Note di Matematica
Note Mat. **42** (2022) no. 2, 19–42.
ISSN 1123-2536, e-ISSN 1590-0932
doi:10.1285/i15900932v42n2p19

On the existence of solutions of a critical elliptic equation involving Hardy potential on compact Riemannian manifolds.

Fatima Zohra Terki
Higher school in Business and Management, Tlemcen, Algeria.
Laboratoire Systèmes Dynamiques et Applications.
fatimazohra113@yahoo.fr

Youssef Maliki
Department of Mathematics, University Aboubekr Belkaïd of Tlemcen, Algeria.
Laboratoire Systèmes Dynamiques et Applications.
malyouc@yahoo.fr

Received: 18.3.2022; accepted: 28.10.2022.

Abstract. In this paper, we are interested in the study of the problem of existence of solutions of an elliptic equation with Hardy potential and critical Sobolev exponent on a compact Riemannian manifold.

Keywords: Riemannian manifolds, Yamabe equation, Hardy potential.

MSC 2022 classification: primary 58J05, secondary 35A01.

1 Introduction

Let (M, g) be a compact $n-$dimensional Riemannian manifold with $n \geq 3$. Denote by $\delta_g > 0$ the injectivity radius of (M, g). For a fixed point $p \in M$, we define (after [12]) on (M, g) a distance function ρ_p as follows

$$\rho_p(x) = \begin{cases} dist_g(p, x), & x \in B(p, \delta_g), \\ \delta_g, & x \in M \setminus B(p, \delta_g). \end{cases} \tag{1.1}$$

Consider on (M, g) the Sobolev space $H_1^2(M)$ consisting of the completion of $\mathcal{C}^\infty(M)$ with respect to the norm

$$||u||^2_{H_1^2(M)} = \int_M (|\nabla u|^2 + u^2) dv_g. \tag{1.2}$$

Let h and f be smooth functions on M. We are interested in studying existence of positive solutions $u \in H_1^2(M)$ of the following equation:

$$\Delta_g u - \frac{h(x)}{\rho_p^2(x)} u = f(x)|u|^{2^*-2}u, x \in M \setminus \{p\}, \qquad (E_{h,f})$$

where $\Delta_g u = -\mathrm{div}(\nabla_g u)$ is the Laplacian of u and $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent.

Equation $(E_{h,f})$ is reminiscent of the famous prescribed scalar curvature equation that involves the term $\frac{n-2}{4(n-1)} Scal_g$, where $Scal_g$ is the scalar curvature of g, instead of the Hardy potential $\frac{h}{\rho_p^2}$. It arises in the study of the conformal deformation of the metric g to a prescribed scalar curvature and it has been extensively studied; we refer the reader to the books [3] and [10] for a compendium on this equation and the related topics.

When the function ρ_p is of power $0 < \gamma < 2$ and $f(x) = 1$, equation $(E_{h,f})$ appears as a case of equations that arise in the study of conformal deformation to constant scalar curvature of metrics which are smooth only in some geodesic ball $B(p, \delta) \subset M$; it is a kind of a singular Yamabe problem that has been formulated and studied in [12, 13].

In this paper, we are interested in the study of existence of weak solutions of equation $(E_{h,f})$ on the compact Riemannian manifold (M, g) approaching the variational method. More precisely, we prove the existence of a function $u \in H_1^2(M) \setminus \{0\}$ such that

$$\int_M (\nabla u.\nabla v - \frac{h}{\rho_p^2} uv) dv_g - \int_M f|u|^{2^*-2} uv dv_g = 0, \forall v \in H_1^2(M).$$

Notice at first that equation $(E_{h,f})$ contains the critical exponents 2^* in the nonlinear term and also the critical Hardy potential $\frac{1}{\rho_p^2}$ which render the study of existence of solutions via the variational method difficult due to the lack of compactness in the inclusion of the Sobolev space $H_1^2(M)$ in the Lebesgue spaces $L_{2^*}(M)$ and $L_2(M, \rho_p^2)$ (the space of functions $u : M \to \mathbb{R}$ such that $\int_M \frac{u^2}{\rho_p^2} dv_g$ is finite). For this reason, an analysis of blow-up phenomena is needed in order to determine compactness levels of Palais-Smale sequences. Here, we should note that the singular term plays an important role in the blow-up analysis. In fact, it can be shown as in [14], that it interferes in the decomposition of the energy of Palais-Smale sequences and then in the determination of critical compactness level.

The nearest equation to $(E_{h,f})$ is possibly the one studied in [15] on the Euclidean space $\mathbb{R}^n$. In [15], the author considered on the Euclidean space $\mathbb{R}^n$

the equation

$$\Delta_{\mathbb{R}^n} u - \frac{\lambda}{|x|^2} u = K|u|^{2^*-2}u, x \in \mathbb{R}^n \setminus \{0\},$$

where $0 < \lambda < \frac{(n-2)^2}{4}$ is a constant and K is some function defined on $\mathbb{R}^n$. The author, obtained some existence results to this equation after having established decomposition formulas for Palais-Smale sequences. At some extent, the decomposition result obtained in [15] is relevant to our case where a similar decomposition formulas can be obtained as it has been shown in [14].

Another important result is the classification of positive solutions, obtained in [18], of the equation

$$\Delta_{\mathbb{R}^n} u - \frac{\lambda}{|x|^2} u = |u|^{2^*-2}u, x \in \mathbb{R}^n \setminus \{0\}, \tag{1.3}$$

where $0 < \lambda < \frac{(n-2)^2}{4}$ is a constant. The author proved that for each $0 < \lambda < \frac{(n-2)^2}{4}$, equation (1.3) has one parameter family of positive radially symmetric solutions

$$U_{\lambda,w}(x) = w^{\frac{2-n}{2}} U_\lambda\left(\frac{x}{w}\right), w > 0, x \in \mathbb{R}^n, \tag{1.4}$$

where

$$U_\lambda(x) = (n(n-2))^{\frac{n-2}{4}} \left(\frac{a_\lambda\, |x|^{a_\lambda - 1}}{1+|x|^{2a_\lambda}}\right)^{\frac{n}{2}-1}, x \in \mathbb{R}^n,$$

with

$$a_\lambda = \sqrt{1 - \frac{4\lambda}{(n-2)^2}}. \tag{1.5}$$

Moreover, the family functions $U_\lambda(x)$ satisfies

$$\begin{aligned} \inf_{u \in D^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} \left(|\nabla u|^2 - \lambda \frac{u^2}{|x|^2}\right) dx}{\left(\int_{\mathbb{R}^n} |u|^{2^*} dx\right)^{\frac{2}{2^*}}} &= \frac{\int_{\mathbb{R}^n} \left(|\nabla U_{\lambda,w}|^2 - \lambda \frac{U_{\lambda,w}^2}{|x|^2}\right) dx}{\left(\int_{\mathbb{R}^n} |U_{\lambda,w}|^{2^*} dx\right)^{\frac{2}{2^*}}} \\ &= \frac{\left(1 - \lambda(\frac{2}{n-2})^2\right)^{\frac{n-1}{n}}}{K^2(n,2)}. \end{aligned} \tag{1.6}$$

with $K(n,2)$ is the best constant appearing in the Euclidean Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u^{2^*}| dx\right)^{\frac{2}{2^*}} \leq K^2(n,2) \int_{\mathbb{R}^n} |\nabla u|^2 dx. \tag{1.7}$$

Its value is calculated in [3, 17, 9] and is given by

$$K(n,2)=\sqrt{\frac{4}{n(n-2)(w_n)^{\frac{n}{2}}}}, \qquad (1.8)$$

where w_n is the volume of the unit sphere $S^n \subset \mathbb{R}^{n+1}$.

Clearly, when $\lambda = 0$, we meet the classification result, obtained in [4] (see also [7, 8, 17]), of positive solutions of the unperturbed equation

$$\Delta_{\mathbb{R}^n} u = |u|^{\frac{4}{n-2}}u, x \in \mathbb{R}^n, \qquad (1.9)$$

by the family of functions

$$U_w(x) = w^{\frac{2-n}{2}} U\left(\frac{x}{w}\right), x \in \mathbb{R}^n, \qquad (1.10)$$

where

$$U(x) = (n(n-2))^{\frac{n-2}{4}} \left(\frac{1}{1+|x|^2}\right)^{\frac{n}{2}-1}, x \in \mathbb{R}^n.$$

In the case of the prescribed scalar curvature equation, the family functions U_w have been utilised as test functions to ensure that the minimum of the corresponding variational setting is under the critical energy level of compactness (see [3, 9]). In the same spirit, we will use the family of functions $U_{\lambda,w}$ for the same goal. As such, we prove the existence result formulated in theorem 2 below.

2 Notation, preliminaries and statement of the main result.

Let (M, g) be a compact Riemannian manifold. By the Rellich-Kondrakov theorem (see [9]), the space $H_1^2(M)$ is compactly embedded in $L_q(M)$ for $q < 2^* = \frac{2n}{n-2}$, and continuously embedded in $L_{2^*}(M)$. Moreover, on the space $H_1^2(M)$, the following optimal Sobolev inequality holds (see [10, Theorem 4.6]): for any function $u \in H_1^2(M)$, there exists a positive constant B such that

$$||u||^2_{L_{2^*}(M)} \le K^2(n,2)||\nabla u||^2_{L_2(M)} + B||u||^2_{L_2(M)}, \qquad (2.11)$$

with $K(n, 2)$ is given by (1.8).

Denote by $L_2(M, \rho_p^2)$ the space of functions on M such that $\int_M \frac{u^2}{\rho_p^2} dv_g < \infty$. This space is a Banach space endowed with the norm

$$||u||^2_{L_2(M,\rho_p^2)} = \int_M \frac{u^2}{\rho_p^2} dv_g.$$

In [12, Theorem 1.2], it is shown that the Sobolev space $H_1^2(M)$ is continuously embedded in $L_2(M, \rho_p^2)$ and the following Hardy inequality on $H_1^2(M)$ holds: for every $\varepsilon > 0$ there exists a positive constant $A(\varepsilon)$ such that for any $u \in H_1^2(M)$,

$$\int_M \frac{u^2}{\rho_p^2} dv_g \le (K^2(n,2,-2) + \varepsilon) \int_M |\nabla u|^2 dv_g + A(\varepsilon) \int_M u^2 dv_g, \tag{2.12}$$

where $K(n,2,-2) = \frac{2}{n-2}$ is the best constant in the Euclidean Hardy inequality

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \le (\frac{2}{n-2})^2 \int_{\mathbb{R}^n} |\nabla u|^2 dx.$$

We will use the notation $K(n,2,-2)$ to denote the number $\frac{2}{n-2}$, where $n \ge 3$ is the dimension of the manifold M.

Besides, always in [12, Lemma 1.1], it is proved that if u is supported in $B(p,\delta)$, $0 < \delta < \delta_g$, a geodesic ball of center p and radius δ, then

$$\int_{B(p,\delta)} \frac{u^2}{\rho_p^2} dv_g \le K_\delta^2(n,2,-2) \int_{B(p,\delta)} |\nabla u|^2 dv_g,$$

with $K_\delta(n,2,-2)$ goes to $K(n,2,-2)$ when δ goes to 0.

Let $\delta > 0$ be a constant and $q \in M$ be a point. A geodesic ball of center q and radius δ is denoted by $B(q,\delta)$ and a Euclidean ball of center 0 and radius δ is denoted simply by $B(\delta)$. The exponential map at a point $q \in M$, which is denoted by $\exp_q^{-1}$, is a local diffeomorphism of class C^∞: $\exp_q^{-1} : B(q,\delta) \subset M \longrightarrow B(\delta) \subset \mathbb{R}^n$, which defines a geodesic normal system on the manifold (M,g).

For $\delta > 0$, we denote by η_δ a smooth cut-off function in $\mathbb{R}^n$, which satisfies $0 \le \eta_\delta \le 1$, $\eta_\delta \equiv 1$ in $B(\delta)$, $\eta_\delta \equiv 0$ in $\mathbb{R}^n \setminus B(2\delta)$.

For $q \in M$ and $0 < \delta < \delta_g$, where δ_g stands for the injectivity radius, we denote by $\eta_{q,\delta}$ the smooth cut-off function in M defined by $\eta_{q,\delta}(x) = \eta_\delta(\exp_q^{-1}(x))$, $x \in M$.

Now, let $J_{h,f}$ be the functional defined on $H_1^2(M)$ by

$$J_{h,f}(u) = \frac{1}{2} \int_M (|\nabla u|^2 - \frac{h}{\rho_p^2} u^2) dv_g - \frac{1}{2^*} \int_M f|u|^{2^*} dv_g. \tag{2.13}$$

The functional $J_{h,f}$ is a C^2 functional on $H_1^2(M)$. Its (Fréchet) derivative at $u \in H_1^2(M)$ is given by

$$DJ_{h,f}(u).v = \int_M (\nabla u.\nabla v - \frac{h}{\rho_p^2} uv) dv_g - \int_M f|u|^{2^*-2} uv dv_g, v \in H_1^2(M).$$

A Palais-Smale sequence of $J_{h,f}$ at a level $\beta \in \mathbb{R}$ is defined as a sequence $u_m \in H_1^2(M)$ such that $J_{h,f}(u_m) \to \beta$ and $DJ_{f,h}(u_m).v \to 0$, $\forall v \in H_1^2(M)$.

A function $u \in H_1^2(M)$ is called a critical point of $J_{h,f}$ if it satisfies $DJ_{h,f}(u)\varphi = 0$, $\forall \varphi \in H_1^2(M)$ and a constant $\beta \in \mathbb{R}$ is called a critical level of $J_{h,f}$ if there exists a critical point u of $J_{h,f}$ such that $\beta = J_{h,f}(u)$. Obviously, a critical point $u \in H_1^2(M)$ of $J_{h,f}$ coincides with a weak solution of equation $(E_{h,f})$.
In what follows, we say that the functional

$$\mathcal{I}(u) = \int_M (|\nabla u|^2 - \frac{h}{\rho_p^2} u^2) dv_g, u \in H_1^2(M). \tag{2.14}$$

is coercive if there exists a constant $\lambda > 0$ such that for all $u \in H_1^2(M)$,

$$\mathcal{I}(u) \geq \lambda \|u\|^2_{H_1^2(M)}.$$

Suppose that the function f is positive on M, the function h satisfies

$$0 < h(p) < \frac{1}{K^2(n, 2, -2)},$$

and that the operator $\mathcal{I}$ is coercive. Set

$$\mu = \inf_{u \in H_1^2(M) \setminus \{0\}} \frac{\int_M \left(|\nabla u|^2 - h \frac{u^2}{\rho_p^2} \right) dv_g}{\left(\int_M f |u|^{2^*} dv_g \right)^{\frac{2}{2^*}}}. \tag{2.15}$$

Notice that by coercivity of $\mathcal{I}$, positivity of the function f and Sobolev inequality (2.11), we have that $\mu > 0$.

Now, let us take $f \equiv 1$. In [12, Theorem 4.1], it has been proven that equation $(E_{h,f})$(for $f \equiv 1$) admits a weak solution if

$$\mu < \frac{1 - h(p) K^2(n, 2, -2)}{K^2(n, 2)}.$$

It seems that the level

$$\frac{1 - h(p) K^2(n, 2, -2)}{K^2(n, 2)},$$

is not the critical one since the existence of solutions of $(E_{h,f})$, for $f \equiv 1$, can be extended to the greater level

$$\frac{\left(1 - h(p) K^2(n, 2, -2)\right)^{\frac{n-1}{n}}}{K^2(n, 2)}.$$

This can be easily deduced from Proposition (1) below after, of course, replacing the function f by the constant function $f \equiv 1$. In fact, Proposition (1) extends

Theorem 4.1 in [12] to non constant functions f and, more importantly, to a wider range of compactness levels

$$\frac{\left(1-h(p)K^2(n,2,-2)\right)^{\frac{n-1}{n}}}{\left(\sup_{x\in M} f(x)\right)^{\frac{n-2}{n}} K^2(n,2)}.$$

Now, for $\alpha \in [0,\infty[$, let h_α be a family of continuous functions on M and consider

$$\mu_\alpha = \inf_{u\in H_1^2(M)\setminus\{0\}} \frac{\int_M \left(|\nabla u|^2 - h_\alpha \frac{u^2}{\rho_p^2}\right) dv_g}{\left(\int_M f|u|^{2^*} dv_g\right)^{\frac{2}{2^*}}}.$$

Suppose that the following conditions, referred to as by $\mathcal{H}$, are satisfied

$$(\mathcal{H})\begin{cases} \text{a- } 0 < h_\alpha(p) < \frac{1}{K^2(n,2,-2)}, \forall\alpha\in[0,\infty[. \\ \text{b- } \mu_\alpha < \frac{1-h_\alpha(p)K^2(n,2,-2)}{\left(\sup_{x\in M} f(x)\right)^{\frac{n-2}{n}} K^2(n,2)}, \forall\alpha\in[0,\infty[. \\ \text{c- } |h_\alpha(x)| \le C, \text{ for a constant } C>0, \forall x\in M \text{ and } \forall\alpha\in[0,\infty[. \\ \text{d- There exists a function } h_\infty \text{ such that } \sup_M |h_\alpha - h_\infty| \to 0, \\ \text{ as } \alpha\to\infty. \end{cases}$$

In [14], the authors considered on a compact Riemannian manifold (M,g), the following equations

$$\Delta_g u - \frac{h_\alpha(x)}{\rho_p^2} u = f(x)|u|^{2^*-2}u, \alpha\in(0,\infty). \qquad (E_\alpha)$$

Let conditions $\mathcal{H}$ be satisfied and let u_α be a sequence of weak solutions of (E_α) with $\int_M f|u_\alpha|^{2^*} dv_g \le C, C>0$. In [14], the authors proved that, under conditions $\mathcal{H}$, there exist k sequences $R_m^i > 0, R_m^i \underset{m\to\infty}{\to} 0$, ℓ sequences $\mathcal{T}_m^j > 0, \mathcal{T}_m^j \underset{m\to\infty}{\to} 0$, converging sequences of points $x_m^j \to x_o^j \neq p$ in M, a solution $u_o \in H_1^2(M)$ of the limiting equation

$$\Delta_g u - \frac{h_\infty(x)}{\rho_p^2} u = f(x)|u|^{2^*-2}u, x\in M\setminus\{p\},$$

and functions $v_i, \nu_j \in D^{1,2}(\mathbb{R}^n)$ respectively nontrivial solutions of the equations

$$\Delta_{\mathbb{R}^n} u - \frac{h_\infty(p)}{|x|^2} u = f(p)|u|^{2^*-2}u, x\in\mathbb{R}^n\setminus\{0\}, \qquad (2.16)$$

and

$$\Delta_{\mathbb{R}^n} u = |u|^{2^*-2} u, \tag{2.17}$$

such that, up to a subsequence, u_α satisfies

$$\begin{aligned} u_\alpha &= u_o + \sum_{i=1}^{k} (R_\alpha^i)^{\frac{2-n}{n}} \eta_{p,\delta}(x) v_i((R_\alpha^i)^{-1} \exp_p^{-1}(x)) \\ &+ \sum_{j=1}^{l} (\mathcal{T}_\alpha^j)^{\frac{2-n}{n}} \eta_{x_\alpha^j,\delta}(x) f(x_o^j)^{\frac{2-n}{4}} \nu_j((r_\alpha^j)^{-1} \exp_{x_\alpha^j}^{-1}(x)) + \mathcal{W}_\alpha, \\ &\quad \text{with } \mathcal{W}_\alpha \to 0 \text{ in } H_1^2(M), \end{aligned}$$

and

$$J_{h_\alpha,f}(u_\alpha) = J_{h_\infty,f}(u_o) + \sum_{i=1}^{k} J_\infty(v_i) + \sum_{j=1}^{l} f(x_o^j)^{\frac{2-n}{2}} J(\nu_j) + o(1).$$

Here, J and J_∞ are the following functionals

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^n} |u|^{2^*} dx, \tag{2.18}$$

and

$$J_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 dx - \frac{h_\infty(p)}{2} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx - \frac{1}{2^*} f(p) \int_{\mathbb{R}^n} |u|^{2^*} dx. \tag{2.19}$$

By replacing the sequence h_α by a function h(a constant sequence in α) and the sequence of solutions u_α of (E_α) by simply a Palais-Smale sequence u_m of the functional $J_{h,f}$ and mimicking what has been done in [14], with absolutely no changes, we get that the Palais-Smale sequence u_m satisfies the decomposition laws as above. Note that, in this case, condition $\mathcal{H}$ is reduced to the condition

$$0 < h(p) < \frac{1}{K^2(n,2,-2)}.$$

In fact, the condition

$$\mu < \frac{1 - h(p) K^2(n,2,-2)}{(\sup_M f(x))^{\frac{n-2}{2}} K^2(n,2)},$$

it is not needed as we take simply a Palais-Smale sequence which replaces the sequence of solutions u_α.
In a very precise way, we get that the following theorem holds:

Theorem 1. *Let (M,g) be a compact Riemannian manifold with $dim(M) = n \geq 3$. Let f be a smooth positive function on M and h a smooth function on M such that on the point $p \in M$, it satisfies $0 < h(p) < \frac{1}{K^2(n,2,-2)}$.*
Let u_m be a Palais-Smale sequence of the functional $J_{h,f}$ at a level β. Then, there exist $k \in \mathbb{N}$, sequences $R_m^i > 0$, $R_m^i \underset{m\to\infty}{\to} 0$, $\ell \in \mathbb{N}$ sequences $\mathcal{T}_m^j > 0$, $\mathcal{T}_m^j \underset{m\to\infty}{\to} 0$, converging sequences $x_m^j \to x_o^j \neq p$ in M, a weak solution $u \in H_1^2(M)$ of $(E_{h,f})$, solutions $v_i \in D^{1,2}(\mathbb{R}^n)$ of

$$\Delta_{\mathbb{R}^n} u - \frac{h(p)}{|x|^2} u = f(p)|u|^{2^*-2} u, x \in \mathbb{R}^n \setminus \{0\}, \tag{2.20}$$

and nontrivial solutions $\nu_j \in D^{1,2}(\mathbb{R}^n)$ of (2.17) such that up to a subsequence

$$\begin{aligned} u_m &= u + \sum_{i=1}^{k} (R_m^i)^{\frac{2-n}{n}} \eta_{p,\delta}(x) v_i((R_m^i)^{-1} \exp_p^{-1}(x)) \\ &+ \sum_{j=1}^{\ell} (\mathcal{T}_m^i)^{\frac{2-n}{n}} \eta_{x_m^j,\delta}(x) f(x_o^j)^{\frac{2-n}{4}} \nu_j((\tau_m^j)^{-1} \exp_{x_m^j}^{-1}(x)) + \mathcal{W}_m, \\ &\quad \text{with } \mathcal{W}_m \to 0 \text{ in } H_1^2(M), \end{aligned} \tag{2.21}$$

and

$$J_{h,f}(u_m) = J_{h,f}(u) + \sum_{i=1}^{k} J_\infty(v_i) + \sum_{j=1}^{l} f(x_o^j)^{\frac{2-n}{2}} J(\nu_j) + o(1). \tag{2.22}$$

Proof. The proof is the same as that of theorem 3.1 in [14]. QED

The above theorem will be used in order to determine regions in which a Palais-Smale sequence of $J_{h,f}$ is relatively compact and then a condition of existence of a weak solution of equation $(E_{h,f})$ is derived. After that, a test function, constructed from the functions defined by (1.4), will be employed in order to ensure that, under some assumptions, the condition of existence is satisfied. Doing so, we prove the following main result

Theorem 2. *Let (M,g) be a compact Riemannian manifold of dimension n and $Scal_g$ its scalar curvature. Let p be a fixed point of M and ρ_p the distance function defined by (1.1). Let h and f be two smooth functions on M such that f is positive, $\sup_{x\in M} f(x) = f(p)$ and h is such that the operator $\mathcal{I}$ (defined by (2.14)) is coercive. Then, under the following conditions:*

(1) $0 < h(p) < \frac{(n-2)^2}{4}$,

(2) $n > \frac{2}{a} + 2$, *where* $a = \sqrt{1 - \frac{4h(p)}{(n-2)^2}}$,

(3) $0 \leq Scal_g(p) < 3 \min\left(\frac{(\Delta_g f)(p)}{f(p)}, \frac{(\Delta_g h)(p)}{h(p)}\right)$,

there exists a positive weak solution of $(E_{h,f})$.

3 Proof of the main result.

In this section, we prove theorem 2. We look for a weak solution of equation $(E_{h,f})$ as a critical point of the functional $J_{h,f}$. Let us introduce the Nehari manifold for the functional $J_{h,f}$

$$\mathcal{N}_{h,f} = \{u \in H_1^2(M) \setminus \{0\}, DJ_{h,f}(u).u = 0\}.$$

We suppose that the function f is positive on M and the function h is such that the operator $\mathcal{I}$ (defined by (2.14)) is coercive. It can be easily checked that for each $u \in H_1^2(M) \setminus \{0\}$, the function

$$\Phi(u) = \left(\frac{\int_M (|\nabla u|^2 - \frac{h}{\rho_p^2} u^2) dv_g}{\int_M f|u|^{2^*} dv_g}\right)^{\frac{n-2}{4}} u, \tag{3.23}$$

belongs to $\mathcal{N}_h$ and that

$$J_{h,f}(\Phi(u)) = \max_{t>0} J_{h,f}(tu). \tag{3.24}$$

Put $G_{h,f}(u) = DJ_{h,f}(u)u, u \in H_1^2(M) \setminus \{0\}$. Denote by $\nabla J_{h,f}(u)$ the gradient of the functional $J_{h,f}(u)$ at $u \in H_1^2(M)$ defined by $(\nabla J_{h,f}(u), v) = DJ_{f,h}(u).v$, $\forall v \in H_1^2(M)$.

The projection of $\nabla J_{h,f}$ on the tangent space $T_u\mathcal{N}_{h,f}$ denoted by $\nabla_{\mathcal{N}_{h,f}} J_{h,f}(u)$ is given by (see [1])

$$\nabla_{\mathcal{N}_{h,f}} J_{h,f}(u) = \nabla J_{h,f}(u) - \frac{\nabla J_{h,f}(u) \cdot \nabla G_{h,f}(u)}{||\nabla G_{h,f}(u)||^2} \nabla G_{h,f}(u), u \in \mathcal{N}_{h,f}.$$

A constrained Palais-Smale sequence of $J_{h,f}$ on $\mathcal{N}_{h,f}$ at level β is a sequence u_m such that $\nabla_{\mathcal{N}_{h,f}} J_{h,f}(u_m) \to 0$ and $J_{h,f}(u_m) \to \beta$. The following lemma, whose proof is common (see for example [1]), follows immediately from the expression of $\nabla_{\mathcal{N}_{h,f}} J_{h,f}(u)$.

Lemma 1. *If* u_m *is a constrained Palais-Smale sequence of* $J_{h,f}$ *on* $\mathcal{N}_{h,f}$, *then* u_m *is a Palais-Smale sequence of* $J_{h,f}$ *on* $H_1^2(M)$.

The following lemma indicates the level under which all Palais-Smale sequences of the functional $J_{h,f}$ are relatively compact.

Lemma 2. *Let f and h be smooth functions on M such that f is positive on M and h is such that $0 < h(p) < \frac{1}{K^2(n,2,-2)}$. Let u_m be a Palais-Smale sequence of the functional $J_{h,f}$ at a level β. Then, if*

$$0 < \beta < \frac{\left(1 - h(p)K^2(n,2,-2)\right)^{\frac{n-1}{2}}}{n\left(\sup_{x\in M} f(x)\right)^{\frac{n-2}{2}} K^n(n,2)},$$

the sequence u_m converges strongly in $H_1^2(M)$, up to a subsequence, to a non zero weak solution of $(E_{h,f})$.

Proof. First, note that if $u \in D^{1,2}(\mathbb{R}^n) \setminus \{0\}$ is a weak solution of equation (2.20), then

$$J_\infty(u) \geq \frac{\left(1 - h(p)K^2(n,2,-2)\right)^{\frac{n-1}{2}}}{n\left(\sup_{x\in M} f(x)\right)^{\frac{n-2}{2}} K^n(n,2)}, \tag{3.25}$$

where J_∞ is the functional defined by (2.19) (with $h_\infty(p) = h(p)$). In fact, if $u \in D^{1,2}(\mathbb{R}^n) \setminus \{0\}$ is a weak solution of equation (2.20), then $v = (f(p))^{\frac{n-2}{4}} u$ is a non trivial weak solution of (1.3) with $\lambda = h(p)$. Thus, by (1.6), we get

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |v|^{2^*} dx\right)^{\frac{2}{n}} &= \frac{\int_{\mathbb{R}^n}(|\nabla v|^2 - \frac{h(p)}{|x|^2} v)dx}{(\int_{\mathbb{R}^n} |v|^{2^*} dx)^{\frac{2}{2^*}}} \\
&\geq \frac{(1 - h(p)K^2(n,2,-2))^{\frac{n-1}{n}}}{K^2(n,2)}.
\end{aligned}$$

Hence

$$\begin{aligned}
J_\infty(u) = \frac{f(p)}{n} \int_{\mathbb{R}^n} |u|^{2^*} dx &= \frac{(f(p))^{1-\frac{n}{2}}}{n} \int_{\mathbb{R}^n} |v|^{2^*} dx \\
&\geq \frac{(1 - h(p)K^2(n,2,-2))^{\frac{n-1}{2}}}{n(\sup_{x\in M} f(x))^{\frac{n-2}{2}} K^n(n,2)}.
\end{aligned}$$

Similarly, if $u \in D^{1,2}(\mathbb{R}^n)$ is a weak solution of (2.17), then

$$J(u) \geq \frac{1}{nK^n(n,2)}, \tag{3.26}$$

with J is defined by (2.18).

Now, let u_m be a Palais-Smale sequence of $J_{h,f}$ at level β. By theorem 1, there is a critical point u of $J_{h,f}$, such that identities (2.21) and (2.22) are satisfied. Suppose that $u \equiv 0$. Then, by (3.25) and (3.26), we get easily that

$$\beta \geq \frac{(1 - K^2(n,2,-2)h(p))^{\frac{n-1}{2}}}{n\left(\sup_{x\in M} f(x)\right)^{\frac{n-2}{2}} K^n(n,2)},$$

which contradicts the hypothesis of the lemma. Thus, all the v_i and ν_j in (2.21) must be zero functions and $u \neq 0$. Hence, u_m converges, up to a subsequence, strongly to a non zero weak solution u. QED

The following proposition is the first step towards the proof of theorem 2.

Proposition 1. *Let f and h be smooth functions on M such that f is positive on M and h is such that $0 < h(p) < \frac{1}{K^2(n,2,-2)}$ and that the operator $\mathcal{I}$ (defined by (2.14)) is coercive.*

If

$$\mu < \frac{\left(1 - h(p)K^2(n,2,-2)\right)^{\frac{n-1}{n}}}{\left(\sup_{x\in M} f(x)\right)^{\frac{n-2}{n}} K^2(n,2)}, \tag{3.27}$$

where μ is defined by (2.15), then there exists a positive weak solution of equation $(E_{h,f})$.

Proof. First, we note that the functional $J_{h,f}$ is bounded from below on the Nehari manifold $\mathcal{N}_{h,f}$. Then, the variational principle of Ekeland (see for example [11]) gives a Palais-Smale sequence $u_n \in \mathcal{N}_{h,f}$ of $J_{h,f}$ at level $\beta = \inf_{u\in\mathcal{N}_{h,f}} J_{h,f}(u)$.

Now, let $u \in H_1^2(M), u \neq 0$, then we have

$$J_{h,f}(\Phi(u)) = \frac{1}{n}\int_M f|\Phi(u)|^{2^*} dv_g = \frac{1}{n}\left(\frac{\int_M (|\nabla u|^2 - h\frac{u^2}{\rho_p^2})dv_g}{(\int_M f|u|^{2^*} dv_g)^{\frac{2}{2^*}}}\right)^{\frac{n}{2}}, \tag{3.28}$$

where $\Phi(u)$ is defined by (3.23). Of course, if $u \in \mathcal{N}_{h,f}$, we have

$$J_{h,f}(u) = J_{h,f}(\Phi(u)) = \frac{1}{n}\left(\frac{\int_M (|\nabla u|^2 - h\frac{u^2}{\rho_p^2})dv_g}{(\int_M f|u|^{2^*} dv_g)^{\frac{2}{2^*}}}\right)^{\frac{n}{2}}.$$

Then, by coercivity of the functional $\mathcal{I}$, positivity of the function f and Sobolev inequality (2.11), we get that $\beta \geq c > 0$, for some positive constant c.

On the other hand, since $\Phi(u) \in \mathcal{N}_{h,f}$, then

$$J_{h,f}(\Phi(u)) = \frac{1}{n}\left(\frac{\int_M (|\nabla u|^2 - h\frac{u^2}{\rho_p^2})dv_g}{(\int_M f|u|^{2^*}dv_g)^{\frac{2}{2^*}}}\right)^{\frac{n}{2}} \geq \beta, \forall u \in H_1^2(M) \setminus \{0\}.$$

Since u is arbitrary in $H_1^2(M) \setminus \{0\}$, we get by definition of μ that

$$\mu \geq (n\beta)^{\frac{2}{n}},$$

and since by assumption,

$$\mu < \frac{\left(1 - h(p)K^2(n,2,-2)\right)^{\frac{n-1}{n}}}{(\sup_{x\in M} f(x))^{\frac{n-2}{n}} K^2(n,2)},$$

we get that

$$0 < \beta < \frac{\left(1 - h(p)K^2(n,2,-2)\right)^{\frac{n-1}{2}}}{n\,(\sup_{x\in M} f(x))^{\frac{n-2}{2}} K^n(n,2)}.$$

Thus, it follows by lemma 2 that there exists a subsequence of u_n, still denoted by u_n, that converges strongly, in $H_1^2(M)$, to a non zero weak solution u_o of $(E_{h,f})$. Furthermore, this solutions satisfies $\beta = J_{h,f}(u_o) = \inf_{u\in\mathcal{N}_{h,f}} J_{h,f}(u)$.

Finally, to see that u_o is positive, we proceed as follows: since $|u_o| \in H_1^2(M)\setminus \{0\}$ and $|\nabla|u_o|| = |\nabla u_o|$ *a.e* (see [9, Proposition 5.1.9]), by (3.28), we remark that

$$J_{h,f}(\Phi(|u_o|)) = J_{h,f}(\Phi(u_o)) = J_{h,f}(u_o) = \inf_{u\in\mathcal{N}_{h,f}} J_{h,f}(u).$$

Since $\Phi(|u_o|) = |u_o| \in \mathcal{N}_{h,f}$, this implies that the solution is positive. QED

A simple consequence of the above proposition is the following corollary:

Corollary 1. *Let f and h be smooth functions on M. Suppose that f and h satisfy the following condition.*

(1) f is positive, $\mathcal{I}$ is coercive and $0 < h(p) < \frac{1}{K^2(n,2,-2)}$,

(2) $0 < Vol(B(p,\delta_g)) - \int_M h dv_g < \delta_g^2$ and

$$\left(\frac{\sup_{x\in M} f(x)}{\int_{M\setminus B(p,\delta_g)} f dv_g}\right)^{\frac{2}{2^*}} \leq \frac{\left(1 - h(p)K^2(n,2,-2)\right)^{\frac{n-1}{n}}}{K^2(n,2)}.$$

Then, there exists a positive weak solution of $(E_{h,f})$.

Proof. Take the function $u(x) = \rho_p(x)$. Then, $u \in H_1^2(M)\backslash\{0\}$ as ρ_p is Lipschitz function (see [9, Proposition 5.1.8]). Moreover, u satisfies $|\nabla u| = 1$ *a.e* on $B(p, \delta_g)$ (see [9, Corollary 4.1.5]). Thus, by definition of μ we have

$$\begin{aligned}\mu &\leq \frac{\int_M |\nabla \rho_p|^2 dv_g - \int_M h dv_g}{(\int_M f|\rho_p|^{2^*} dv_g)^{\frac{2}{2^*}}} \\ &\leq \frac{Vol(B(p,\delta_g)) - \int_M h dv_g}{\delta_g^2(\int_{M\backslash B(p,\delta_g)} f dv_g)^{\frac{2}{2^*}}} \\ &< \frac{1}{(\int_{M\backslash B(p,\delta_g)} f dv_g)^{\frac{2}{2^*}}}.\end{aligned}$$

Thus, if condition 2 of the corollary is satisfied, then we get

$$\mu < \frac{\left(1 - h(p)K(n,2,-2)^2\right)^{\frac{n-1}{n}}}{(\sup_{x\in M} f(x))^{\frac{n-2}{n}} K^2(n,2)},$$

and the conclusion follows from proposition 1. QED

Another important step towards the proof of theorem 2 is lemma 4 below. In this lemma, we test the quotient

$$Q_{h,f}(u) = \frac{\int_M (|\nabla u|^2 - h\frac{u^2}{\rho_p^2}) dv_g}{(\int_M f|u|^{2^*} dv_g)^{\frac{2}{2^*}}},$$

on some family functions that we define just below. Then, we derive conditions under which inequality (3.27) of proposition 1 holds.

Let $0 < \delta < \frac{\delta_g}{2}$ be a constant and let φ be a smooth cut-off function defined on $\mathbb{R}$ such that $0 \leq \varphi < 1, \varphi \equiv 1$ on $(-\delta, \delta)$ and $\varphi \equiv 0$ on $\mathbb{R} \setminus (-2\delta, 2\delta)$.

Given $\varepsilon \in (0,1)$ and $0 < \varepsilon_o < \delta$ a small fixed constant. Consider on M the functions

$$\phi_\varepsilon(x) = C(n,a)\varphi(\rho_p(x))\left(\frac{\varepsilon^a}{((\rho_p(x))^{1-a}(\varepsilon^{2a} + (\rho_p(x))^{2a})}\right)^{\frac{n-2}{2}},$$

where

$$C(n,a) = (a^2 n(n-2))^{\frac{n-2}{4}}, \tag{3.29}$$

and $a = \sqrt{1 - h(p)K^2(n,2,-2)}$ with, of course, the condition that $0 < h(p) < \frac{1}{K^2(n,2,-2)}$.

Let us, first, prove the following lemma

Lemma 3. *For each $\varepsilon \in (0,1)$, the function $\phi_\varepsilon(x)$ belongs to the Sobolev space $H_1^2(M)$.*

Proof. Let $j \in \mathbb{N}^*$ and consider the functions $g_j : \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined by

$$g_j(t) = C(n,a) \begin{cases} \varphi(t)\left(\frac{\varepsilon^a}{(t^{1-a}(\varepsilon^{2a}+t^{2a})}\right)^{\frac{n-2}{2}}, & t \geq \frac{1}{j} \\ \left(\frac{\varepsilon^a}{j^{a-1}(\varepsilon^{2a}+j^{-2a})}\right)^{\frac{n-2}{2}}, & 0 \leq t \leq \frac{1}{j}. \end{cases}$$

For each $j \in \mathbb{N}^*$, the function g_j is differentiable on $\mathbb{R}^+ \setminus \{\frac{1}{j}\}$ and $|g_j'(t)| \leq C_j$ for $t \in \mathbb{R}^+ \setminus \{\frac{1}{j}\}$ and C_j is some positive constant. Then, g_j is a Lipschitz function on $\mathbb{R}^+ \setminus \{\frac{1}{j}\}$.

Let $t_1 \in [0, \frac{1}{j}[$ and $t_2 \in]\frac{1}{j}, \infty[$. Then,

$$\begin{aligned} &|g_j(t_1) - g_j(t_2)| \\ &= C(n,a)|\left(\frac{\varepsilon^a}{j^{a-1}(\varepsilon^{2a}+j^{-2a})}\right)^{\frac{n-2}{2}} - \varphi(t_2)\left(\frac{\varepsilon^a}{t_2^{1-a}(\varepsilon^{2a}+t_2^{2a})}\right)^{\frac{n-2}{2}}| \\ &\leq \frac{C(n,a)\varepsilon^{a\frac{n-2}{2}}}{(j^{a-1}(\varepsilon^{2a}+j^{-2a}))^{n-2}}|\left(t_2^{1-a}(\varepsilon^{2a}+t_2^{2a})\right)^{\frac{n-2}{2}} - \varphi(t_2)\left(j^{a-1}(\varepsilon^{2a}+j^{-2a})\right)^{\frac{n-2}{2}}| \\ &\leq \frac{C(n,a)\varepsilon^{a\frac{n-2}{2}}}{(j^{a-1}(\varepsilon^{2a}+j^{-2a}))^{n-2}}|\varphi(t_1)\left(t_2^{1-a}(\varepsilon^{2a}+t_2^{2a})\right)^{\frac{n-2}{2}} - \varphi(t_2)\left(t_1^{1-a}(\varepsilon^{2a}+t_1^{2a})\right)^{\frac{n-2}{2}}|. \end{aligned}$$

Since the function $\gamma(t) = \varphi(t_1+t_2-t)\left(t^{1-a}(\varepsilon^{2a}+t^{2a})\right)^{\frac{n-2}{2}}, t \in [0,\infty[$ is a Lipschitz function, we get that there exists a positive constant $C_\varepsilon > 0$ such that

$$|g_j(t_1) - g_j(t_2)| \leq C_\varepsilon |t_2 - t_1|,$$

which means that the function $g_j(t)$ is a Lipschitz function on $[0,\infty[$ for each $j \in \mathbb{N}^*$.

Now, put $\phi_{\varepsilon,j}(x) = g_j(\rho_p(x))$. It is not difficult to see that $\phi_{\varepsilon,j} \in L_2(M)$. Then, since $\rho_p \in H_1^2(M)$, and g_j is a Lipschitz function for each $j \in \mathbb{N}^*$, we get by [9, Proposition 5.1.9], that $\phi_{\varepsilon,j} \in H_1^2(M)$.

Finally, simple calculations show that $\phi_{\varepsilon,j} \to \phi_\varepsilon$ in $H_1^2(M)$ as $j \to \infty$. Thus, $\phi_\varepsilon \in H_1^2(M)$. QED

Lemma 4. *For $n > \frac{2}{a} + 2$, we have the following expansion*

$$\begin{aligned} & Q_{h,f}(\phi_\varepsilon) \\ = \; & \frac{(\int_M |\nabla \phi_\varepsilon|^2 - \frac{h(x)}{(\rho_p(x))^2}\phi_\varepsilon^2)dv_g}{(\int_M f|\phi_\varepsilon|^{2^*}dv_g)^{\frac{2}{2^*}}} \qquad (3.30) \\ = \; & \frac{(1-h(p)K^2(n,2,-2))^{\frac{n-1}{n}}}{f(p)^{\frac{2}{2^*}}K^2(n,2)} - \frac{1}{n}\int_0^\infty \frac{t^{an+1}}{(1+t^{2a})^n}dt \\ & \left[\frac{\int_{\mathbb{R}^n}(|\nabla U|^2 - h(p)\frac{U^2}{|x|^2})dx}{2^*(\int_{\mathbb{R}^n} f(p)|U|^{2^*}dx)^{\frac{2}{2^*}+1}}\left((\Delta_g f)(p) - \frac{1}{3}f(p)Scal_g(p)\right)C(n,a)w_{n-1} + \right. \\ & \frac{1}{2(\int_{\mathbb{R}^n} f(p)|U|^{2^*})^{\frac{2}{2^*}}dx}\left(\frac{1}{3}Scal_g(p)C_1(n,a) + \right. \\ & \left.\left.\left((\Delta_g h)(p) - \frac{1}{3}h(p)Scal_g(p)\right)C_2(n,a)\right)\right]\varepsilon^2 + o(\varepsilon^2). \end{aligned}$$

where $C_1(n,a)$ and $C_2(n,a)$ are given respectively by (3.38) *and* (3.39)*.*

Proof. First, for $\alpha, \beta \in \mathbb{R}$ such that $2a\beta - 1 > \alpha > 0$, let us define the integral

$$I_\beta^\alpha = \int_0^\infty \frac{r^\alpha}{(1+r^{2a})^\beta}dr. \qquad (3.31)$$

Then, by integration by part we obtain,

$$I_\beta^\alpha = \frac{2a\beta}{\alpha+1}I_{\beta+1}^{\alpha+2a}. \qquad (3.32)$$

On the other hand

$$I_\beta^\alpha = \int_0^\infty \frac{r^\alpha(1+r^{2a})}{(1+r^{2a})^{\beta+1}}dr = I_{\beta+1}^\alpha + I_{\beta+1}^{\alpha+2a}.$$

Then, for $2a\beta > \alpha + 1$, we obtain

$$I_\beta^\alpha = \frac{2a\beta}{2a\beta - (\alpha+1)}I_{\beta+1}^\alpha.$$

But, by (3.32) we have for $\alpha > 2a$

$$I_\beta^{\alpha-2a} = \frac{2a\beta}{\alpha - 2a + 1}I_{\beta+1}^\alpha.$$

then, we get for $2a\beta - 1 > \alpha > 2a$

$$I^{\alpha}_{\beta} = \frac{\alpha - 2a + 1}{2a\beta - (\alpha + 1)} I^{\alpha - 2a}_{\beta}. \tag{3.33}$$

Now, in order to have the development of $Q_{h,f}(\phi_\varepsilon)$ we have to develop each one of the terms $\int_M |\nabla \phi_\varepsilon|^2 dv_g$, $\int_M h \frac{\phi_\varepsilon^2}{\rho_p^2} dv_g$ and $\left(\int_M f |\phi_\varepsilon|^{2^*} dv_g\right)^{-\frac{2^*}{2}}$.

3.1 Development of $\int_M |\nabla \phi_\varepsilon|^2 dv_g$.

Consider a geodesic normal coordinate system around p. Let $|g|$ denote the determinant of the matrix formed of the components of the metric g in this system. Let G be the function

$$G(r) = \frac{1}{w_{n-1}} \int_{S^{n-1}} \sqrt{|g|} d\sigma,$$

where $d\sigma$ denotes the volume element on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ and w_{n-1} is the volume of the standard sphere S^{n-1}. Then, we have the expansions (see for example [9, pages 283-284]).

$$dv_g = \left(1 - \frac{1}{6} Ric_{ij}(p) x_i x_j + o(r^2)\right) dx, \tag{3.34}$$

and

$$G(r) = 1 - \frac{1}{6n} Scal_g(p) r^2 + o(r^2), \tag{3.35}$$

where $(x_1, ..., x_n)$ are the coordinates of x in the geodesic normal system, $r = d_g(p, x)$ and $Ric_{ij}(p)$ is the Ricci curvature at p.

Now, we have

$$\begin{aligned}
&\int_M |\nabla \phi_\varepsilon|^2 dv_g \\
= \;& C(n,a)^2 \varepsilon^{a(n-2)} \int_M |\nabla \left(\varphi(\rho_p(x)) \left(\frac{1}{\rho_p(x)^{1-a}(\varepsilon^{2a} + \rho_p(x)^{2a})} \right)^{\frac{n}{2}-1} \right) |^2 dv_g \\
= \;& C(n,a)^2 \varepsilon^{a(n-2)} \left[\int_{B(p,2\delta)} | \frac{n-2}{2} \varphi(\rho_p(x)) \frac{(a-1)\varepsilon^{2a} - (1+a)\rho_p(x)^{2a}}{\rho_p(x)^a \left(\rho_p(x)^{1-a}(\varepsilon^{2a} + \rho_p(x)^{2a})\right)^{\frac{n}{2}}} \right. \\
+ \;& \left. \varphi'(\rho_p(x)) \left(\frac{1}{\rho_p(x)^{1-a}(\varepsilon^{2a} + \rho_p(x)^{2a})} \right)^{\frac{n}{2}-1} |^2 . |\nabla \rho_p(x)|^2 dv_g \right].
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_M |\nabla\phi_\varepsilon|^2 dv_g = C(n,a)^2 w_{n-1}\varepsilon^{a(n-2)}\\
&\left[(\frac{n-2}{2})^2\int_0^\delta \frac{r^{a(n-2)-1}((a-1)\varepsilon^{2a}-(1+a)r^{2a})^2}{(\varepsilon^{2a}+r^{2a})^n}G(r)dr\right.\\
+&\quad (\frac{n-2}{2})^2\int_\delta^{2\delta} \frac{r^{a(n-2)-1}((a-1)\varepsilon^{2a}-(1+a)r^{2a})^2}{(\varepsilon^{2a}+r^{2a})^n}(\varphi(r))^2G(r)dr\\
+&\quad (\frac{n-2}{2})^2\int_\delta^{2\delta} \frac{r^{a(n-2)+1}}{(\varepsilon^{2a}+r^{2a})^{n-2}}(\varphi'(r))^2G(r)dr\\
+&\quad \left.(n-2)\int_\delta^{2\delta} \frac{r^{a(n-2)}((a-1)\varepsilon^{2a}-(1+a)r^{2a})}{(\varepsilon^{2a}+r^{2a})^{n-1}}\varphi(r)\varphi'(r)G(r)dr\right]
\end{aligned}$$

The functions $G(r)$, φ and φ' are bounded in $[0,2\delta]$, then we have for $n > \frac{2}{a}+2$,

$$\begin{aligned}
&\varepsilon^{a(n-2)}\int_\delta^{2\delta} \frac{r^{a(n-2)-1}((a-1)\varepsilon^{2a}-(1+a)r^{2a})^2}{(\varepsilon^{2a}+r^{2a})^n}(\varphi(r))^2G(r)dr = o(\varepsilon^2), \varepsilon\to 0\\
&\varepsilon^{a(n-2)}\int_\delta^{2\delta} \frac{r^{a(n-2)+1}}{(\varepsilon^{2a}+r^{2a})^{n-2}}(\varphi'(r))^2G(r)dr = o(\varepsilon^2), \varepsilon\to 0\\
&\varepsilon^{a(n-2)}\int_\delta^{2\delta} \frac{r^{a(n-2)}((a-1)\varepsilon^{2a}-(1+a)r^{2a})}{(\varepsilon^{2a}+r^{2a})^{n-1}}\varphi(r)\varphi'(r)G(r)dr = o(\varepsilon^2), \varepsilon\to 0.
\end{aligned}$$

We have also for $n > \frac{2}{a}+2$,

$$\begin{aligned}
&\varepsilon^{a(n-2)}\int_\delta^{\infty} \frac{r^{a(n-2)-1}((a-1)\varepsilon^{2a}-(1+a)r^{2a})^2}{(\varepsilon^{2a}+r^{2a})^n}dr = o(\varepsilon^2), \varepsilon\to 0,\\
&\varepsilon^{a(n-2)}\int_\delta^{\infty} \frac{r^{a(n-2)-1}((a-1)\varepsilon^{2a}-(1+a)r^{2a})^2}{(\varepsilon^{2a}+r^{2a})^n}r^2dr = o(\varepsilon^2), \varepsilon\to 0.
\end{aligned}$$

Then, by using the expansion (3.35), we obtain

$$\begin{aligned}
&\int_M |\nabla\phi_\varepsilon|^2 dv_g\\
=&\quad (\frac{n-2}{2})^2 C(n,a)^2 w_{n-1}\varepsilon^{a(n-2)}\left[\int_0^\infty \frac{r^{a(n-2)-1}((a-1)\varepsilon^{2a}-(1+a)r^{2a})^2}{(\varepsilon^{2a}+r^{2a})^n}dr\right.\\
-&\quad \frac{1}{6n}Scal_g(p)\int_0^\infty \frac{r^{a(n-2)+1}((a-1)\varepsilon^{2a}-(1+a)r^{2a})^2}{(\varepsilon^{2a}+r^{2a})^n}dr\\
+&\quad \left.\int_0^\infty \frac{r^{a(n-2)-1}((a-1)\varepsilon^{2a}-(1+a)r^{2a})^2}{(\varepsilon^{2a}+r^{2a})^n}o(r^2)dr + o(\varepsilon^2)\right].
\end{aligned}$$

Hence, after making the change of variable $r = \varepsilon t$, we obtain

$$
\begin{aligned}
&\int_M |\nabla \phi_\varepsilon|^2 dv_g \\
= \ & (\frac{n-2}{2})^2 C(n,a)^2 w_{n-1} \left[\int_0^\infty \frac{t^{a(n-2)-1}((1-a)+(1+a)t^{2a})^2}{(1+t^{2a})^n} dt \right. \\
- \ & \left. \frac{1}{6n} Scal_g(p) \varepsilon^2 \int_0^\infty \frac{t^{a(n-2)+1}((a-1)-(1+a)t^{2a})^2}{(1+t^{2a})^n} dt + o(\varepsilon^2) \right],
\end{aligned}
$$

By using (3.33), we get for $n > \frac{2}{a} + 2$,

$$
\begin{aligned}
&\int_0^\infty \frac{t^{a(n-2)+1}((1-a)+(1+a)t^{2a})^2}{(1+t^{2a})^n} dt \\
= \ & (1-a)^2 I_n^{a(n-2)+1} + 2(1-a^2) I_n^{an+1} + (1+a)^2 I_n^{a(n+2)+1} \\
= \ & \left[(1-a)^2 \frac{an-2}{a(n-2)+2} + 2(1-a^2) + (1+a)^2 \frac{an+2}{a(n-2)-2} \right] I_n^{an+1},
\end{aligned}
$$

and by observing that

$$
C(n,a)^2 (\frac{n-2}{2})^2 w_{n-1} \int_0^\infty \frac{t^{a(n-2)+1}((a-1)-(1+a)t^{2a})^2}{(1+t^{2a})^n} dt = \int_{\mathbb{R}^n} |\nabla U|^2 dx,
$$

where

$$
U(x) = C(n,a) \left(\frac{|x|^{a-1}}{(1+|x|^{2a})} \right)^{\frac{n-2}{2}}, x \in \mathbb{R}^n, \tag{3.36}
$$

we get for $n > \frac{2}{a} + 2$,

$$
\int_M |\nabla \phi_\varepsilon|^2 dv_g = \int_{\mathbb{R}^n} |\nabla U|^2 dx - \frac{1}{6n} Scal_g(p) C_1(n,a) I_n^{an+1} \varepsilon^2 + o(\varepsilon^2), \tag{3.37}
$$

with

$$
\begin{aligned}
C_1(n,a) \ = \ & (\frac{n-2}{2})^2 C(n,a)^2 w_{n-1} \left[(1-a)^2 \frac{an-2}{a(n-2)+2} + 2(1-a^2) + \right. \\
& \left. (1+a)^2 \frac{an+2}{a(n-2)-2} \right].
\end{aligned} \tag{3.38}
$$

3.2 Development of $\int_M h\frac{\phi_\varepsilon^2}{\rho_p^2}dv_g$.

First, by choosing δ small we can write for $x \in B(p,2\delta)$

$$h(x)(\varphi(\rho_p(x))^2 = h(p) + (\nabla_i h)(p)x_i + \frac{1}{2}(\nabla_{ij}h)(p)x_ix_j + o(r^2).$$

By using the expansion (3.34), we get

$$\begin{aligned}
&\int_M \frac{h(x)}{(\rho_p(x))^2}\phi_\varepsilon^2 dv_g \\
= \quad & C(n,a)^2\varepsilon^{a(n-2)}\left[h(p)\int_{B(2\delta)}\frac{1}{|x|^2}\left(\frac{|x|^{a-1}}{\varepsilon^{2a}+|x|^{2a}}\right)^{n-2}dx\right. \\
+ \quad & (\nabla_i h)(p)\int_{B(2\delta)}\frac{1}{|x|^2}\left(\frac{|x|^{a-1}}{\varepsilon^{2a}+|x|^{2a}}\right)^{n-2}x_idx \\
+ \quad & \left(\frac{1}{2}(\nabla_{ij}h)(p) - \frac{1}{6}h(p)Ric_{ij}(p)\right)\int_{B(2\delta)}\frac{1}{|x|^2}\left(\frac{|x|^{a-1}}{\varepsilon^{2a}+|x|^{2a}}\right)^{n-2}x_ix_jdx \\
+ \quad & \left.\int_{B(2\delta)}\frac{1}{|x|^2}\left(\frac{|x|^{a-1}}{\varepsilon^{2a}+|x|^{2a}}\right)^{n-2}o(r^2)dx\right] + o(\varepsilon^2),
\end{aligned}$$

Using the fact that $\int_{S^{n-1}} x_i d\sigma = 0$ and $\int_{S^{n-1}} x_ix_j d\sigma = \begin{cases} \frac{w_{n-1}}{n}r^2, & \text{i=j} \\ 0, & i \neq j \end{cases}$,

we get

$$\begin{aligned}
&\int_M \frac{h(x)}{(\rho_p(x))^2}\phi_\varepsilon^2 dv_g \\
= \quad & C(n,a)^2\varepsilon^{a(n-2)}w_{n-1}\left[h(p)\int_0^{2\delta}\left(\frac{r^{a-1}}{\varepsilon^{2a}+r^{2a}}\right)^{n-2}r^{n-3}dr\right. \\
+ \quad & \left(\frac{1}{2n}(\Delta_g h)(p) - \frac{1}{6n}h(p)Scal_g(p)\right)\int_0^{2\delta}\left(\frac{r^{a-1}}{\varepsilon^{2a}+r^{2a}}\right)^{n-2}r^{n-1}dr \\
+ \quad & \left.\int_0^{2\delta}\frac{1}{r^2}\left(\frac{r^{a-1}}{\varepsilon^{2a}+r^{2a}}\right)^{n-2}r^{n-3}o(r^2)dr\right],
\end{aligned}$$

Note that , for $n > 2+\frac{2}{a}$, we have

$$\varepsilon^{a(n-2)}\int_{2\delta}^{\infty}\left(\frac{r^{a-1}}{\varepsilon^{2a}+r^{2a}}\right)^{n-2}r^{n-1}ds = o(\varepsilon^2), \varepsilon \to 0,$$

and

$$\varepsilon^{a(n-2)}\int_{2\delta}^{\infty}\left(\frac{r^{a-1}}{\varepsilon^{2a}+r^{2a}}\right)^{n-2}r^{n-3}ds = o(\varepsilon^2), \varepsilon \to 0.$$

Thus, after the change of variable $r = \varepsilon t$, we get

$$\begin{aligned}
&\int_M \frac{h(x)}{(\rho_p(x))^2}\phi_\varepsilon^2 dv_g \\
= \; & C(n,a)^2 w_{n-1}\left[h(p)\int_0^\infty \left(\frac{t^{a-1}}{1+t^{2a}}\right)^{n-2} t^{n-3}dt\right. \\
+ \; & \left.\left(\frac{1}{2n}(\Delta_g h)(p) - \frac{1}{6n}h(p)Scal_g(p)\right)\varepsilon^2 \int_0^\infty \left(\frac{t^{a-1}}{1+t^{2a}}\right)^{n-2} t^{n-1}dr + o(\varepsilon^2)\right]
\end{aligned}$$

Noting that

$$\int_{\mathbb{R}^n} \frac{U^2}{|x|^2}dx = C(n,a)^2 w_{n-1}\int_0^\infty \left(\frac{t^{a-1}}{1+t^{2a}}\right)^{n-2} t^{n-3}dt,$$

we get then

$$\begin{aligned}
&\int_M \frac{h(x)}{(\rho_p(x))^2}\phi_\varepsilon^2 dv_g = h(p)\int_{\mathbb{R}^n}\frac{U^2}{|x|^2}dx + c(n,a)^2 w_{n-1} \\
&\left(\frac{1}{2n}(\Delta_g h)(p) - \frac{1}{6n}h(p)Scal_g(p)\right)\varepsilon^2 \int_0^\infty \left(\frac{t^{a-1}}{1+t^{2a}}\right)^{n-2} t^{n-1}dr + o(\varepsilon^2) \\
= \; & h(p)\int_{\mathbb{R}^n}\frac{U^2}{|x|^2}dx + c(n,a)^2 w_{n-1} \\
&\left(\frac{1}{2n}(\Delta_g h)(p) - \frac{1}{6n}h(p)Scal_g(p)\right)\varepsilon^2 \int_0^\infty \frac{t^{a(n-2)+1}(1+t^{2a})^2}{(1+t^{2a})^n}dt + o(\varepsilon^2).
\end{aligned}$$

But, for $n > \frac{2}{a} + 2$, we have

$$\begin{aligned}
\int_0^\infty \frac{t^{a(n-2)+1}(1+t^{2a})^2}{(\varepsilon^{2a}+t^{2a})^n}dt &= I_n^{a(n-2)+1} + 2I_n^{an+1} + I_n^{a(n+2)+1} \\
&= \left(2 + \frac{an-2}{a(n-2)+2} + \frac{an+2}{a(n-2)-2}\right) I_n^{an+1}
\end{aligned}$$

Then, for $n > \frac{2}{a} + 2$, put

$$C_2(n,a) = c(n,a)^2 w_{n-1}\left(2 + \frac{an-2}{a(n-2)+2} + \frac{an+2}{a(n-2)-2}\right), \tag{3.39}$$

we get

$$\begin{aligned}
&\int_M \frac{h(x)}{(\rho_p(x))^2}\phi_\varepsilon^2 dv_g = h(p)\int_{\mathbb{R}^n}\frac{U^2}{|x|^2}dx \\
+ \; & \left(\frac{1}{2n}(\Delta_g h)(p) - \frac{1}{6n}h(p)Scal_g(p)\right) C_2(n,a) I_n^{an+1}\varepsilon^2 + o(\varepsilon^2).
\end{aligned} \tag{3.40}$$

3.3 Development of $(\int_M f|u|^{2^*}dv_g)^{-\frac{2^*}{2}}$.

First, suppose that $\max_{x\in M} f(x) = f(p)$, then we write

$$f(\varphi(\rho_p(x))^{2^*} = f(p) + \frac{1}{2}(\nabla_{ij}f)(p)x_ix_j + o(r^2).$$

By using (3.34), we get

$$\begin{aligned}
&\int_M f|\phi_\varepsilon|^{2^*}dv_g \\
= \quad & C(n,a)^{2^*}\varepsilon^{an}\int_{B(p,2\delta)} f(x)\eta_{p,\delta}^{2^*}\left(\frac{\rho_p(x)^{a-1}}{\varepsilon^{2a}+(\rho_p(x))^{2a}}\right)^n dv_g \\
= \quad & C(n,a)^{2^*}\varepsilon^{an}\left[f(p)\int_{B(2\delta)}\left(\frac{|x|^{a-1}}{\varepsilon^{2a}+|x|^{2a}}\right)^n dx\right. \\
+ \quad & \left(\frac{1}{2}(\nabla_{ij}f)(p) - \frac{1}{6}f(p)Ric(p)\right)\int_{B(2\delta)}\left(\frac{|x|^{a-1}}{\varepsilon^{2a}+|x|^{2a}}\right)^n x_ix_jdx \\
+ \quad & \left.\int_{B(2\delta)}\left(\frac{|x|^{a-1}}{\varepsilon^{2a}+|x|^{2a}}\right)^n o(r^2)dx\right] + o(\varepsilon^2).
\end{aligned}$$

Using the fact that $\int_{S^{n-1}} x_ix_jd\sigma = \begin{cases} \frac{w_{n-1}}{n}r^2, & \text{i=j} \\ 0, & i \neq j \end{cases}$, we get

$$\begin{aligned}
&\int_M f|\phi_\varepsilon|^{2^*}dv_g \\
= \quad & C(n,a)^{2^*}\varepsilon^{an}w_{n-1}\left[f(p)\int_0^{2\delta}\left(\frac{r^{a-1}}{\varepsilon^{2a}+r^{2a}}\right)^n r^{n-1}dr\right. \\
+ \quad & \left(\frac{1}{2n}(\Delta_g f)(p) - \frac{1}{6n}f(p)Scal_g(p)\right)\int_0^{2\delta}\left(\frac{r^{a-1}}{\varepsilon^{2a}+r^{2a}}\right)^n r^{n+1}dr \\
+ \quad & \left.\int_0^{2\delta}\left(\frac{r^{a-1}}{\varepsilon^{2a}+r^{2a}}\right)^n o(r^2)dr\right]
\end{aligned}$$

Noting that for $n > \frac{2}{a}$, we have

$$\varepsilon^{an}\int_{2\delta}^{\infty}\left(\frac{r^{a-1}}{\varepsilon^{2a}+r^{2a}}\right)^n r^{n-1}dr = o(\varepsilon^2), \varepsilon \to 0,$$

and

$$\varepsilon^{an}\int_{2\delta}^{\infty}\left(\frac{r^{a-1}}{\varepsilon^{2a}+r^{2a}}\right)^n r^{n+1}dr = o(\varepsilon^2), \varepsilon \to 0.$$

we obtain

$$\int_M f|\phi_\varepsilon|^{2^*}dv_g = f(p)\int_{\mathbb{R}^n}|U(x)|^{2^*}dx$$
$$+ \quad C(n,a)^{2^*}w_{n-1}\left(\frac{1}{2n}(\Delta_g f)(p)-\frac{1}{6n}f(p)Scal_g(p)\right)I_n^{an+1}\varepsilon^2+o(\varepsilon^2).$$

and then

$$\left(\int_M f|\phi_\varepsilon|^{2^*}dv_g\right)^{-\frac{2}{2^*}} = \left(f(p)\int_{\mathbb{R}^n}|U(x)|^{2^*}dx\right)^{-\frac{2}{2^*}}[1- \tag{3.41}$$
$$\left.\frac{C(n,a)w_{n-1}}{2^*nf(p)\int_{\mathbb{R}^n}|U(x)|^{2^*}dx}\left((\Delta_g f)(p)-\frac{1}{3}f(p)Scal_g(p)\right)I_n^{an+1}\varepsilon^2\right]+o(\varepsilon^2).$$

3.4 Development of $Q_{h,f}(\phi_\varepsilon)$.

Using the expansions (3.37), (3.40) and (3.41), we get

$$Q_{h,f}(\phi_\varepsilon)$$
$$= \frac{\int_M\left(|\nabla\phi_\varepsilon|^2-\frac{h(x)}{(\rho_p(x))^2}\phi_\varepsilon^2\right)dv_g}{\left(\int_M f|\phi_\varepsilon|^{2^*}dv_g\right)^{\frac{2}{2^*}}}$$
$$= \frac{\int_{\mathbb{R}^n}\left(|\nabla U|^2-h(p)\frac{U^2}{|x|^2}\right)dx}{\left(\int_{\mathbb{R}^n}f(p)|U|^{2^*}dx\right)^{\frac{2}{2^*}}} - I_n^{an+1}\left[\frac{\int_{\mathbb{R}^n}\left(|\nabla U|^2-h(p)\frac{U^2}{|x|^2}\right)dx}{2^*n\left(\int_{\mathbb{R}^n}f(p)|U|^{2^*}dx\right)^{\frac{2}{2^*}+1}}\right.$$
$$\left((\Delta_g f)(p)-\frac{1}{3}f(p)Scal_g(p)\right)C(n,a)w_{n-1}+\frac{1}{2n\left(\int_{\mathbb{R}^n}f(p)|U|^{2^*}dx\right)^{\frac{2}{2^*}}}$$
$$\left.\left(\frac{1}{3}Scal_g(p)C_1(n,a)+\left((\Delta_g h)(p)-\frac{1}{3}h(p)Scal_g(p)\right)c_2(n,a)\right)\right]\varepsilon^2+o(\varepsilon^2).$$

Now, using the fact that

$$\frac{\int_{\mathbb{R}^n}\left(|\nabla U|^2-h(p)\frac{U^2}{|x|^2}\right)dx}{\left(\int_{\mathbb{R}^n}|U|^{2^*}dx\right)^{\frac{2}{2^*}}} = \frac{(1-h(p)K^2(n,2,-2))^{\frac{n-1}{n}}}{K^2(n,2)},$$

we finally get the expansion (3.30). QED

Now, we are in position to prove theorem 2.

Proof of theorem 2. By proposition 1, there exists a positive weak solution of $(E_{h,f})$ if condition (3.27) is satisfied and by lemma 4, this condition is satisfied under conditions (1), (2) and (3) of the theorem. QED

Acknowledgements. The authors are very grateful to the referee who did not skimp on the valuable advice and guidelines that greatly improved the article.

References

[1] A. Ambrosetti, A. Malchiodi, Nonlinear analysis and semilinear elliptic problems, Cambridge studies in advanced mathematics 104, Cambridge Univ. Press 2007.

[2] T. Aubin: *Problemes isoperimetriques et espaces de Sobolev*, J. Diff. Geom. 11(1976), 573–598.

[3] T. Aubin, Some nonlinear problems in Riemannian geometry, Springer Monographs in Mathematics 1998.

[4] L. A. Caffarelli, B. Gidas, J. Spruck: *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math., 42(1989), 271–297.

[5] O. Druet, E. Hebbey, F. Robert, Blow-up theory for elliptic PDEs in Riemannian geometry, Princeton University Press 2004.

[6] V. Felli, A. Pistoia: *Existence of blowing-up solutions for a nonlinear elliptic equation with Hardy potential and critical growth*, Communications in Partial Differential Equations, 31(1)(2006), 21–56

[7] B. Gidas, W. M. Ni, L. Nirenberg: *Symmetry and related properties via the maximum principle*, Comm. Math. Phys., 68(1979), 209–243.

[8] B. Gidas, W.M. Ni, L. Nirenberg: *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^n$, Math. Anal. and Applic., Part A, Advances in Math. Suppl. Studies 7A, ed. L. Nachbin, Academic Press (1981), 209–243.

[9] E. Hebey, Introduction à l'analyse non linéaire sur les variétés. Diderot 1997.

[10] E. Hebey, Nonlinear analysis on manifolds: Sobolv spaces and Inequalities, Courant lecture notes 5(2000).

[11] O. Kavian, Introduction à la théorie des points critiques et applications aux problemes elliptiques. Springer-Verlag 1993.

[12] F. Madani, *Le problème de Yamabe avec singularités*, Bull. Sci. Math., 132(2008), 5757–591.

[13] F. Madani, Le problème de Yamabe avec singularités et la conjecture de Hebey-Vaugon. Thesis, Université Piérre et Marie Curie 2009.

[14] Y. Maliki, F. Z. Terki: *A Struwe type decomposition result for a singular elliptic equation on compact Riemannian manifolds*, Anal. Theory App., 34(2018), n. 1, 17–35.

[15] D. Smet: *Nonlinear Schrödinger equations with Hardy potential and critical nonlinearaties*, Trans. Amer. Math. Soc., 357(2005), 2909–2938.

[16] M. Struwe: *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, math. Z., 187(1987), 511–517.

[17] G. Talenti: *Best constants in Sobolev inequality*, Ann. Mat. Pura Appl., 110(1976), 353–372.

[18] S. Terracini: *On positive entire solutions to a class of equations with a singular coefficient and critical exponent*, Advan. Diff. Equa., 2(1996), 241–264.