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B_3 BLOCK REPRESENTATIONS OF DIMENSION 6 AND BRAID REVERSIONS

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Abstract. We construct a family of six dimensional block representations of the braid group B_3 on three strings. We show that these representations can be used to distinguish braids of knots with 9 and 10 crossings from their reversed braids.

Keywords: Braid groups, knots, invertible

MSC 2022 classification: primary 20F36

1 Introduction

Gauss was the first mathematician who studied knots mathematically in the 1800s. Reidemeister and Alexander (around 1930), were able to make significant progress in knot theory, which has been a very dynamic branch of topology especially after the discovery of the Jones polynomial in 1984 and its connections with quantum field theory, as well as some concrete applications in the study of enzymes acting on DNA strands [7].
The reverse of an oriented knot K is defined as the same knot with the opposite orientation. Vertibility seems to be very difficult to detect. The connection between knot theory and braid theory was discovered in 1923 by Alexander. He proved that every oriented knot or link is isotopic to a closed braid [1].
In [6], Lieven Le Bruyn introduced some simple representations of the braid group B_3 that are able to distinguish the braids of the following knots from their reversed braids, 6_3, 7_5, 8_7, 8_9, 8_{10} and 8_{17} which is the unique non-invertible

knot with minimal number of crossings. All these knots have at most 8 crossings and are closures of 3-string braids. The braid $b = \sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^2$ is the braid whose closure is the knot 8_{17}, and the braid $b' = \sigma_2^2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-2}$ is the reversed braid of b. It turns out that the trace of the braid b is different from the trace of the reversed braid b' for sufficiently large B_3-representations. Bruce Westbury discovered 12-dimensional representations of B_3 that are able to detect a braid from its reversed braid [10].
In section 2, we state some essential definitions and theorems. In section 3, we present some basic results about detecting vertibility and separating braids of some knots from their reversed braids using representations of B_3, the braid group on three strings. In [6], Lieven Le Bruyn was able to detect reversions using simple representations of B_3. In fact, the author in [6] discovered a 6-dimensional representation of B_3 which distinguishes a braid of the knot 8_{17} from its reversed braid. The knots inspected by Lieven were $6_3, 7_5, 8_7, 8_9, 8_{10}$, and 8_{17}. In section 4, we construct a family of block representations of B_3 of dimension 6. In section 5, we prove that some complex specializations of the block representations, constructed in section 4, are able to distinguish braids from their reversed braids of the above knots, in addition to some knots of 9 and 10 crossings like 9_6, 9_9 and 10_5. Note that, even if 3-braid α is not conjugate to the reversed braid α', this does not mean that the closure of α is non-invertible. In fact, 8_{17} is the only non-invertible knot with minimal number of crossings, and braid corresponding to this knot is indeed separated by its reversed braid. In order to apply Theorem 2, section 2, to show that the closure of 3-braid α is non-invertible, one needs to show (i) that α is not conjugate to α' and (ii) that α is not conjugate to certain type of braids (flypes). In this paper, section 5, we work on the separation of braids from their reversed braids and we construct a table of some knots with their braids listed there.

2 Definitions and theorems

Definition 1. [2] The braid group on n strings, B_n, is the abstract group with the presentation

$$B_n = \langle \sigma_1, \cdots, \sigma_{n-1} | \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1 \text{ and } \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \text{for } i = 1, \cdots, n-2\rangle.$$

Definition 2. [4] A knot K is the image of a homeomorphism of a unit circle S^1 into $\mathbb{R}^3$ considered up to continuous deformations (ambient isotopies) in the following sense.

Two knots K_1 and K_2 are equivalent (isotopic) if there exists a continuous mapping $H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that

(1) For every $t \in [0,1]$ the mapping $x \mapsto H(x,t)$ is a homeomorphism of $\mathbb{R}^3$ onto $\mathbb{R}^3$.

(2) $H(x,0) = x$ for all $x \in \mathbb{R}^3$.

(3) $H(K_1,1) = K_2$.

Such mapping H is called ambient isotopy.

Definition 3. A link is a finite union of pairwise disjoint knots, which are called the components of the link.

A closed braid is a braid in which the corresponding ends of its strings are connected in pairs. This means that every braid can be closed up to be a knot or a link. Now, we have the following theorem.

Theorem 1 (Alexander's Theorem [1])**.** *Every knot or link can be represented as a closed braid.*

Every knot or Link may be closure of many braids even with different number of strings. However, Markov's theorem gives necessary and sufficient conditions for the closures of two braids to give equivalent knots or links [1]. One of the sufficient conditions is conjugation. That is, if two braids are conjugate then their closures are equivalent links. For example, the braids $\sigma_1^{-1}\sigma_2\sigma_1^{-3}\sigma_2^3$ and $\sigma_2^3\sigma_1^{-1}\sigma_2\sigma_1^{-3}$ are associated with the same knot 8_9.

Definition 4. [5] The minimal number of strings needed in braid to represent a knot or link K is called the braid index of K.

Definition 5. The reverse of a braid of the form $\sigma_1^{n_1}\sigma_2^{m_1}\sigma_1^{n_2}\sigma_2^{m_2}\cdots\sigma_1^{n_k}\sigma_2^{m_k}$ is the braid $\sigma_2^{m_k}\sigma_1^{n_k}\cdots\sigma_2^{m_2}\sigma_1^{n_2}\sigma_2^{m_1}\sigma_1^{n_1}$, where $n_1, m_1, \cdots, n_k, m_k$ are integers.

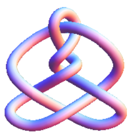

Figure 1. Knot 7_5

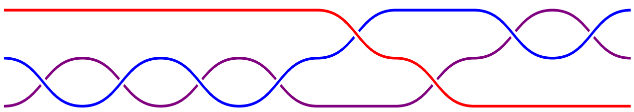

Figure 2. Representative braid of 7_5: $\sigma_1^4\sigma_2\sigma_1^{-1}\sigma_2^2$

Definition 6. [4] A knot is said to be invertible if it can be deformed continuously to itself, but with the orientation reversed.

Definition 7. [3] A knot of braid index 3 is said to admit a flype if its associated braids are conjugate to a braid of the form

$$\sigma_1^a \sigma_2^b \sigma_1^c \sigma_2^\epsilon$$

for some integers a, b, c, $\epsilon = \pm 1$.

A flype is a knot move consisting of turning a part of a knot, a tangle, by $180°$.

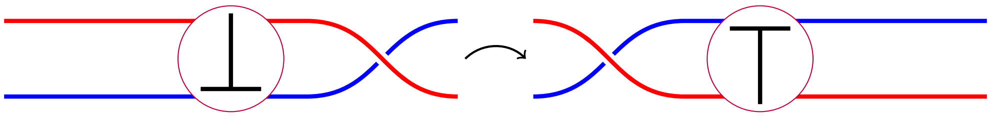

Figure 3. Flype

Definition 8. A flype is said to be non-degenerate when its associated braid $\sigma_1^a \sigma_2^b \sigma_1^c \sigma_2^\epsilon$ and its reverse $\sigma_2^\epsilon \sigma_1^c \sigma_2^b \sigma_1^a$ are in distinct conjugacy classes.

The braid $b = \sigma_1^{-1} \sigma_2^2 \sigma_1^{-1} \sigma_2$ is a non-degenerate flype of minimal length.

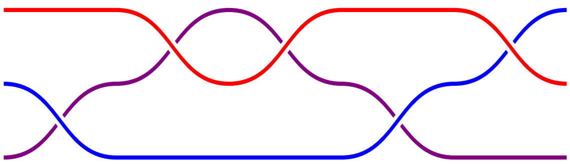

Figure 4. The braid $b = \sigma_1^{-1} \sigma_2^2 \sigma_1^{-1} \sigma_2$

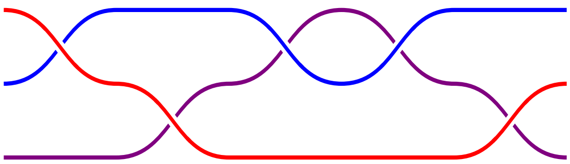

Figure 5. The reversed braid $b' = \sigma_2 \sigma_1^{-1} \sigma_2^2 \sigma_1^{-1}$

Theorem 2. *[3] Let $\mathcal{K}$ be a link of braid index 3 with oriented 3-braid representative K. Then $\mathcal{K}$ is non-invertible if and only if K and its reverse braid K' are in distinct conjugacy classes, and the class of K does not contain a representative which admits a non-degenerate flype.*

3 Basic results

There is an infinite family of non-invertible knots, [8]. The knot 8_{17}, which is the closure of the braid $\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^2\sigma_1^{-2}\sigma_2$, is the unique non-invertible knot with a minimal number of crossings. The following table gives the numbers of non-invertible and invertible knots according to their number of crossings up to 16 [7].

Number of crossings	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Non-invertible	0	0	0	0	0	1	2	33	187	1144	6919	38118	226581	1309875
Invertible	1	1	2	3	7	20	47	132	365	1032	3069	8854	267121	78830

Notice that some non-invertible knots, referred to in the table above, are: 8_{17}, 9_{32}, 9_{33}, 10_{67}, 10_{80}, 10_{81}, 10_{83}, for more details see [7].
Based on Theorem 2, one of the conditions, needed to check knot-vertibilty of a knot of braid index 3, is that to distinguish a braid associated to it from its reversed braid.
Imre Tuba and Hans Wenzl introduced a complete classification of all simple B_3-representations of dimension ≤ 5 [9]. It is mentioned there that none of these representations can detect a braid from its reversed braid; the minimal dimension must be six. Bruce Westbury found a representation of dimension 12 that is able to detect a braid from its reversed braid by taking traces. The question, which was raised after that, was about determining the minimal dimension of a B_3-representation which is able to detect a braid from its reversed braid.
Lieven Le Bruyn proposed a general method to solve the separation problems for three string braids [6]. In fact, he succeeded to solve Westbury's separation problem using simple representations of B_3 of dimension 6. A specific representation of B_3 is given by the matrices

$$\sigma_1 = \begin{pmatrix} p+1 & p-1 & p-1 & p-1 & -p+1 & -p+1 \\ -2p-1 & -1 & -2p-1 & 2p+1 & -2p-1 & 2p+1 \\ p+2 & p+2 & -p & -p-2 & -p-2 & p+2 \\ -p-2 & -3p & p+2 & -p+2 & 3p & -p-2 \\ p-1 & -p+1 & 3p+3 & -p+1 & 3p+1 & -3p-3 \\ -3 & -2p-1 & 2p+1 & 3 & 2p+1 & -2p-3 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} p+1 & p-1 & p-1 & -p+1 & p-1 & p-1 \\ -2p-1 & -1 & -2p-1 & -2p-1 & 2p+1 & -2p-1 \\ p+2 & p+2 & -p & p+2 & p+2 & -p-2 \\ p+2 & 3p & -p-2 & -p+2 & 3p & -p-2 \\ -p+1 & p-1 & -3p-3 & -p+1 & 3p+1 & -3p-3 \\ 3 & 2p+1 & -2p-1 & 3 & 2p+1 & -2p-3 \end{pmatrix},$$

where p is a primitive third root of unity. This representation distinguishes a braid of the knot 8_{17} from its reversed braid because they have different traces. More precisely, $Tr(b) = -7128p - 1092$, whereas $Tr(b') = 7128p + 6036$ (see [6]). Lieven Le Bruyn constructed Zariski dense family of simple B_3-representations, which distinguish braids from their reversed braids on the list of knots, having at most 8 crossings, and which are closures of 3-string braids. The knots inspected by Lieven were $6_3, 7_5, 8_7, 8_9, 8_{10}$ (which are 'flypes') and 8_{17}.

4 Constructing block representations of B_3 of dimension 6

In this section, we construct representations of the braid group B_3 of dimension six. Let A, B, C and D be 3×3 non-zero matrices. Let ρ be a mapping from B_3 to $M_6(\mathbb{C})$, the vector space of 6×6 matrices over the complex vector space $\mathbb{C}$. This mapping is given by

$$\rho(\sigma_1) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad \rho(\sigma_2) = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix},$$

where σ_1 and σ_2 are the generators of B_3.

Proposition 1. *The mapping $\rho : B_3 \to GL(6, \mathbb{C})$ defines a representation of the braid group B_3 if and only if $\det(\rho(\sigma_i)) \neq 0$ $(i = 1, 2)$ and the matrices A, B, C and D satisfy the following relations.*

$$A^2B - BCB - ABD + BD^2 = 0 \tag{4.1}$$

$$CA^2 - DCA - CBC + D^2C = 0 \tag{4.2}$$

Proof. Recall that the generators σ_1 and σ_2 of the braid group B_3 satisfy the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. Then $\rho(\sigma_1\sigma_2\sigma_1) = \rho(\sigma_2\sigma_1\sigma_2)$. This implies that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A & -B \\ -C & D \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A & -B \\ -C & D \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} A^3 - BCA - ABC + BDC & A^2B - BCB - ABD + BD^2 \\ \\ CA^2 - DCA - CBC + D^2C & CAB - DCB - CBD + D^3 \end{pmatrix}$$

$$= \begin{pmatrix} A^3 - BCA - ABC + BDC & -A^2B + BCB + ABD - BD^2 \\ \\ -CA^2 + DCA + CBC - D^2C & CAB - DCB - CBD + D^3 \end{pmatrix}.$$

So, $A^2B - BCB - ABD + BD^2 = 0$ and $CA^2 - DCA - CBC + D^2C = 0$ $\boxed{QED}$

5 Knots and detecting inversion among their braids

In this section, we distinguish braids from their reversed braids of some knots using the representations constructed in the previous section. We take the matrices A, B, C, D, mentioned in Section 4, as follows:

$$A = \begin{pmatrix} a & a-2 & a-2 \\ -2a+1 & d & -2a+1 \\ f & f & g \end{pmatrix}, \quad B = \begin{pmatrix} a-2 & -a+2 & -a+2 \\ 2a-1 & -2a+1 & 2a-1 \\ -f & -f & f \end{pmatrix}$$

and

$$D = \begin{pmatrix} -a+3 & 3a-3 & -a-1 \\ -a+2 & 3a-2 & -3a \\ 3 & 2a-1 & -2a-1 \end{pmatrix},$$

where a, d, f and g are complex numbers.

The matrices $\rho(\sigma_1)$ and $\rho(\sigma_2)$ are

$$\rho(\sigma_1) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and } \rho(\sigma_2) = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix},$$

where the matrices A, B, C, D satisfy Equations 4.1 and 4.2. The complex numbers a, d, f, g are chosen to have the matrices of σ_1 and σ_2 invertible. In order to solve for the matrix C in Equation 4.1, we require that B is an invertible matrix

In the next proposition, we give values to d, g, f, all in terms of a; which guarantees the invertibility of the matrix B. This will be done in a way that the map ρ is a representation of the braid group B_3 in $GL(6, \mathbb{C})$.

Proposition 2. *ρ is a representation of B_3 if $a \notin \{-1, 0, 2, \frac{1}{2}\}$ and either one of the following conditions holds true.*

1) $d = 1 + 2a,\ g = -1 \pm ia\sqrt{3},\ f \neq 0$

2) $d = 1 + 2a,\ g = 3a - 1,\ f = 1 + a$

3) $(d = 1 - a + ia\sqrt{3},\ g = -1 - ia\sqrt{3},\ f \neq 0)$ *or*
$(d = 1 - a - ia\sqrt{3},\ g = -1 + ia\sqrt{3},\ f \neq 0)$

4) $(d = 1 - a + ia\sqrt{3},\ g = -1 + ia\sqrt{3},\ f = 1 + a)$ *or*
$(d = 1 - a - ia\sqrt{3},\ g = -1 - ia\sqrt{3},\ f = 1 + a)$

5) $d = 1 - a \pm ia\sqrt{3},\ g = 3a - 1,\ f = 1 + a.$

Proof. The determinant of B is $4(a - 2)(2a - 1)f \neq 0$. So B is invertible. Substituting $C = B^{-1}A^2 - B^{-1}ABDB + D^2B^{-1}$ in Equation 4.2, we get 9 equations with 4 unknown complex numbers a, d, f, g. We fix a and we solve for the numbers d, f, g. Using Mathematica software, we solve the system of 9 equations to get the solutions stated above.
The invertibility of the matrices $\rho(\sigma_1)$ and $\rho(\sigma_2)$ comes from the fact that the determinant of $\rho(\sigma_i)$ is $-64a^6$ once we assign values to d, g and f as in the second condition of the Proposition 2. Also, the determinant of $\rho(\sigma_i)$ $(i = 1, 2)$ is $32(1 \pm i\sqrt{3})a^6$ under the conditions 1, 3, 4, 5 of Proposition 2. Since $a \neq 0$, it follows that $\det(\rho(\sigma_i)) \neq 0$ $(i = 1, 2)$. Therefore $\rho(\sigma_1)$ and $\rho(\sigma_2)$ are invertible. $\boxed{QED}$

Remark 1. Let p be a primitive third root of unity. By taking $a = p + 1$ and $f = p + 2$ in condition 3 of Proposition 2, we get the representation in section 3, which Lieven Le Bruyn used to distinguish the braid describing the knot 8_{17} from its reveresd braid.

Theorem 3. *The representations constructed in Proposition 2 are able to distinguish braids from their reversed braid on the list of several knots which are the closures of three string braids, and which some of them have more than 8 crossings.*

Proof. Consider the representation ρ of B_3 given in Proposition 2 with the condition $d = 1 - a - ia\sqrt{3}, g = -1 + ia\sqrt{3}, f \neq 0$. The representation ρ becomes

$$\rho(\sigma_1) = \begin{pmatrix} a & a-2 & a-2 & a-2 & -a+2 & -a+2 \\ -2a+1 & 1-a-ia\sqrt{3} & -2a+1 & 2a-1 & -2a+1 & 2a-1 \\ f & f & -1+ia\sqrt{3} & -f & -f & f \\ -1-a & 1+(-2+i\sqrt{3})a & 1+a & 3-a & -3+3a & -1-a \\ \frac{2-2a-a^2}{-2+a} & ia\sqrt{3} & \frac{[(3+i\sqrt{3})(1+a)+(3-i\sqrt{3})f]a}{2f} & -a+2 & 3a-2 & -3a \\ \frac{4-a-2a^2}{-2+a} & -1+a(-1+i\sqrt{3}) & \frac{2+[1-i\sqrt{3}+(3-i\sqrt{3})f]a-(1+i\sqrt{3})a^2}{2f} & 3 & 2a-1 & -2a-1 \end{pmatrix}$$

and

$$\rho(\sigma_2) = \begin{pmatrix} a & a-2 & a-2 & -a+2 & a-2 & a-2 \\ -2a+1 & 1-a-ia\sqrt{3} & -2a+1 & -2a+1 & 2a-1 & -2a+1 \\ f & f & -1+ia\sqrt{3} & f & f & -f \\ 1+a & -1-(-2+i\sqrt{3})a & -1-a & 3-a & -3+3a & -1-a \\ \frac{-2+2a+a^2}{-2+a} & -ia\sqrt{3} & -\frac{[(3+i\sqrt{3})(1+a)+(3-i\sqrt{3})f]a}{2f} & -a+2 & 3a-2 & -3a \\ \frac{-4+a+2a^2}{-2+a} & 1-a(-1+i\sqrt{3}) & -\frac{2+[1-i\sqrt{3}+(3-i\sqrt{3})f]a-(1+i\sqrt{3})a^2}{2f} & 3 & 2a-1 & -2a-1 \end{pmatrix}$$

Next, we take three different values of a and f. Thus we obtain three different representations of B_3. We show that these representations are able to distinguish the braids of the following knots: $6_3, 7_5, 8_7, 8_9, 8_{10}, 8_{17}, 9_6, 9_9, 10_5$ from their reversed braids. The author in [6] succeeded to distinguish some knots up to 8 crossings. Our representations were able to recognize knots with 9 and 10 crossings that are distinguished from their reversed braids. More precisely, we perform the following calculations as shown in the table below.

Knot	Braid word w	$Tr(\rho(w)) - Tr(\rho(w'))$		
		$a = 2 - 3i, f = 7.3$	$a = 1.5 + i,\ f = 6 - 4.2i$	$a = 1 + 3i,\ f = 10.2 + 10.3i$
6_3	$\sigma_1^{-1}\sigma_2^2\sigma_1^{-2}\sigma_2$	$170.17 + 47.15i$	$201.38 - 11.75i$	$427.9 + 123.1i$
7_5	$\sigma_1^4\sigma_2\sigma_1^{-1}\sigma_2^2$	$1.96 \times 10^7 + 1.52 \times 10^7 i$	$-419.54 - 142.8i$	$2.3 \times 10^6 + 2.8 \times 10^7 i$
8_7	$\sigma_1^4\sigma_2^{-2}\sigma_1\sigma_2^{-1}$	$3624.8 + 23139i$	$-9244 - 3706.1i$	$73,847.2 - 58,855.3i$
8_9	$\sigma_1^{-1}\sigma_2\sigma_1^{-3}\sigma_2^3$	$170.17 + 47.15i$	$201.38 - 11.75i$	$427.9 + 123.1i$
8_{10}	$\sigma_1^{-1}\sigma_2^2\sigma_1^{-2}\sigma_2^3$	$3624.8 + 23139i$	$-9244 - 3706.1i$	$73,847.2 - 58,855.3i$
8_{17}	$\sigma_1^{-1}\sigma_2^2\sigma_1^{-1}\sigma_2^2\sigma_1^{-2}\sigma_2$	$-524.9 - 581.1i$	$-459 + 182.3i$	$-510.4 - 653.8i$
9_6	$\sigma_1^2\sigma_2^2\sigma_1^5\sigma_2^{-1}$	$7.5 \times 10^8 - 3.2 \times 10^9 i$	$-1.6 \times 10^7 - 1.5 \times 10^7 i$	$-5.2 \times 10^9 - 3.1 \times 10^9 i$
9_9	$\sigma_1^3\sigma_2^{-1}\sigma_1^4\sigma_2^2$	$7.5 \times 10^8 - 3.2 \times 10^9 i$	$-1.6 \times 10^7 - 1.5 \times 10^7 i$	$-5.2 \times 10^9 - 3.1 \times 10^9 i$
10_5	$\sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2^6$	$-3.5 \times 10^6 + 2.6 \times 10^6 i$	$308,285 + 628,064i$	$-1.7 \times 10^6 - 2.9 \times 10^7 i$

Here, w' denotes the reverse of the braid w. $\boxed{QED}$

Conflict of Interest. On behalf of all authors, the corresponding author states that there is no conflict of interest.

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