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Moduli space of extrinsic circular trajectories on real hypersurfaces of type (A_2) in a complex hyperbolic space

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Abstract. We study extrinsic shapes of trajectories for Sasakian magnetic fields on real hypersurfaces of type (A_2) in a complex hyperbolic space $\mathbb{C}H^n$ and investigate a relationship between trajectories for Kähler magnetic fields on $\mathbb{C}H^n$ and these extrinsic shapes.

Keywords: Sasakian magnetic fields, real hypersurfaces of type (A_2), extrinsic circular trajectories, congruent, structure torsion, principal torsion

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Introduction

A closed 2-form on a Riemannian manifold is said to be a magnetic field (see [11], for example). When a magnetic field is closely related with the geometric structure on the underlying manifold, trajectories of charged particles under the influence of this magnetic field should show some properties of the underlying manifold. Since geodesics are trajectories for the null 2-form, the second author proposed to study curve theoretic properties of trajectories to investigate Riemannian manifolds with some geometric structures in [1] and its sequels.

In this paper, we study trajectories on real hypersurfaces of type (A_2) in a complex hyperbolic space. This real hypersurface admits an almost contact metric structure induced by the complex structure on the ambient space. Associated with this almost contact metric structure we have Sasakian magnetic fields, and we are interested in properties of their trajectories. When we study

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submanifolds from curve theoretic point of view, it is one of natural ways to investigate how curves on submanifolds can be seen in the ambient space. For example, geodesics on a standard sphere can be seen as circles in a Euclidean space. In the preceding paper [4], the second author studied conditions on trajectories on a real hypersurface of type (A_2) to be seen as circles in the ambient complex hyperbolic space. But he dropped a condition in Lemma 3 of [4], which is a condition that trajectories turn to be circles of complex torsion ± 1. Fortunately, he investigated trajectories which can be seen as circles of complex torsion not ± 1, his error did not give any influence on the results in that paper.

In this paper, we first correct the error. This gives us a new point of view. We take the moduli space of extrinsic circular trajectories, the set of all congruence classes of trajectories which can be seen as circles in the ambient space. Then, we have a canonical map of this space into the moduli space of circles on a complex hyperbolic space. As we see in [4], into the subset of congruence classes of circles of complex torsion not ± 1, this map is one-to-one. But into the subset of congruence classes of circles of complex torsion ± 1, we show in this paper that it is infinite-to-one. By giving a proof of congruence theorem on trajectories on real hypersurface of type (A_2) which was announced in [4], we show the detail of the canonical map into this subset.

1 Sasakian magnetic fields on real hypersurfaces

A real hypersurface M in a Kähler manifold $\widetilde{M}$ with complex structure J and Riemannian metric $\langle\ ,\ \rangle$ admits an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ induced by J. By use of a unit normal vector field $\mathcal{N}$ on M in $\widetilde{M}$, we define the vector field ξ by $\xi = -J\mathcal{N}$, the 1-form η by $\eta(v) = \langle v, \xi\rangle$ and the $(1,1)$-tensor field ϕ by $\phi v = Jv - \eta(v)\mathcal{N}$. The Riemannian metric $\langle\ ,\ \rangle$ is induced by the metric on $\widetilde{M}$. We call ξ and ϕ the characteristic vector field and the characteristic tensor field, respectively.

On this almost contact metric manifold M, we have a natural closed 2-form $\mathbb{F}_\phi$ defined by $\mathbb{F}_\phi(v, w) = \langle v, \phi w\rangle$ for tangent vectors $v, w \in T_pM$ at an arbitrary point $p \in M$. We call its constant multiple $\mathbb{F}_\kappa = \kappa\mathbb{F}_\phi$ ($\kappa \in \mathbb{R}$) a *Sasakian magnetic field*. Sometime, it is also called a contact magnetic field. A smooth curve γ parameterized by its arclength is said to be a *trajectory* for $\mathbb{F}_\kappa$ if it satisfies the differential equation $\nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma$. Since trajectories for Sasakian magnetic fields are closely related with the almost contact metric structure, and as they are geodesics when $\kappa = 0$, the authors consider that some properties on the underlying manifold should reflect on properties of trajectories.

In this paper we study trajectories on a real hypersurface of type (A_2) in a complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature c. A real hypersurface of type (A) in $\mathbb{C}H^n(c)$ is either a horosphere or a tube around totally geodesic totally complex $\mathbb{C}H^\ell(c)$ ($\ell = 0, \dots, n-1$). Horospheres, geodesic spheres which are the case $\ell = 0$ and tubes around $\mathbb{C}H^{n-1}$ are collectively called real hypersurfaces of type (A_1), and other tubes around $\mathbb{C}H^\ell$ ($\ell = 1, \dots, n-2$) are called of type (A_2). For a real hypersurface M of type (A), its shape operator A_M and the characteristic tensor field ϕ are simultaneously diagonalizable, that is, they satisfy $A_M\phi = \phi A_M$. Its characteristic vector is a principal curvature vector at each point. For a tube $M = T_\ell(r)$ of radius r around $\mathbb{C}H^\ell$, which is a real hypersurface of type (A_2) in $\mathbb{C}H^n(c)$, its principal curvature is $\delta_M = \sqrt{|c|}\coth\sqrt{|c|}r$. The subbundle T^0M of the tangent bundle TM which consists of tangent vectors orthogonal to ξ is divided into two subbundles V_λ, V_μ of principal curvature vectors whose principal curvatures are

$$\lambda_M = \big(\sqrt{|c|}/2\big)\coth(\sqrt{|c|}r/2) \quad \text{and} \quad \mu_M = \big(\sqrt{|c|}/2\big)\tanh(\sqrt{|c|}r/2).$$

For a trajectory γ for a Sasakian magnetic field $\mathbb{F}_\kappa$ on a real hypersurface M of type (A), we set $\rho_\gamma = \langle\dot\gamma, \xi_\gamma\rangle$ and call it its *structure torsion*. We find that it is constant along γ, because A_M is symmetric and ϕ is skew-symmetric. In fact, by using Gauss and Weingarten formulae which are $\widetilde\nabla_X Y = \nabla_X Y + \langle A_M X, Y\rangle\mathcal{N}$ and $\widetilde\nabla_X\mathcal{N} = -A_M X$ for arbitrary vector fields X, Y tangent to M, we have

$$\rho'_\gamma = \langle\kappa\phi\dot\gamma, \xi_\gamma\rangle + \langle\dot\gamma, JA_M\dot\gamma\rangle = \langle\dot\gamma, \phi A_M\dot\gamma\rangle = -\langle A_M\phi\dot\gamma, \dot\gamma\rangle,$$

hence have $2\rho'_\gamma = \langle\dot\gamma, (\phi A_M - A_M\phi)\dot\gamma\rangle = 0$. When M is a real hypersurface of type (A_2), we denote by $\mathrm{Proj}_\lambda : TM \to V_\lambda$, $\mathrm{Proj}_\mu : TM \to V_\mu$ the projections onto the subbundles of principal curvature vectors. For a trajectory γ, we set $\omega_\gamma = \|\mathrm{Proj}_\lambda(\dot\gamma)\|$ and call it its *principal torsion*. Since a real hypersurface M in $\mathbb{C}H^n(c)$ is of type (A) if and only if its shape operator satisfies

$$(\nabla_X A_M)Y = -\frac{c}{4}\big\{\langle\phi X, Y\rangle\xi + \eta(Y)\phi X\big\}$$

for arbitrary vector fields X, Y on M (see [9], for example), we have

$$\begin{aligned}\frac{d}{dt}\langle A_M\dot\gamma, \dot\gamma\rangle &= \langle(\nabla_{\dot\gamma}A_M)\dot\gamma, \dot\gamma\rangle + \langle\kappa A_M\phi\dot\gamma, \dot\gamma\rangle + \langle A_M\dot\gamma, \kappa\phi\dot\gamma\rangle \\ &= -\frac{c\rho_\gamma}{2}\langle\phi\dot\gamma, \dot\gamma\rangle + \kappa\langle(A_M\phi - \phi A_M)\dot\gamma, \dot\gamma\rangle = 0.\end{aligned}$$

As we have $\langle A_M\dot\gamma, \dot\gamma\rangle = \lambda_M\omega_\gamma^2 + \mu_M(1 - \omega_\gamma^2 - \rho_\gamma^2) + \delta_M\rho_\gamma^2$, we find that ω_γ is also constant along γ. Thus, we see that structure and principal torsions should be important invariants for trajectories for Sasakian magnetic fields. We note that Maeda and the second author used in [7] the invariant $\langle A_M\dot\gamma, \dot\gamma\rangle$.

2 Congruency of trajectories

In this section we show that structure and principal torsions for trajectories on real hypersurfaces of type (A_2) in a complex hyperbolic space give their classification. We say that two smooth curves γ_1, γ_2 parameterized by their arclengths on a Riemannian manifold M to be *congruent* to each other (in strong sense) if there exists an isometry φ satisfying $\gamma_2(t) = \varphi \circ \gamma_1(t)$ for all t.

In [2], the second author showed that on a real hypersurface of type (A_1) in $\mathbb{C}H^n$ two trajectories γ_1, γ_2 for Sasakian magnetic fields $\mathbb{F}_{\kappa_1}, \mathbb{F}_{\kappa_2}$ respectively are congruent to each other if and only if one of the following conditions holds:

i) $|\rho_{\gamma_1}| = |\rho_{\gamma_2}| = 1$,

ii) $|\rho_{\gamma_1}| = |\rho_{\gamma_2}| < 1$, $|\kappa_1| = |\kappa_2|$ and $\kappa_1\rho_{\gamma_1} = \kappa_2\rho_{\gamma_2}$

We hence study congruency of trajectories on real hypersurfaces of type (A_2). The following was given in [4] without its proof.

Proposition 1. *Let γ_1, γ_2 be trajectories for Sasakian magnetic fields $\mathbb{F}_{\kappa_1}$, $\mathbb{F}_{\kappa_2}$ respectively on a real hypersurface $M = T_\ell(r)$ of type* (A_2) *in $\mathbb{C}H^n(c)$. They are congruent to each other if and only if one of the following conditions holds*:

i) $|\rho_{\gamma_1}| = |\rho_{\gamma_2}| = 1$,

ii) $|\rho_{\gamma_1}| = |\rho_{\gamma_2}| < 1$, $|\kappa_1| = |\kappa_2|$, $\kappa_1\rho_{\gamma_1} = \kappa_2\rho_{\gamma_2}$ *and* $\omega_{\gamma_1} = \omega_{\gamma_2}$.

In order to show this theorem we briefly recall more on real hypersurfaces of type (A_2). Let $\langle\!\langle\ ,\ \rangle\!\rangle$ denote the Hermitian product on $\mathbb{C}^{n+1}$ given by

$$\langle\!\langle z, w\rangle\!\rangle = -z_o\bar{w}_0 + z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$$

for $z = (z_0, \dots, z_n),\ w = (w_0, \dots, w_n) \in \mathbb{C}^{n+1}$. Let $\varpi : H_1^{2n+1}[1] \to \mathbb{C}H^n(-4)$ be a Hopf fibration of an anti-de Sitter space $H_1^{2n+1}[1] = \{z = (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid \langle\!\langle z, z\rangle\!\rangle = -1\}$. At a point $z \in H_1^{2n+1}$, the horizontal and vertical subspaces $\mathcal{H}_z,\ \mathcal{V}_z$ of the tangent space $T_zH_1^{2n+1}$ are given as

$$\mathcal{H}_z = \{(z, v) \in T_z\mathbb{C}^{n+1} \mid \langle\!\langle z, v\rangle\!\rangle = 0\},$$
$$\mathcal{V}_z = \{(z, \sqrt{-1}az) \in T_z\mathbb{C}^{n+1} \mid a \in \mathbb{R}\},$$

where $z = (z_0, \dots, z_n),\ v = (v_0, \dots, v_n) \in \mathbb{C}^{n+1}$. A real hypersurface $T_\ell(r)$ of type (A_2) in $\mathbb{C}H^n(-4)$ is expressed as $T_\ell(r) = \varpi(\widehat{M})$ with $\widehat{M} = H_1^{2\ell+1}[\cosh r] \times S^{2(n-\ell)-1}[\sinh r]$, where

$$H_1^{2\ell+1}[\cosh r] = \{w = (w_0, \dots, w_\ell) \in \mathbb{C}^{\ell+1} \mid -|w_0|^2 + |w_1|^2 + \cdots + |w_\ell|^2 = -\cosh^2 r\},$$

$$S^{2(n-\ell)-1}[\sinh r] \\ = \{w = (w_{\ell+1}, \dots, w_n) \in \mathbb{C}^{n-\ell} \mid |w_{\ell+1}|^2 + \cdots + |w_n|^2 = \sinh^2 r\}.$$

At a point $\varpi(z) = \varpi\big((z_f, z_s)\big) \in T_\ell(r)$ with $(z_f, z_s) \in \widehat{M} \subset \mathbb{C}^{\ell+1} \times \mathbb{C}^{n-\ell}$, a unit normal $\mathcal{N}_{\varpi(z)}$ is given by $\mathcal{N}_{\varpi(z)} = d\varpi_z\big(\widehat{\mathcal{N}}_z\big)$ with

$$\widehat{\mathcal{N}}_z = \big(z, (-\tanh r z_f, -\coth r z_s)\big) \in \big(T_z\widehat{M}\big)^{\perp} \cap \mathcal{H}_z.$$

The subspaces $V_\lambda\big(\varpi(z)\big)$, $V_\mu\big(\varpi(z)\big)$ of principal curvature vectors are expressed as $V_\lambda\big(\varpi(z)\big) = d\varpi_z\big(\widehat{V}_\lambda(z)\big)$, $V_\mu\big(\varpi(z)\big) = d\varpi_z\big(\widehat{V}_\mu(z)\big)$ with

$$\widehat{V}_\lambda(z) = \big\{(z, v) \in \mathcal{H}_z \cap \big(\{0\} \times T_{z_s} S^{2(n-\ell)-1}\big) \mid \langle v, \hat{J}\widehat{\mathcal{N}}_z\rangle = 0\big\},$$
$$\widehat{V}_\mu(z) = \big\{(z, v) \in \mathcal{H}_z \cap \big(T_{z_f} H_1^{2\ell+1} \times \{0\}\big) \mid \langle v, \hat{J}\widehat{\mathcal{N}}_z\rangle = 0\big\}.$$

We here construct isometries of $T_\ell(r)$ satisfying some conditions. It is well known that every isometry φ of $T_\ell(r)$ is equivariant. That is, denoting an isometric embedding by $\iota : T_\ell(r) \to \mathbb{C}H^n(c)$ we have an isometry $\tilde{\varphi}$ of $\mathbb{C}H^n(c)$ satisfying $\tilde{\varphi} \circ \iota = \iota \circ \varphi$. In the next lemma, we consider $T_\ell(r)$ to be a subset of $\mathbb{C}H^n(c)$ through an isometric embedding.

Lemma 1 (cf. [5]). *Let $x, y \in T_\ell(r)$ be arbitrary points. For arbitrary principal curvature vectors $u \in V_\lambda(x), v \in V_\mu(x), u' \in V_\lambda(y), v' \in V_\mu(y)$ with $\|u\| = \|v\| = \|u'\| = \|v'\| = 1$, there are a holomorphic isometry $\tilde{\varphi}^+$ and an anti-holomorphic isometry $\tilde{\varphi}^-$ of $\mathbb{C}H^n(c)$ satisfying*

i) $\tilde{\varphi}^\pm\big(T_\ell(r)\big) = T_\ell(r)$ *and* $\tilde{\varphi}^\pm(x) = y$,

ii) $(d\tilde{\varphi}^\pm)_x(u) = u'$, $(d\tilde{\varphi}^\pm)_x(v) = v'$,

iii) $(d\tilde{\varphi}^\pm)_x(\xi_x) = \pm\xi_y$.

Proof. We may suppose $c = -4$. For the sake of simplicity, we only treat the case $n = 3$. In this case, the 1-dimensional complex subspaces $\widehat{V}_\lambda(z), \widehat{V}_\mu(z)$ at $z = (z_0, z_1, z_2, z_3) \in \widehat{M}$ $(\subset \mathbb{C}^4)$ are spanned by the unit vectors

$$\hat{\eta}_z = \Big(z, \frac{1}{\sinh r}(0, 0, -\bar{z}_3, \bar{z}_2)\Big), \quad \hat{\zeta}_z = \Big(z, \frac{1}{\cosh r}(\bar{z}_1, \bar{z}_0, 0, 0)\Big),$$

respectively. We take a point $z^0 = (\cosh r, 0, \sinh r, 0) \in \widehat{M}$. For arbitrary $\alpha, \beta \in$

$\mathbb{C}$ with $|\alpha| = |\beta| = 1$, the matrix

$$A_{\alpha,\beta} = \begin{pmatrix} \dfrac{z_0}{\cosh r} & \dfrac{\beta\bar{z}_1}{\cosh r} & 0 & 0 \\ \dfrac{z_1}{\cosh r} & \dfrac{\beta\bar{z}_0}{\cosh r} & 0 & 0 \\ 0 & 0 & \dfrac{z_2}{\sinh r} & -\dfrac{\alpha\bar{z}_3}{\sinh r} \\ 0 & 0 & \dfrac{z_3}{\sinh r} & \dfrac{\alpha\bar{z}_2}{\sinh r} \end{pmatrix}$$

satisfies

$$A^*_{\alpha,\beta}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} A_{\alpha,\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

hence induces a holomorphic isometry $\hat{\varphi}^+_{\alpha,\beta}$ of $\mathbb{C}^4$ with respect to the Hermitian product $\langle\!\langle\ ,\ \rangle\!\rangle$ which satisfies the following properties:

i) $\hat{\varphi}^+_{\alpha,\beta}(z^0) = z$ and $\hat{\varphi}^+_{\alpha,\beta}\big(H^7_1[1]\big) = H^7_1[1]$, $\hat{\varphi}^+_{\alpha,\beta}\big(\widehat{M}\big) = \widehat{M}$,

ii) $\big(d\hat{\varphi}^+_{\alpha,\beta}\big)_{z^0}\big(\langle \tilde{J}\widehat{\mathcal{N}}_{z^0}\rangle^\perp \cap \mathcal{H}_{z^0}\big) = \langle \hat{J}\widehat{\mathcal{N}}_z\rangle^\perp \cap \mathcal{H}_z$,

iii) $\big(d\hat{\varphi}^+_{\alpha,\beta}\big)_{z^0}\big(\widehat{\mathcal{N}}_{z^0}\big) = \widehat{\mathcal{N}}_z$

iv) $\big(d\hat{\varphi}^+_{\alpha,\beta}\big)_{z^0}(\hat{\eta}_{z^0}) = \alpha\hat{\eta}_z$, $\big(d\hat{\varphi}^+_{\alpha,\beta}\big)_{z^0}(\hat{\zeta}_{z^0}) = \beta\hat{\zeta}_z$,

where $\langle \hat{J}\widehat{\mathcal{N}}_z\rangle$ denotes the real vector subspace spanned by $\hat{J}\widehat{\mathcal{N}}_z$.

Let $\hat{\tau} : \mathbb{C}^4 \to \mathbb{C}^4$ denote the isometry which is defined by $(z_0, z_1, z_2, z_3) \mapsto (\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3)$. Then the isometry $\hat{\varphi}^-_{\alpha,\beta} = \hat{\varphi}^+_{\alpha,\beta}\circ\hat{\tau}$ is anti-holomorphic and satisfies the above conditions i) – iv). Thus, we can obtain isometries $\tilde{\varphi}^\pm$ of $\mathbb{C}H^3(-4)$ satisfying desirable conditions. QED

Proof of Proposition 1. The "only if" part. Suppose that γ_1, γ_2 are congruent to each other. Then, there is an isometry φ of $T_\ell(r)$ satisfying $\gamma_2(t) = \varphi\circ\gamma_1(t)$ for all t. We take an isometry $\tilde{\varphi}$ of $\mathbb{C}H^n(c)$ satisfying $\tilde{\varphi}\circ\iota = \iota\circ\varphi$ with an isometric embedding $\iota : T_\ell(r) \to \mathbb{C}H^n(c)$. This isometry is either holomorphic or anti-holomorphic, i.e. $d\tilde{\varphi}\circ J = \pm J\circ d\tilde{\varphi}$. Since $\tilde{\varphi}$ is induced from an isometry of a complex Euclidean space $\mathbb{C}^{n+1}_1$ which preserves $\varpi^{-1}\big(T_\ell(r)\big) = H^{2\ell+1}_1\times S^{2(n-\ell)-1}$, we see that $d\tilde{\varphi}$ preserves the unit normal $\mathcal{N}$. We hence find

$$\begin{aligned}
\rho_{\gamma_2} &= \langle\dot{\gamma}_2, -J\mathcal{N}_{\varphi\circ\gamma_1}\rangle = \langle d\varphi(\dot{\gamma}_1), -Jd\tilde{\varphi}(\mathcal{N}_{\gamma_1})\rangle = \langle\dot{\gamma}_1, \pm\xi_{\varphi\circ\gamma_1}\rangle = \pm\rho_{\gamma_1},\\
\kappa_2\phi\dot{\gamma}_2 &= \nabla_{\dot{\gamma}_2}\dot{\gamma}_2 = (d\varphi)\big(\nabla_{\dot{\gamma}_1}\dot{\gamma}_1\big) = (d\varphi)(\kappa_1\phi\dot{\gamma}_1)\\
&= \kappa_1(d\tilde{\varphi})\big(J\dot{\gamma}_1 - \rho_{\gamma_1}\mathcal{N}_{\gamma_1}\big) = \kappa_1\big\{\pm J(d\varphi)(\dot{\gamma}_1) - \rho_{\gamma_1}\mathcal{N}_{\varphi\circ\gamma_1}\big\} = \pm\kappa_1\phi\dot{\gamma}_2
\end{aligned}$$

In particular, we have $|\kappa_1|\sqrt{1-\rho_{\gamma_1}^2} = |\kappa_2|\sqrt{1-\rho_{\gamma_2}^2}$. Thus, when $|\rho_{\gamma_1}| = |\rho_{\gamma_2}| < 1$, we find $|\kappa_1| = |\kappa_2|$ and $\kappa_1\rho_{\gamma_1} = \kappa_2\rho_{\gamma_2}$ hold. Moreover, we consider the curves $\iota\circ\gamma_1,\ \iota\circ\gamma_2$ on $\mathbb{C}H^n(c)$, which are usually also denoted as γ_1,γ_2 for the sake of simplicity. These curves on $\mathbb{C}H^n(c)$ are also congruent to each other. By Gauss formula, we have $\widetilde{\nabla}_{\dot\gamma_i}\dot\gamma_i = \kappa_i\phi\dot\gamma_i + \langle A_M\dot\gamma_i,\dot\gamma_i\rangle\mathcal{N}$, hence the square of the geodesic curvature of $\iota\circ\gamma_i$ is given by $\kappa_i^2(1-\rho_{\gamma_i}^2)+\langle A_M\dot\gamma_i,\dot\gamma_i\rangle^2$. As $\langle A_M\dot\gamma_i,\dot\gamma_i\rangle = \lambda_M\omega_{\gamma_i}^2 + \mu_M(1-\omega_{\gamma_i}^2-\rho_{\gamma_i}^2)+\delta_M\rho_{\gamma_i}^2 > 0$ and $\lambda_M > \mu_M$, we find $\omega_{\gamma_1} = \omega_{\gamma_2}$.

Next we study the "if" part. Since trajectories of structure torsion ± 1 are geodesics, we find that they are congruent by Lemma 1. If γ_1,γ_2 satisfy the condition ii), we have $\|\mathrm{Proj}_\lambda(\dot\gamma_1)\| = \|\mathrm{Proj}_\lambda(\dot\gamma_2)\|$ and $\|\mathrm{Proj}_\mu(\dot\gamma_1)\| = \|\mathrm{Proj}_\mu(\dot\gamma_2)\|$. Thus, by Lemma 1, we have isometries φ^+ and φ^- of $T_\ell(r)$ which are restrictions of a holomorphic isometry $\tilde\varphi^+$ and an anti-holomorphic isometry $\tilde\varphi^-$ of $\mathbb{C}H^n(c)$, respectively, and that satisfy

$$\varphi^\pm\big(\gamma_1(0)\big) = \gamma_2(0),\ (d\varphi^\pm)_{\gamma_1(0)}\big(\xi_{\gamma_1(0)}\big) = \pm\xi_{\gamma_2(0)},$$
$$(d\varphi^\pm)_{\gamma_1(0)}\big(\dot\gamma_1(0)-\rho_{\gamma_1}\xi_{\gamma_1(0)}\big) = \dot\gamma_2(0)-\rho_{\gamma_2}\xi_{\gamma_2(0)}.$$

We take φ^+ when $\rho_{\gamma_1} = \rho_{\gamma_2} \neq 0$ or when $\rho_{\gamma_1} = \rho_{\gamma_2} = 0$ and $\kappa_1\kappa_2 \geq 0$, and take φ^- when $\rho_{\gamma_1} = -\rho_{\gamma_2} \neq 0$ or when $\rho_{\gamma_1} = \rho_{\gamma_2} = 0$ and $\kappa_1\kappa_2 < 0$. We then have

$$\begin{aligned}(d\varphi^\pm)_{\gamma_1(0)}\big(\dot\gamma_1(0)\big) &= (d\varphi^\pm)_{\gamma_1(0)}\big((\dot\gamma_1(0)-\rho_{\gamma_1}\xi_{\gamma_1(0)})+\rho_{\gamma_1}\xi_{\gamma_1(0)}\big)\\ &= \big(\dot\gamma_2(0)-\rho_{\gamma_2}\xi_{\gamma_2(0)}\big) \pm \rho_{\gamma_1}\xi_{\gamma_2(0)} = \dot\gamma_2(0).\end{aligned}$$

We define smooth curves σ^+,σ^- by $\sigma^\pm = \varphi^\pm\circ\gamma_1$. They satisfy

$$\nabla_{\dot\sigma^\pm}\dot\sigma^\pm = (d\varphi^\pm)\big(\nabla_{\dot\gamma_1}\dot\gamma_1\big) = (d\varphi^\pm)\big(\kappa_1\phi\dot\gamma_1\big) = \pm\kappa_1\phi(d\varphi^\pm)\dot\gamma_1 = \kappa_2\phi\dot\sigma^\pm.$$

Hence, corresponding to the signatures of ρ_{γ_1} and ρ_{γ_2}, the curves γ_2 and one of σ^+,σ^- satisfy the same differential equation. As they have the same initial conditions, we find that $\gamma_2 = \sigma^+$ or $\gamma_2 = \sigma^-$. Hence γ_1,γ_2 are congruent to each other. QED

We denoet by $\mathcal{F}_\kappa = \mathcal{F}_\kappa\big(T_\ell(r)\big)$ the set of all congruence classes of trajectory for $\mathbb{F}_\kappa$ on $T_\ell(r)$. Proposition 1 shows that it is set theoretically identified with a half disk $D = \{(\rho,\omega)\mid \rho^2+\omega^2\leq 1,\ \omega\geq 0\}$ when $\kappa\neq 0$. If we denote by $\mathcal{F} = \mathcal{F}\big(T_\ell(r)\big)$ the set of all congruence classes of trajectory for some Sasakian magnetic field on $T_\ell(r)$, then it is set theoretically identified with $[0,\infty)\times D/\sim$ or with $\mathbb{R}\times\Omega/\backsim$, where $\Omega = \{(\rho,\omega)\mid \rho^2+\omega^2\leq 1,\ \rho\geq 0,\ \omega\geq 0\}$. Here, for $(\kappa_1,\rho_1,\omega_1)$, $(\kappa_2,\rho_2,\omega_2)\in[0,\infty)\times D$, we set $(\kappa_1,\rho_1,\omega_1)\sim(\kappa_2,\rho_2,\omega_2)$ if and only if one of the following condition holds:

i) $\kappa_1 = \kappa_2$, $\rho_1 = \rho_2$, $\omega_1 = \omega_2$,

ii) $\kappa_1 = \kappa_2 = 0$, $\rho_1 = -\rho_2$, $\omega_1 = \omega_2$,

iii) $|\rho_1| = |\rho_2| = 1$.

For $(\kappa_1, \rho_1, \omega_1)$, $(\kappa_2, \rho_2, \omega_2) \in \mathbb{R} \times \Omega$, we set $(\kappa_1, \rho_1, \omega_1) \backsim (\kappa_2, \rho_2, \omega_2)$ if and only if one of the following condition holds:

i) $\kappa_1 = \kappa_2$, $\rho_1 = \rho_2$, $\omega_1 = \omega_2$,

ii) $\kappa_1 = -\kappa_2$, $\rho_1 = \rho_2 = 0$, $\omega_1 = \omega_2$,

iii) $\rho_1 = \rho_2 = 1$.

3 Extrinsic circular trajectories

A smooth curve γ parameterized by its arclength is said to be a *circle* if there exist a unit vector field Y along γ and a nonnegative constant k_γ satisfying $\nabla_{\dot\gamma}\dot\gamma = k_\gamma Y$, $\nabla_{\dot\gamma} Y = -k_\gamma \dot\gamma$ (see [10]). The constant k_γ is called the *geodesic curvature* of γ. When $k_\gamma = 0$, it is a geodesic. On a Kähler manifold with complex structure J, for each non-geodesic circle γ, we set $\tau_\gamma = \langle \dot\gamma, Y \rangle / k_\gamma$ and call it its *complex torsion*.

Let $\iota : T_\ell(r) \to \mathbb{C}H^n(c)$ be an isometric embedding of a real hypersurface $T_\ell(r)$ of type (A_2). We say a trajectory on $T_\ell(r)$ to be *extrinsic circular* if its extrinsic shape $\iota \circ \gamma$ is a circle on $\mathbb{C}H^n(c)$. In the preceding paper [4], the second author gave a condition that trajectories to be extrinsic circular. But as he dropped a condition, we here correct it.

Proposition 2. *Let γ be a trajectory for a Sasakian magnetic field $\mathbb{F}_\kappa$ on a real hypersurface $M = T_\ell(r)$ of type* (A_2) *in a complex hyperbolic space $\mathbb{C}H^n(c)$. It is extrinsic circular if and only if one of the following conditions holds*:

1) $\rho_\gamma = \pm 1$,

2) $\kappa\rho_\gamma = \lambda_M \omega_\gamma^2 + \mu_M(1 - \omega_\gamma^2 - \rho_\gamma^2) + \delta_M \rho_\gamma^2$,

3) $\omega_\gamma = 0$ *and* $\kappa + \lambda_M \rho_\gamma = 0$,

4) $\omega_\gamma^2 + \rho_\gamma^2 = 1$ *and* $\kappa + \mu_M \rho_\gamma = 0$.

Corresponding to each case, the geodesic curvature k_γ and the complex torsion τ_γ of the extrinsic shape of γ are given as follows:

1) $k_\gamma = \delta_M$, $\tau_\gamma = \mp 1$,

2) $k_\gamma = |\kappa|$, $\tau_\gamma = -\mathrm{sgn}(\kappa)$,

3) $k_\gamma = \sqrt{\kappa^2 - 2\kappa\mu_M\rho_\gamma + \mu_M^2}$, $\tau_\gamma = -(\kappa - 2\kappa\rho_\gamma^2 + \mu_M\rho_\gamma)/k_\gamma$,

4) $k_\gamma = \sqrt{\kappa^2 - 2\kappa\lambda_M\rho_\gamma + \lambda_M^2}$, $\tau_\gamma = -(\kappa - 2\kappa\rho_\gamma^2 + \lambda_M\rho_\gamma)/k_\gamma$.

Here, $\mathrm{sgn}(\kappa)$ *denotes the signature of* κ*, and in the case* $k_\gamma = 0$*, we ignore* τ_γ

Proof. By use of Gauss and Weingarten formulae, we have

$$\begin{aligned}
\widetilde{\nabla}_{\dot\gamma}\dot\gamma &= \nabla_{\dot\gamma}\dot\gamma + \langle A_M\dot\gamma, \dot\gamma\rangle\mathcal{N} \\
&= \kappa\phi\dot\gamma + \langle A_M\dot\gamma, \dot\gamma\rangle\mathcal{N} = \kappa J\dot\gamma + \big(\langle A_M\dot\gamma, \dot\gamma\rangle - \kappa\rho_\gamma\big)\mathcal{N}, \\
\widetilde{\nabla}_{\dot\gamma}&\Big(\kappa J\dot\gamma + \big(\langle A_M\dot\gamma, \dot\gamma\rangle - \kappa\rho_\gamma\big)\mathcal{N}\Big) \\
&= -\kappa^2\dot\gamma - \kappa\big(\langle A_M\dot\gamma, \dot\gamma\rangle - \kappa\rho_\gamma\big)\xi - \big(\langle A_M\dot\gamma, \dot\gamma\rangle - \kappa\rho_\gamma\big)A_M\dot\gamma \\
&= -\big\{\kappa^2(1-\rho_\gamma^2) + \langle A_M\dot\gamma, \dot\gamma\rangle^2\big\}\dot\gamma \\
&\qquad + \big(\langle A_M\dot\gamma, \dot\gamma\rangle - \kappa\rho_\gamma\big)\big\{\big(\langle A_M\dot\gamma, \dot\gamma\rangle + \kappa\rho_\gamma\big)\dot\gamma - \kappa\xi - A_M\dot\gamma\big\}.
\end{aligned}$$

Thus, the extrinsic shape of γ is a circle if and only if the vector

$$\big(\langle A_M\dot\gamma, \dot\gamma\rangle - \kappa\rho_\gamma\big)\big\{\big(\langle A_M\dot\gamma, \dot\gamma\rangle + \kappa\rho_\gamma\big)\dot\gamma - \kappa\xi - A_M\dot\gamma\big\}$$

vanishes. The equality $\langle A_M\dot\gamma, \dot\gamma\rangle - \kappa\rho_\gamma = 0$ leads us to the second condition, which the second author dropped in [4]. When $\big(\langle A_M\dot\gamma, \dot\gamma\rangle + \kappa\rho_\gamma\big)\dot\gamma - \kappa\xi - A\dot\gamma$ vanishes, by decomposing it to principal vectors, it is equivalent to the equalities

$$\big(\langle A_M\dot\gamma, \dot\gamma\rangle + \kappa\rho_\gamma - \lambda_M\big)\mathrm{Proj}_\lambda(\dot\gamma) = 0, \tag{3.1}$$

$$\big(\langle A_M\dot\gamma, \dot\gamma\rangle + \kappa\rho_\gamma - \mu_M\big)\mathrm{Proj}_\mu(\dot\gamma) = 0, \tag{3.2}$$

$$\big(\langle A_M\dot\gamma, \dot\gamma\rangle + \kappa\rho_\gamma - \delta_M\big)\rho_\gamma - \kappa = 0. \tag{3.3}$$

As $\lambda_M > \mu_M$, the condition $\lambda_M = \langle A_M\dot\gamma, \dot\gamma\rangle + \kappa\rho_\gamma = \mu_M$ does not stand. Hence at least one of the vectors $\mathrm{Proj}_\lambda(\dot\gamma)$ and $\mathrm{Proj}_\mu(\dot\gamma)$ is null. When both of them are null, which is the case $\rho_\gamma = \pm 1$, as the equality (3.3) holds, we obtain the first condition. When $\mathrm{Proj}_\lambda(\dot\gamma)$ is null and $\mathrm{Proj}_\mu(\dot\gamma)$ is not null, the equality (3.2) turns to $\rho_\gamma\{(\delta_M - \mu_M)\rho_\gamma + \kappa\} = 0$. When $\rho_\gamma = 0$, by (3.3) we have $\kappa = 0$, hence this case is contained in the third condition. When $(\delta_M - \mu_M)\rho_\gamma + \kappa = 0$, we find that (3.3) also holds. As $\delta_M = \lambda_M + \mu_M$, we obtain the third condition. Similarly, we have the fourth condition when $\mathrm{Proj}_\lambda(\dot\gamma)$ is not null and $\mathrm{Proj}_\mu(\dot\gamma)$ is null. Since we have

$$k_\gamma^2 = \kappa^2(1-\rho^2) + \langle A_M\dot\gamma, \dot\gamma\rangle^2, \qquad \tau_\gamma = -\big\{\kappa(1-\rho_\gamma^2) + \langle A_M\dot\gamma, \dot\gamma\rangle\rho_\gamma\big\}/k_\gamma,$$

we get the conclusion. *QED*

The readers should confer the conditions on extrinsic circular trajectories on totally η-umbilic real hypersurfaces in $\mathbb{C}H^n$ (see [6]).

4 Moduli space of extrinsic circular trajectories

Let $\mathcal{E}\big(T_\ell(r)\big)$ denote the set of all congruence classes of extrinsic circular trajectories for some Sasakian magnetic fields on a real hypersurface $M = T_\ell(r)$ of type (A_2) in $\mathbb{C}H^n(c)$. We shall call this the *moduli space* of extrinsic circular trajectories on $T_\ell(r)$. Needless to say, it is a subset of $\mathcal{F}\big(T_\ell(r)\big)$. Corresponding to the conditions in Proposition 2, we classify extrinsic circular trajectories into four classes and denote the sets of their congruence classes by $\mathcal{E}_{I}\big(T_\ell(r)\big)$, $\mathcal{E}_{II}\big(T_\ell(r)\big)$, $\mathcal{E}_{III}\big(T_\ell(r)\big)$ and $\mathcal{E}_{IV}\big(T_\ell(r)\big)$. Then $\mathcal{E}_{I}\big(T_\ell(r)\big)$ consists of one point. As the second condition holds for $\kappa = \delta_M$ and $\rho_\gamma = 1$, we see $\mathcal{E}_{I}\big(T_\ell(r)\big) \subset \mathcal{E}_{II}\big(T_\ell(r)\big)$. On the other hand, if we substitute the equalities in the third condition into that of the second, we have $\mu_M + 2\lambda_M\rho_\gamma^2 = 0$, which is a contradiction. Hence we have $\big(\mathcal{E}_{II}\big(T_\ell(r)\big) \cap \mathcal{E}_{III}\big(T_\ell(r)\big)\big) \setminus \mathcal{E}_{I}\big(T_\ell(r)\big) = \emptyset$. Here, we should note that the third condition holds when $\kappa = \pm\lambda_M$ and $\rho_\gamma = \mp 1$. This means $\mathcal{E}_{I}\big(T_\ell(r)\big) \subset \mathcal{E}_{III}\big(T_\ell(r)\big)$. Similarly, we have $\mathcal{E}_{I}\big(T_\ell(r)\big) \subset \mathcal{E}_{IV}\big(T_\ell(r)\big)$ and $\big(\mathcal{E}_{II}\big(T_\ell(r)\big) \cap \mathcal{E}_{IV}\big(T_\ell(r)\big)\big) \setminus \mathcal{E}_{I}\big(T_\ell(r)\big) = \emptyset$.

Let $\mathcal{M} = \mathcal{M}\big(\mathbb{C}H^n(c)\big)$ denote the moduli space of circles, the set of all congruence classes of circles, on $\mathbb{C}H^n(c)$. Since two circles σ_1, σ_2 on $\mathbb{C}H^n(c)$ are congruent to each other if and only if they satisfy one of the following conditions (see [8]):

i) they are geodesics, (i.e. $k_{\sigma_1} = k_{\sigma_2} = 0$),

ii) they satisfy $k_{\sigma_1} = k_{\sigma_2} > 0$ and $|\tau_{\sigma_1}| = |\tau_{\sigma_2}|$.

Therefore, the set $\mathcal{M}$ is expressed as $[0,\infty) \times [0,1]/\sim$, where $(k,\tau) \sim (k',\tau')$ if and only if either $k = k' = 0$ or $(k,\tau) = (k',\tau')$. As we have a natural mapping $\Phi : \mathcal{E}\big(T_\ell(r)\big) \ni [\gamma] \to [(k_\gamma, |\tau_\gamma|)] \in \mathcal{M}$, we study its properties in this section.

At first we study extrinsic circular geodesics, the case $\kappa = 0$. When $\rho_\gamma = \pm 1$, as we have $\nabla_{\dot\gamma}\dot\gamma = 0$, we see that γ is a geodesic on $T_\ell(r)$, hence is an extrinsic circular geodesic. We suppose a geodesic γ satisfies $|\rho_\gamma| < 1$. Since $\lambda_M\omega_\gamma^2 + \mu_M(1 - \omega_\gamma^2 - \rho_\gamma^2) + \delta_M\rho_\gamma^2 > 0$, this geodesic can not satisfy the second condition in Proposition 2. If this geodesic satisfies either the third or fourth conditions in Proposition 2, we have $\rho_\gamma = 0$. Thus, we obtain the following which is an extension of a result in §3 of [3].

Theorem 1. *A geodesic γ on a real hypersurface $T_\ell(r)$ of type* (A_2) *in $\mathbb{C}H^n(c)$ is extrinsic circular if and only if its tangent vector is principal curvature vector. The extrinsic geodesic curvature k_γ and complex torsion τ_γ are as follows*:

(1) *When $\rho_\gamma = \pm 1$, hence $\omega_\gamma = 0$, we have $k_\gamma = \sqrt{|c|}\coth\sqrt{|c|}r$, $\tau_\gamma = \mp 1$.*

(2) *When $\omega_\gamma = 0$ and $\rho_\gamma = 0$, we have $k_\gamma = \big(\sqrt{|c|}/2\big)\tanh(\sqrt{|c|}r/2)$, $\tau_\gamma = 0$.*

(3) *When* $\omega_\gamma = 1$, *hence* $\rho_\gamma = 0$, *we have* $k_\gamma = \big(\sqrt{|c|}/2\big)\coth(\sqrt{|c|}r/2)$, $\tau_\gamma = 0$.

On the other hand, we study whether a trajectory γ for $\mathbb{F}_\kappa$ can be seen as geodesic in $\mathbb{C}H^n(c)$. When $\rho_\gamma = \pm 1$, it is not extrinsic geodesic. Thus we have $|\rho_\gamma| < 1$. When γ satisfies either the third or the fourth condition in Proposition 2, we find $k_\gamma > 0$. When γ satisfies the second condition in Proposition 2, as $\lambda_M\omega_\gamma^2 + \mu_M(1-\omega_\gamma^2-\rho_\gamma^2) + \delta_M\rho_\gamma^2 > 0$, we find $\kappa \neq 0$ and that it is not extrinsic geodesic. We hece have the following.

Proposition 3. *No trajectories on a real hypersurface* $T_\ell(r)$ *of type* (A_2) *in* $\mathbb{C}H^n(c)$ *can be seen as geodesics on* $\mathbb{C}H^n(c)$.

In the rest of this section we study non-geodesic trajectories. We already studied in [4] extrinsic circular trajectories satisfying either the third or the fourth conditions in Proposition 2. In terms of the map Φ, we showed that it induces an injective correspondence of $\big(\mathcal{E}_{III}\big(T_\ell(r)\big)\cup\mathcal{E}_{IV}\big(T_\ell(r)\big)\big)\setminus\mathcal{E}_I\big(T_\ell(r)\big)$ into $(0,\delta_M)\times[0,1) \subset \mathcal{M}$. We therefore study extrinsic circular trajectories satisfying the second condition in Proposition 2.

First, we study extrinsic circular trajectories for a Sasakian magnetic field of given strength. We hence study the set $\mathcal{E}_{II}\big(T_\ell(r)\big)\bigcap\mathcal{F}_\kappa$. In the first place, we classify circular trajectories by their structure torsions.

Theorem 2. *Let* $M = T_\ell(r)$ *be a real hypersurface of type* (A_2) *in* $\mathbb{C}H^n(c)$. *We take an arbitrary nonzero constant* κ *and a constant* ρ *with* $|\rho| < 1$. *Then the number of congruence classes of non-geodesic trajectories for* $\mathbb{F}_\kappa$ *of structure torsion* ρ *whose extrinsic shapes are circles of geodesic curvature* $|\kappa|$ *is* 1 *if and only if one of the following conditions holds*:

i) $|\kappa| = \sqrt{|c|}$ *and* $\rho = \tanh(\sqrt{|c|}\,r/2)$,

ii) $\sqrt{|c|} < |\kappa| < \sqrt{c}\coth(\sqrt{|c|}\,r)$ *and*
$\frac{1}{\sqrt{|c|}}\big(\kappa - \sqrt{\kappa^2+c}\big)\tanh\frac{\sqrt{|c|}\,r}{2} \le \rho \le \frac{1}{\sqrt{|c|}}\big(\kappa + \sqrt{\kappa^2+c}\big)\tanh\frac{\sqrt{|c|}\,r}{2}$,

iii) $\kappa = \sqrt{|c|}\coth(\sqrt{|c|}\,r)$ *and* $\tanh^2\big(\sqrt{|c|}\,r/2\big) \le \rho < 1$,

iv) $\kappa > \sqrt{|c|}\coth(\sqrt{|c|}\,r)$ *and*
$\frac{1}{\sqrt{|c|}}\big(\kappa - \sqrt{\kappa^2+c}\big)\tanh\frac{\sqrt{|c|}\,r}{2} \le \rho \le \frac{1}{\sqrt{|c|}}\big(\kappa - \sqrt{\kappa^2+c}\big)\coth\frac{\sqrt{|c|}\,r}{2}$.

v) $\kappa = -\sqrt{c}\coth(\sqrt{|c|}\,r)$ *and* $-1 < \rho \le -\tanh^2\big(\sqrt{|c|}\,r/2\big)$,

vi) $\kappa < -\sqrt{c}\coth(\sqrt{|c|}\,r)$ *and*
$\frac{1}{\sqrt{|c|}}\big(\kappa + \sqrt{\kappa^2+c}\big)\coth\frac{\sqrt{|c|}\,r}{2} \le \rho \le \frac{1}{\sqrt{|c|}}\big(\kappa + \sqrt{\kappa^2+c}\big)\tanh\frac{\sqrt{|c|}\,r}{2}$.

Otherwise, it is 0.

Proof. We take a trajectory γ for $\mathbb{F}_\kappa$ on $T_\ell(r)$. When it satisfies the first condition in Proposition 2, then γ is a geodesic. If we suppose that the extrinsic shape of γ satisfying the third condition in Proposition 2 has $k_\gamma = |\kappa|$, then we find $2\kappa\rho_\gamma = \mu_M$. Since we have $\kappa + \lambda_M\rho_\gamma = 0$ by the third condition, we have $-2\lambda\rho_\gamma^2 = \mu_M$, which is a contradiction. Thus, in this case $k_\gamma \neq |\kappa|$. Similarly, we can show that the geodesic curvature of the extrinsic shape of a circular trajectory satisfying the fourth condition in Proposition 2 does not coincide with $|\kappa|$.

We only need to study trajectories satisfying the second condition in Proposition 2. Therefore, we need to find the solution for the equation $(\lambda_M - \mu_M)\omega^2 = \kappa\rho - \mu_M - \lambda_M\rho^2$ on ω. Since $0 \leq \omega^2 \leq 1 - \rho^2$, it has a solution if and only if the following inequalities hold:

$$-\lambda_M\rho^2 + \kappa\rho - \mu_M \geq 0, \tag{4.1}$$

$$\mu_M\rho^2 - \kappa\rho + \lambda_M \geq 0, \tag{4.2}$$

We consider them as quadratic inequalities on ρ. In view of the discriminant of (4.1), we find that it does not hold when $|\kappa| < \sqrt{|c|}$. When $|\kappa| \geq \sqrt{|c|}$, we find that (4.1) holds if and only if

$$\frac{1}{2\lambda_M}\bigl(\kappa - \sqrt{\kappa^2 + c}\bigr) \leq \rho \leq \frac{1}{2\lambda_M}\bigl(\kappa + \sqrt{\kappa^2 + c}\bigr),$$

and find that (4.2) holds if and only if either

$$\rho \leq \frac{1}{2\mu_M}\bigl(\kappa - \sqrt{\kappa^2 + c}\bigr) \quad \text{or} \quad \rho \geq \frac{1}{2\mu_M}\bigl(\kappa + \sqrt{\kappa^2 + c}\bigr).$$

We note that the behavior of $\kappa + \sqrt{\kappa^2 + c}$ with respect to κ is monotone decreasing when $\kappa < 0$ and is monotone increasing when $\kappa > 0$, and that the behavior of $\kappa - \sqrt{\kappa^2 + c}$ with respect to κ is monotone increasing when $\kappa < 0$ and is monotone decreasing when $\kappa > 0$. Since we have

$$\frac{1}{2\mu_M}\bigl(\delta_M - \sqrt{\delta_M^2 + c}\bigr) = 1 = \frac{1}{2\lambda_M}\bigl(\delta_M + \sqrt{\delta_M^2 + c}\bigr),$$
$$\frac{1}{2\mu_M}\bigl(-\delta_M + \sqrt{\delta_M^2 + c}\bigr) = -1 = \frac{1}{2\lambda_M}\bigl(-\delta_M - \sqrt{\delta_M^2 + c}\bigr),$$

considering the condition $|\rho| < 1$, we get the assertion. QED

As a consequence of this theorem, the map $\Phi : \mathcal{E}_{II}\bigl(T_\ell(r)\bigr) \to \mathcal{M}$ has a different property between on $\mathcal{E}_{II}\bigl(T_\ell(r)\bigr)$ and on other part.

Corollary 1. *The image of the restricted map* $\Phi : \mathcal{E}_{II}\big(T_\ell(r)\big) \to \mathcal{M}$ *is* $[\sqrt{|c|},\infty)\times\{1\}$*. Onto each point of the subset* $(\sqrt{|c|},\infty)\times\{1\}$ *in* $\mathcal{M}$*, it is infinite-to-one, and onto the point* $(\sqrt{|c|},1)$ *in* $\mathcal{M}$ *it is one-to-one.*

In the second, we classify circular trajectories by their principal torsions.

Theorem 3. *Let* $M = T_\ell(r)$ *be a real hypersurface of type* (A_2) *in* $\mathbb{C}H^n(c)$*. We take an arbitrary nonzero constant* κ *and a constant* ω *with* $0 \le \omega \le 1$*. Then the number of congruence classes of non-geodesic trajectories for* $\mathbb{F}_\kappa$ *of principal torsion* ω *whose extrinsic shapes are circles of geodesic curvature* $|\kappa|$ *is as follows.*

(1) *When* $|\kappa| < |c|$*, it is* 0 *for every* ω*.*

(2) *When* $|c| \le |\kappa| < \sqrt{|c|}\coth\big(\sqrt{|c|}\,r\big)$*,*

  (a) *if* $0 \le \omega^2 < |c|^{-1}(\kappa^2+c)\sinh^2\big(\sqrt{|c|}\,r/2\big)$*, it is* 2*,*
  (b) *if* $\omega^2 = |c|^{-1}(\kappa^2+c)\sinh^2\big(\sqrt{|c|}\,r/2\big)$*, it is* 1*,*
  (c) *otherwise, it is* 0*.*

(3) *When* $|\kappa| = \sqrt{|c|}\coth\big(\sqrt{|c|}\,r\big)$*,*

  (a) *if* $\omega = 0$ *or if* $\omega = 1/\{2\cosh\big(\sqrt{|c|}\,r/2\big)\}$*, it is* 1*,*
  (b) *if* $0 < \omega < 1/\{2\cosh\big(\sqrt{|c|}\,r/2\big)\}$*, it is* 2*,*
  (c) *otherwise, it is* 0*.*

(4) *When* $\sqrt{|c|}\coth\big(\sqrt{|c|}\,r\big) < |\kappa| < \sqrt{|c|}\coth^2\big(\frac{\sqrt{|c|}\,r}{2}\big)\Big/\sqrt{2\coth^2\big(\frac{\sqrt{|c|}\,r}{2}\big)-1}$*,*

  (a) *if* $0 \le \omega^2 < 1 - |c|^{-1}\big(|\kappa| - \sqrt{\kappa^2+c}\big)^2\coth^2\big(\sqrt{|c|}\,r/2\big)$ *or if* $\omega^2 = |c|^{-1}(\kappa^2+c)\sinh^2\big(\sqrt{|c|}\,r/2\big)$*, it is* 1*,*
  (b) *if* $1-|c|^{-1}\big(|\kappa|-\sqrt{\kappa^2+c}\big)^2\coth^2\big(\frac{\sqrt{|c|}\,r}{2}\big) \le \omega^2 < |c|^{-1}(\kappa^2+c)\sinh^2\big(\frac{\sqrt{|c|}\,r}{2}\big)$*, it is* 2*,*
  (c) *otherewise, it is* 0*.*

(5) *When* $|\kappa| \ge \sqrt{|c|}\coth^2\big(\frac{\sqrt{|c|}\,r}{2}\big)\Big/\sqrt{2\coth^2\big(\frac{\sqrt{|c|}\,r}{2}\big)-1}$*,*

  (a) *if* $0 \le \omega^2 \le 1 - |c|^{-1}\big(|\kappa| - \sqrt{\kappa^2+c}\big)^2\coth^2\big(\sqrt{|c|}\,r/2\big)$*, it is* 1*,*
  (b) *otherewise, it is* 0*.*

Proof. As we see in the proof of Theorem 2 we only need to consider the second case in Proposition 2. We take a function $f_\kappa(\rho) = (-\lambda_M\rho^2 + \kappa\rho - \mu_M)/(\lambda_M - \mu_M)$, and study the crossing points of the two graphs of functions $y = f_\kappa(\rho)$ and $y = \omega^2$ in the domain $\{(\rho, y) \in \mathbb{R}^2 \mid |\rho| < 1,\ \rho^2 + y \le 1,\ y \ge 0\}$. By Proposition 1, we are enough to study the case $\kappa > 0$.

When $0 < \kappa < \sqrt{|c|}$, the quadratic function $y = f_\kappa(\rho)$ is negative. When $\kappa \ge \sqrt{|c|}$, we need to study in the interval

$$\frac{1}{2\lambda_M}\left(\kappa - \sqrt{\kappa^2 + c}\right) \le \rho \le \min\left\{\frac{1}{2\lambda_M}\left(\kappa + \sqrt{\kappa^2 + c}\right), \frac{1}{2\mu_M}\left(\kappa - \sqrt{\kappa^2 + c}\right)\right\},$$

because we need $0 \le f_\kappa(\rho) \le 1 - \rho^2$. We note that $\kappa + \sqrt{\kappa^2 + c}$ and $\kappa - \sqrt{\kappa^2 + c}$ are monotone increasing and monotone decreasing with erspect to κ respectively, and that they coincide with each other when $\kappa = \delta_M$. Hence, the graph of the function $y = f_\kappa(\rho)$ passes through the point $(1, 0)$ when $\kappa = \delta_M$. We also note that the vertex $\left(\frac{\kappa}{2\lambda_M}, \frac{\kappa^2+c}{4\lambda_M(\lambda_M-\mu_M)}\right)$ of the graph of the function $y = f_\kappa(\rho)$ lies on the graph of the function $y = 1 - \rho^2$ when $\kappa^2 = 4\lambda_M^3/(2\lambda_M - \mu_M)$ $(> \delta_M^2)$. Therefore, we study the cases i) $\sqrt{|c|} \le \kappa < \delta_M$, ii) $\kappa = \delta_M$, iii) $\delta_M < \kappa < 2\sqrt{\lambda_M^3/(2\lambda_M - \mu_M)}$ and iv) otherwise. Since we have

$$f_\kappa\big((\kappa - \sqrt{\kappa^2 + c}\,)/(2\mu_M)\big) = 1 - (\kappa - \sqrt{\kappa^2 + c}\,)^2/(4\mu_M^2)$$

when $\kappa \ge \delta_M$, we can get the conclusion by considering the graph of the function $y = f_\kappa(\rho)$. QED

Remark 1. Our proof shows that the structure torsions of extrinsic circular trajectories in Theorem 3 satisfy as follows.

1) In the cases (2-a) and (3-b), two sturucture torsions ρ_-, ρ_+ satisfy $\left(\kappa - \sqrt{\kappa^2 + c}\,\right)/(2\lambda_M) < \rho_- < \kappa/(2\lambda_M) < \rho_+ < \left(\kappa + \sqrt{\kappa^2 + c}\,\right)/(2\lambda_M)$.

2) In the case (4-b), they satisfy $\left(|\kappa| - \sqrt{\kappa^2 + c}\,\right)/(2\lambda_M) < |\rho_-| < |\kappa|/(2\lambda_M) < |\rho_+| < \left(|\kappa| - \sqrt{\kappa^2 + c}\,\right)/(2\mu_M)$.

3) In the case (2-b), the structure torsion ρ satisfies $\left(|\kappa| - \sqrt{\kappa^2 + c}\,\right)/(2\lambda_M) < |\rho| < |\kappa|/(2\lambda_M)$.

4) In the case (5-a), the structure torsion ρ satisfies $\left(|\kappa| - \sqrt{\kappa^2 + c}\,\right)/(2\lambda_M) < |\rho| \le \left(|\kappa| - \sqrt{\kappa^2 + c}\,\right)/(2\mu_M)$.

We here give some figures on $\mathcal{E}_{II}\big(T_\ell(r_0)\big)$ in the moduli space $\mathcal{F}_\kappa\big(T_\ell(r_0)\big)$ of trajectories for $\mathbb{F}_\kappa$, where $c = -4$ and $\coth r_0 = 2$. In this case, we have

$\sqrt{|c|}\coth\big(\sqrt{|c|}\,r_0\big) = 5/2$. In those figures, points are as follows:

$$\mathrm{A} = \left(\frac{\kappa - \sqrt{\kappa^2-4}}{4}, 0\right), \quad \mathrm{B} = \left(\frac{\kappa + \sqrt{\kappa^2-4}}{4}, 0\right), \quad \mathrm{C} = \left(\frac{\kappa}{4}, \sqrt{\frac{\kappa^2-4}{12}}\right),$$
$$\mathrm{D} = \left(\kappa + \sqrt{\kappa^2-4}, \sqrt{1-\big(\kappa - \sqrt{\kappa^2-4}\,\big)^2}\right).$$

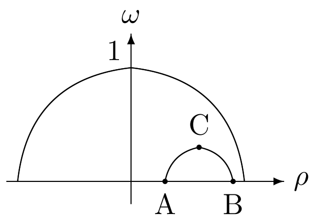

Figure 1. $\kappa = 7/3$

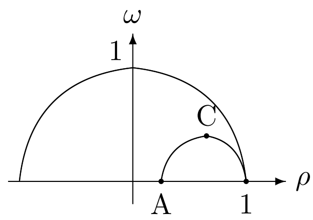

Figure 2. $\kappa = 5/2$

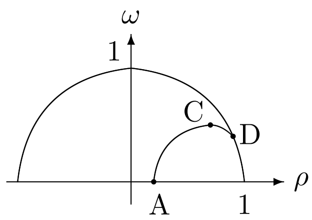

Figure 3. $\kappa = 11/4$

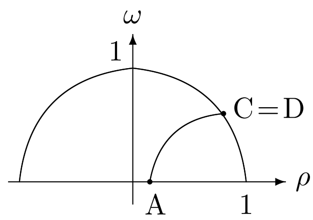

Figure 4. $\kappa = 8/\sqrt{7}$

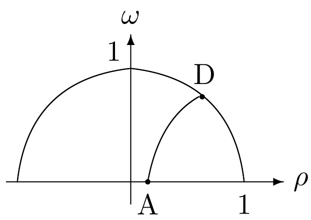

Figure 5. $\kappa = 4$

Next, we study the range of strengths of Sasakian magnetic fields having circular trajectories. By the proof of Theorem 2, particularly by the inequalities (4.1) and (4.2), one can easily obtain the following.

Proposition 4. *Let* $M = T_\ell(r)$ *be a real hypersurface of type* (A_2) *in* $\mathbb{C}H^n(c)$. *We take a constant* ρ *with* $|\rho| < 1$. *Then there is one congruence* *class of non-geodesic trajectories for $\mathbb{F}_\kappa$ of structure torsion ρ whose extrinsic shapes are circles of geodesic curvature $|\kappa|$ if and only if either $\rho > 0$ and*

$$\sqrt{|c|}\Big(\rho\coth\frac{\sqrt{|c|}\,r}{2}+\rho^{-1}\tanh\frac{\sqrt{|c|}\,r}{2}\Big)\le\kappa \le\sqrt{|c|}\Big(\rho^{-1}\coth\frac{\sqrt{|c|}\,r}{2}+\rho\tanh\frac{\sqrt{|c|}\,r}{2}\Big),$$

or $\rho < 0$ and

$$\sqrt{|c|}\Big(\rho^{-1}\coth\frac{\sqrt{|c|}\,r}{2}+\rho\tanh\frac{\sqrt{|c|}\,r}{2}\Big)\le\kappa \le\sqrt{|c|}\Big(\rho\coth\frac{\sqrt{|c|}\,r}{2}+\rho^{-1}\tanh\frac{\sqrt{|c|}\,r}{2}\Big).$$

Otherwise, there are no congruence classes of such trajectories.

Corresponding to the above we have the following.

Proposition 5. *Let $M = T_\ell(r)$ be a real hypersurface of type* (A$_2$) *in $\mathbb{C}H^n(c)$. We take a constant ω with $0 \le \omega \le 1$. Then the number of congruence classes of non-geodesic trajectories for $\mathbb{F}_\kappa$ of principal torsion ω whose extrinsic shapes are circles of geodesic curvature $|\kappa|$ is as follows.*

(1) *When $\omega = 0$,*

   (a) *if $0 < |\kappa| < \sqrt{|c|}$, it is* 0,
   (b) *if $|\kappa| = \sqrt{|c|}$ or if $|\kappa| \ge \sqrt{|c|}\coth(\sqrt{|c|}r)$, it is* 1,
   (c) *if $\sqrt{|c|} < |\kappa| < \sqrt{|c|}\coth(\sqrt{|c|}r)$, it is* 2.

(2) *When $0 < \omega < 1/\sqrt{\cosh^2(\sqrt{|c|}r/2)+1}$,*

   (a) *if $0 < |\kappa| < \sqrt{|c|\{(\omega^2/\sinh^2(\sqrt{|c|}r/2))+1\}}$, it is* 0,
   (b) *if $|\kappa| = \sqrt{|c|\{(\omega^2/\sinh^2(\sqrt{|c|}r/2))+1\}}$ or if $|\kappa| > \sqrt{|c|}\{\coth(\sqrt{|c|}r)-(\omega^2/2)\tanh(\sqrt{|c|}r/2)\}/\sqrt{1-\omega^2}$, it is* 1,
   (c) *otherwise, it is* 2.

(3) *When $1/\sqrt{\cosh^2(\sqrt{|c|}r/2)+1} \le \omega < 1$,*

   (a) *if $|\kappa| \ge \sqrt{|c|}\{\coth(\sqrt{|c|}r)-(\omega^2/2)\tanh(\sqrt{|c|}r/2)\}/\sqrt{1-\omega^2}$, it is* 1,
   (b) *otherwise, it is* 0.

(4) *When* $\omega = 1$*, it is* 0.

Proof. We only need to study trajectories satisfying the second condition in Proposition 2. If $\rho_\gamma = 0$, then we have $(\lambda_M - \mu_M)\omega^2 + \mu_M = 0$, which is a contradiction. Hence we see $\rho_\gamma \neq 0$. First, we suppose $0 < \omega < 1$. We study the function $y = g_\omega(\rho) = \{\lambda_M\omega^2 + \mu_M(1-\omega^2)\}\rho^{-1} + \lambda_M\rho$ on the interval $(0, \sqrt{1-\omega^2}\,]$, and consider crossing points of its graph and the line $y = \kappa$. We put

$$\rho_0 = \sqrt{\omega^2 + (\mu_M/\lambda_M)(1-\omega^2)} = \frac{\sqrt{\omega^2 + \sinh^2(\sqrt{|c|}r/2)}}{\cosh(\sqrt{|c|}r/2)}.$$

When $\omega^2 < 1/\big(\coth^2(\sqrt{|c|}r/2)+1\big)$, we have $\rho_0 < \sqrt{1-\omega^2}$, and find that g_ω is monotone decreasing on the interval $(0, \rho_0)$ and is monotone increasing on the interval $(\rho_0, \sqrt{1-\omega^2}\,]$. When $\omega^2 \geq 1/\big(\coth^2(\sqrt{|c|}r/2)+1\big)$, we find that $\rho_0 \geq \sqrt{1-\omega^2}$ and hence that g_ω is monotone decreasing on the interval $(0, \sqrt{1-\omega^2}\,]$. As we have $\lim_{\rho\downarrow 0} g_\omega(\rho) = \infty$ and

$$g_\omega(\sqrt{1-\omega^2}) = \frac{\sqrt{|c|}\big\{\coth(\sqrt{|c|}r) - (\omega^2/2)\tanh(\sqrt{|c|}r/2)\big\}}{\sqrt{1-\omega^2}},$$

$$g_\omega(\rho_0) = \frac{\sqrt{|c|\big\{\omega^2 + \sinh^2(\sqrt{|c|}r/2)\big\}}}{\sinh(\sqrt{|c|}r/2)},$$

we get the consequence for the case $0 < \omega < 1$. When $\omega = 0$, we study g_ω on $(0, 1)$, that is, the function g_ω can not take the value at $\sqrt{1-\omega^2}$. When $\omega = 1$, we have $\rho_\gamma = 0$. Thus, we get the conclusion. QED

Extrinsic shapes of trajectories satisfying the second condition in Proposition 2 have complex torsions ± 1. Hence, these extrinsic shapes are trajectories for Kähler magnetic fields on $\mathbb{C}H^n(c)$. Here, a Kähler magnetic fields are constant multiples of the Kähler form $\mathbb{B}_J$. A trajectory σ for a Kähler magnetic field $\mathbb{B}_\kappa = \kappa\mathbb{B}_J$ on $\mathbb{C}H^n(c)$, which is a smooth curve parameterized by its arclength and that satisfies $\widetilde{\nabla}_{\dot\sigma}\dot\sigma = \kappa J\dot\sigma$, has the following properties (see [1]):

1) Every trajectory lies on a totally geodesic $\mathbb{C}H^1$;

2) When $|\kappa| > \sqrt{|c|}$, it is closed of length $2\pi/\sqrt{\kappa^2 + c}$;

3) When $|\kappa| \leq \sqrt{|c|}$, it is unbounded and has limit points $\lim_{t\to-\infty}\sigma(t)$, $\lim_{t\to\infty}\sigma(t)$ in the ideal boundary of $\mathbb{C}H^n(c)$;

4) When $\kappa = \pm\sqrt{|c|}$, it has single limit point, and other unbounded trajectories have two distinct limit points.

Corollary 1 shows that unbounded trajectories for Kähler magnetic fields having two distinct limit points can not be extrinsic shapes of trajectories for Sasakian magnetic fields on real hypersurfaces of type (A_2). Since some extrinsic circular trajectories satisfying the third condition in Proposition 2 are unbounded (see Figure 1 in [4]), we may say that this point is remarkable.

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