, \quad k \in \mathbb{N},$$

where $\hat{D}$ and d are positive constants depending on ε and γ. The proof is completed. $\boxed{QED}$

Lemma 6. *Assume that the 2π-periodic function φ is analytic in $\mathbb{C}_-$, all of its derivatives are continuous in $\overline{\mathbb{C}}_-$ and*

$$\sup_{z\in T}\left|\varphi^{(k)}(z)\right| \leq \eta_k, \quad k \in \mathbb{N}\cup\{0\}.$$

The set $A \subset \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right]$ with linear Lebesgue measure zero is the set of all zeros of the function φ with infinity multiplicity in T. If

$$\int_0^w lnt(s) d\mu(A_s) = -\infty,$$

where

$$t(s) = \inf_k \frac{\eta_k s^k}{k!}, \quad k \in \mathbb{N}\cup\{0\}$$

and $\mu(A_s)$ is the linear Lebesgue measure of the s-neighborhood of A and $w \in (0, 2\pi)$ is an arbitrary constant, then $\varphi \equiv 0$ [9].

Theorem 6. *Assume* (4.33). *Then* $D_5 = \emptyset$.

Proof. According to Lemma 6, we write

$$\int_0^w lnt(s)d\mu(D_5, s) > -\infty, \tag{4.40}$$

where $\mu(D_5, s)$ is the Lebesgue measure of the s-neighborhood of D_5, η_k is defined by Lemma 5 and $t(s) = \inf_k \dfrac{\eta_k s^k}{k!}$. We have by substituting η_k into the definition of

$$t(s) = B\hat{D}exp\left\{-\frac{1-\gamma}{\gamma}e^{-1}(ds)^{-\frac{\gamma}{1-\gamma}}\right\}. \tag{4.41}$$

So, we have by using (4.40) and (4.41)

$$\int_0^w s^{-\frac{\gamma}{1-\gamma}}d\mu(D_5, s) \leq -\int_0^w lnt(s)d\mu(D_5, s) < \infty.$$

The last inequality holds for arbitrary s if and only if $\mu(D_5, s) = \emptyset$, i.e., $D_5 = \emptyset$. It completes the proof. $\boxed{QED}$

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