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Recognizing Euclidean Space Forms with Minimal Fundamental Tetrahedra

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Abstract. We completely recognize the topological structure of the ten compact euclidean space forms with special minimal tetrahedra, constructed by face pairings in nice papers of Molnár [8-9]. From these polyhedral descriptions we derive special presentations with two generators for the fundamental groups of the considered manifolds. Our proofs also show that such group presentations completely characterize the euclidean space forms among closed connected 3-manifolds. The results have also didactical importance.

Keywords: Finitely generated group, spine, Seifert manifold, torus bundle, space form, polyhedral representation, Schlegel diagram

MSC 2020 classification: primary 57M12, secondary 57M25

1 Introduction and main results

The face identification procedure is a very classical method for constructing closed connected 3-manifolds and determining their principal topological and algebraic invariants. See, for example, [11], [13], and [16]. In fact, each closed connected 3-manifold can be combinatorially represented as a quotient space of a polyhedral 3-ball via pairwise identifications of its boundary faces. The interior of this 3-ball becomes an open 3-ball in the quotient space. Its triangulated boundary becomes an embedded two-dimensional polyhedron, which is a *spine* of the quotient manifold. This means that the manifold minus an open 3-cell

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collapses onto such a polyhedron.

Molnár [8] constructed a surprising (simply connected) fundamental domain to present the compact euclidean space form, denoted there $E^3/P3_1$. In a sense this domain is a "tetrahedron" as stated by the cited author. It has two faces without common point, which are identified by a translation p, and two faces with two edges in common, which are identified by a 3_1 screw motion. The face pairing transformations generate the crystallographic group (i.e., *the space group*) $G = P3_1$, which acts on the euclidean space E^3 freely (without fixed point). The identified tetrahedron represents the orbit space $E^3/P3_1$, which is a complete connected 3-dimensional compact Riemannian manifold of zero sectional curvature (in short, an *euclidean space form*).

As stated by Molnar in [9, p.429], it is natural to ask which euclidean space forms can be presented by such minimal tetrahedra. These space forms have fundamental groups G with two generators identifying the faces of the above-mentioned tetrahedron. It is shown in the quoted paper that only the space groups $P3_1$, $P4_1$, $P6_1$, Bb, and $Pna2_1$ (notations from [9]) can play the role of G. The corresponding space forms have been presented by special minimal fundamental tetrahedra by Molnar in [8]. In the next section we depict the polyhedral representations by using the symbol Schlegel diagrams taken out from [9].

Our first theorem gives the topological classification of the euclidean space forms mentioned above. To state the result, we first recall the definition of Seifert manifolds, as given in [13] and [14]. Let S be a closed connected surface, k a natural number, b an integer, and $(p_1, q_1), \ldots, (p_k, q_k)$ coprime integer pairs with $|p_i| \geq 2$, for $i = 1, \ldots, k$. Then a (closed connected) *Seifert manifold* represented by the Seifert invariants

$$\Sigma = (S \,|\, b\,(p_1, q_1) \cdots (p_k, q_k))$$

is defined as follows. Let F be the surface S minus $k+1$ disjoint open 2-discs and let N be the orientable $\mathbb{S}^1$-bundle over F. Give N any orientation, pick a section σ of N and choose positive homology bases (μ_0, λ_0), (μ_1, λ_1), $\ldots$, (μ_k, λ_k) on the components of ∂N arising from the punctures of S, with $\mu_i \subset \partial\sigma$ and a fibre as λ_i, for every $i = 0, \ldots, k$. Then Σ is the Dehn filling of N along the simple curves $p_1\mu_1 + q_1\lambda_1, \ldots, p_k\mu_k + q_k\lambda_k$, and $\mu_0 + b\lambda_0$. In our case, the base surface S will be the oriented standard 2-sphere $\mathbb{S}^2$, the annulus A, the real projective plane $\mathbb{R}P^2$, the Klein bottle K, or the torus $T = \mathbb{S}^1 \times \mathbb{S}^1$. We also denote by T_X the torus bundle with monodromy given by the matrix $X \in \mathrm{GL}(2;\mathbb{Z})$. Recall that T_X is nonorientable if and only if $\det X = -1$. If $\pi_1(K) =< x_1, x_2 : x_1^2 x_2^2 = 1 >$, let us denote by $K_{\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)}$ the (nonorientable) Klein bottle bundle over the circle with characteristic map

$$x_1 \to x_1^a x_2^b \qquad\qquad x_2 \to x_1^c x_2^d.$$

Theorem 1. *The euclidean space forms with special minimal tetrahedra constructed by Molnár in* [9] *are homeomorphic to the following fibered manifolds:*

$$\begin{aligned}
E^3/P3_1 &\cong (\mathbb{S}^2 \mid -1\ (3,1)\ (3,1)\ (3,1)) = T_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)} \\
E^3/P4_1 &\cong (\mathbb{S}^2 \mid -1\ (2,1)\ (4,1)\ (4,1)) = T_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)} \\
E^3/P6_1 &\cong (\mathbb{S}^2 \mid -1\ (2,1)\ (3,1)\ (6,1)) = T_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)} \\
E^3/Bb &\cong (A \times \mathbb{S}^1)_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)} = T_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} \cong K_{\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)} \\
E^3/Pna2_1 &\cong K_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)}
\end{aligned}$$

Such euclidean space forms are completely characterized among closed connected 3*-manifolds by special presentations with two generators for their fundamental groups, which correspond to spines.*

The remaining five euclidean space forms have also been described by Molnár in [8] by means of a geometric method presenting each of them by a fundamental domain. The following result recognizes the topological structure of such manifolds.

Theorem 2. *The remaining five euclidean space forms constructed by Molnár in* [8] *by means of polyhedral schemata are homeomorphic to the following fibered manifolds:*

$$E^3/P1 \cong (\,T \mid 0\,) \cong \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$$

$$E^3/Pca2_1 \cong K_{\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)}$$

$$E^3/P2_1 \cong (\mathbb{S}^2 \mid -2\ (2,1)\ (2,1)\ (2,1)\ (2,1)) \cong T_{\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)}$$

$$E^3/Pb \cong (A \times \mathbb{S}^1)_{\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)} = T_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)} \cong K \times \mathbb{S}^1$$

$$E^3/P2_12_12_1 \cong (\mathbb{R}P^2 \mid -1\ (2,1)\ (2,1)).$$

Such euclidean space forms are completely characterized among closed connected 3-manifolds by special presentations with two generators for their fundamental groups, which correspond to spines.

For the proof, we need the following results, proved in [18] (see also [10]) and [1, p.215], respectively.

Theorem 3. *Let M be a closed connected 3-manifold having a spine associated to the finite group presentation*

$$< x, y \, : \, x^m \, = \, (x^p \, y^q)^\ell \, = \, y^n \, >$$

where $|m|$, $|n|$, $|\ell| > 1$, and (m, p) and (n, q) are coprime integer pairs. Then M is homeomorphic to the orientable Seifert space defined by the Seifert invariants

$$(\,\mathbb{S}^2 \,| \, -1 \; (m, p) \; (n, q) \; (\ell, \ell - 1)).$$

Theorem 4. *Let M and M' be closed connected orientable prime 3-manifolds with isomorphic fundamental groups. If M and M' are not lens spaces, then M and M' are homeomorphic.*

2 Proof of Theorem 1

(2.1) *The euclidean space form $E^3/P3_1$.* Set $M_1 = E^3/P3_1$. A polyhedral representation of M_1 is depicted in Figure 1.a, which shows the Schlegel diagram for a combinatorial "tetrahedron as given in [9, Figure 7, p.450] with $s = x$ and $p = y$. The two generators x and y can be expressed by the face pairing generators labeled by the same symbols in Figure 1.a. There are two edge equivalence classes indicated by single and double arrows in the faces. The single arrow class provides the relation

$$yxyxyx^{-2} \, = \, 1. \tag{2.1}$$

The double arrow class provides the relation

$$x^2yx^{-1}yx^{-1}y \, = \, 1. \tag{2.2}$$

See [9, Figure 7], where this presentation defines the group denoted by $P3_1$, that is, $\pi_1(M_1) \cong P3_1$. Thus the fundamental group $\pi_1(M_1)$ has a finite presentation with generators x and y and relations (2.1) and (2.2). By construction, such a presentation is geometric, that is, it corresponds to a spine (or, equivalently, it arises from a Heegaard diagram of genus 2) of the manifold M_1. See [15] and [17]. The first integral homology group of M_1 (Abelianization of $P3_1$) is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}$.

Setting $a = xy$ and $b = x$ (with inverse relations $x = b$ and $y = b^{-1}a$), relation (2.1) becomes $a^3 = b^3$. Relation (2.2) becomes

$$bab^{-2}ab^{-2}a = 1$$

whose inverse relation is

$$a^{-1}b^2a^{-1}b^2a^{-1} = b$$

or, equivalently, $(a^{-1}b^2)^3 = b^3$. Then $\pi_1(M_1)$ admits a finite presentation with generators a and b and relations

$$a^3 = (a^{-1}b^2)^3 = b^3. \tag{2.3}$$

Such a presentation is also geometric. In fact, it is easily seen that the above Tietze transformations on the group presentations correspond to elementary Singer moves on the associated Heegaard diagrams of genus 2 representing the same manifold M_1. See [15] and [17]. Now we can apply Theorem 3 with $m = n = \ell = 3$, $p = -1$ and $q = 2$ by using (2.3). It follows that M_1 is homeomorphic to the Seifert manifold defined by the Seifert invariants

$$(\mathbb{S}^2 \mid -1\ (3,2)\ (3,-1)\ (3,2)).$$

But this space can also be presented by the Seifert invariants

$$(\mathbb{S}^2 \mid -1\ (3,1)\ (3,1)\ (3,1))$$

by using the surgery instructions described in [13, p.147]. See also [13, p.155]. It is known that this euclidean Seifert manifold is homeomorphic to the torus bundle with monodromy given by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. See [14, Theorem 2, p.137].

(2.2) *The euclidean space form $E^3/P4_1$.* Set $M_2 = E^3/P4_1$. A Schlegel diagram for M_2 is drawn in Figure 1.b. This is precisely Figure 2 from [9, p.441], where $s = x$ and $p = y$. As above, we have two edge equivalence classes arising from the matching of single resp. double arrows in the faces. The single arrow class provides the relation

$$yxyxyx^{-1}y^{-1}x^{-1} = 1. \tag{2.4}$$

The double arrow class provides the relation

$$yx^{-2}yx^2 = 1 \tag{2.5}$$

or, equivalently,

$$x^4 = (x^2y)^2. \tag{2.6}$$

This geometric presentation defines the group denoted by $P4_1$ in [9, Figure 2, p.441], that is, $\pi_1(M_2) \cong P4_1$. The first integral homology group of M_2 (Abelianization of $P4_1$) is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}$. Multiplying (2.4) on the left by x yields

$$xyxyxy = x^2yx. \tag{2.7}$$

Substituting $x^2y = y^{-1}x^2$ from (2.5) into the right side of (2.7), we get

$$(xy)^3 = y^{-1}x^3$$

or, equivalently,

$$(xy)^4 = x^4. \tag{2.8}$$

Setting $a = x^2y$ and $b = xy$ (with inverse relations $x = ab^{-1}$ and $y = ba^{-1}b$), relation (2.6) becomes $(ab^{-1})^4 = a^2$. Relation (2.8) becomes $b^4 = (ab^{-1})^4$. Then $\pi_1(M_2) \cong P4_1$ admits a finite presentation with generators a and b and relations

$$a^2 = (ab^{-1})^4 = b^4. \tag{2.9}$$

Such a presentation is also geometric because the above Tietze transformations on the group presentations can be realized by Singer moves on Heegaard diagrams of genus 2, representing the same manifold M_2. So the resulting group presentation from (2.9) corresponds to a spine of M_2. We can now apply Theorem 3 with $m = 2$, $n = \ell = 4$, $p = 1$ and $q = -1$. It follows that M_2 is homeomorphic to the Seifert manifold defined by the Seifert invariants

$$(\,\mathbb{S}^2 \mid\, -1\ (2,1)\ (4,-1)\ (4,3)).$$

But this Seifert space is also represented by the invariants

$$(\,\mathbb{S}^2 \mid\, -1\ (2,1)\ (4,1)\ (4,1))$$

by using the surgery instructions described in [13, p.147]. See also [13, p.155]. It is known that this euclidean Seifert manifold is homeomorphic to the torus bundle with monodromy given by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. See [14, Theorem 2, p.137].

(2.3) *The euclidean space form* $E^3/P6_1$. Set $M_3 = E^3/P6_1$. A Schlegel diagram for M_3 is depicted in Figure 1.c; this was first given in Figure 8 from

[9, p.169], where $s = x$ and $p = y$. There are again two edge equivalence classes which provide the relations

$$yx^{-1}y^{-1}x^{-1}yx^2 = 1 \tag{2.10}$$

and

$$yxy^{-1}xyx^{-2} = 1. \tag{2.11}$$

Setting $a = yx$ and $b = yx^2$ (with inverse relations $x = a^{-1}b$ and $y = ab^{-1}a$), relation (2.10) becomes $ab^{-2}ab = 1$, hence $(ab)^2 = b^3$. Relation (2.11) becomes

$$ba^{-2}bab^{-1}ab^{-1}ab^{-1}a = 1. \tag{2.12}$$

Substituting $ba = a^{-1}b^2$ from the first relation into (2.12) yields

$$ba^{-2}a^{-1}b^2b^{-1}ab^{-1}ab^{-1}a = 1$$

hence

$$ba^{-3}bab^{-1}ab^{-1}a = 1.$$

We are going on like this. Substituting again $ba = a^{-1}b^2$ into the last relation, we get

$$ba^{-4}bab^{-1}a = 1.$$

Reasoning as above, the last relation becomes $ba^{-5}ba = 1$, and then

$$ba^{-5}a^{-1}b^2 = 1,$$

or, equivalently, $a^{-6}b^3 = 1$. Thus $\pi_1(M_3) \cong P6_1$ admits a geometric presentation with generators a and b and relations

$$a^6 \;=\; (ab)^2 \;=\; b^3, \tag{2.13}$$

which corresponds to a spine of M_3. The first integral homology group of M_3 (Abelianization of $P6_1$) is isomorphic to $\mathbb{Z}$. We now apply Theorem 3 with $m = 6$, $n = 3$, $\ell = 2$ and $p = q = 1$. It follows that M_3 is homeomorphic to the Seifert manifold defined by the invariants

$$(\,\mathbb{S}^2 \mid\; -1\ (3,1)\ (6,1)\ (2,1)).$$

It is known that this Seifert space is homeomorphic to the torus bundle with monodromy given by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

(2.4) *The euclidean space form* E^3/Bb. Set $M_4 = E^3/Bb$. A Schlegel diagram for M_4 is reported in Figure 2.a from [9, Figure 3, p.443], where $b = x$ and $p = y$. There are two edge equivalence classes which provide the relations

$$yx^{-1}yxy^{-1}x^{-1}y^{-1}x \;=\; 1 \tag{2.14}$$

and

$$x^2yx^{-2}y^{-1} \;=\; 1. \tag{2.15}$$

Then $\pi_1(M_4) \cong Bb$ has a presentation with generators x and y and relations (2.14) and (2.15).

Let M_4' denote the torus bundle with monodromy given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We prove that M_4 and M_4' have isomorphic fundamental groups. The fundamental group $\pi_1(T)$ of the torus $T = \mathbb{S}^1 \times \mathbb{S}^1$ has the usual presentation with generators x_1 and x_2 and relation $x_1x_2x_1^{-1}x_2^{-1} \;=\; 1$. A presentation for $\pi_1(M_4')$ can be obtained from that of $\pi_1(T)$ by adding the relations induced by the monodromy map φ_* from $H_1(T) \cong \pi_1(T) \cong \mathbb{Z}\oplus\mathbb{Z}$ onto itself with associated matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with respect to the ordered basis (x_1, x_2). More precisely, we have

$$\varphi_*(x_1) = \, x_3^{-1}\, x_1\, x_3 \;=\; x_1^0\, x_2^1 \;=\; x_2$$

$$\varphi_*(x_2) = \, x_3^{-1}\, x_2\, x_3 \;=\; x_1^1\, x_2^0 \;=\; x_1$$

where x_3 represents the fibre of the bundle. Setting $x_1 = y$, $x_2 = z$ and $x_3 = x$, the fundamental group of M_4' has a presentation with generators x, y, z and relations

$$yzy^{-1}z^{-1} \;=\; 1 \tag{2.16}$$

$$x^{-1}yx \;=\; z \tag{2.17}$$

and

$$x^{-1}zx \;=\; y. \tag{2.18}$$

Eliminating the generator z from (2.17) and substituting its expression in terms of x and y into (2.16) gives

$$yx^{-1}yxy^{-1}x^{-1}y^{-1}x \;=\; 1,$$

which is relation (2.14). Substituting (2.17) into (2.18) yields

$$x^{-1}x^{-1}yxx \;=\; y$$

hence

$$x^{-2}yx^2y^{-1} = 1$$

or, equivalently,

$$x^2y^{-1}x^{-2}y = 1,$$

which is relation (2.15). This proves the group isomorphisms

$$\pi_1(M_4) \cong \pi_1(M_4^{'}) \cong Bb.$$

The manifolds M_4 and $M_4^{'}$ are not lens spaces as the integral first homology group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Furthermore, M_4 and $M_4^{'}$ are prime because, otherwise, the fundamental group would be a nontrivial (nonabelian) free product. Such a group has a trivial centre. But the centre of Bb is not trivial as it contains x^2 from (2.15). Of course, M_4 and $M_4^{'}$ are closed, connected nonorientable fibered manifolds. Thus [14, Theorem 4] implies that M_4 and $M_4^{'}$ are homeomorphic. Furthermore, it is known that $M_4^{'}$ is also homeomorphic to the annulus bundle with monodromy given by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, that is,

$$M_4^{'} \cong T_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} \cong (A \times \mathbb{S}^1)_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}.$$

The manifold $M_4^{'}$ is also the Klein bottle bundle over the circle with characteristic map $x_1 \to x_2^{-1}$ and $x_2 \to x_1^{-1}$, i.e., $M_4^{'} \cong K_{\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)}$.

(2.5) *The euclidean space form* $E^3/Pna2_1$. Set $M_5 = E^3/Pna2_1$. A Schlegel diagram for M_5 is reported in Figure 2.b from [9, Figure 5, p.446], where $a = x$ and $n = y$. There are two edge equivalence classes which provide the relations

$$yxyxy^{-1}xy^{-1}x = 1 \tag{2.19}$$

and

$$x^2yx^2y^{-1} = 1. \tag{2.20}$$

Then the fundamental group of M_5 has a geometric presentation with generators x and y and relations (2.19) and (2.20).

Setting $a = x^2y$ and $b = xy$ (with inverse relations $x = ab^{-1}$ and $y = ba^{-1}b$), relation (2.19) becomes

$$ba^{-1}bbab^{-1}b^{-1}ab^{-1}ab^{-1}b^{-1}ab^{-1}ab^{-1} = 1$$

hence

$$bab^{-2}ab^{-1}ab^{-2}a = 1. \tag{2.21}$$

Relation (2.20) becomes

$$aab^{-1}ab^{-1}b^{-1}ab^{-1} = 1$$

hence

$$a^2b^{-1}ab^{-2}ab^{-1} = 1. \tag{2.22}$$

Substituting $ab^{-2}a = ba^{-2}b$ from (2.22) into (2.21) gives

$$bab^{-2}ab^{-1}ba^{-2}b = 1$$

hence

$$b^2ab^{-2}a^{-1} = 1. \tag{2.23}$$

Substituting $ab^{-2} = b^{-2}a$ from (2.23) into (2.22) yields

$$a^2b^{-1}b^{-2}aab^{-1} = 1$$

hence

$$a^2b^{-3}a^2b^{-1} = 1$$

whose inverse relation is equivalent to

$$b^3a^{-2}ba^{-2} = 1. \tag{2.24}$$

Thus $\pi_1(M_5)$ also admits a group presentation with generators a and b and relations (2.23) and (2.24).

Let $M_5^{'}$ be the Klein bottle bundle over the circle with characteristic map $x_1 \to x_2$ and $x_2 \to x_1$, that is,

$$M_5^{'} = K_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)}.$$

We prove that M_5 and $M_5^{'}$ have isomorphic fundamental groups. A presentation for $\pi_1(M_5^{'})$ can be derived from [14, p.91]. See also [14, Theorem 4]. More precisely, if $\pi_1(K) =< x_1, x_2 \,:\, x_1^2x_2^2 = 1 >$, then $\pi_1(M_5^{'})$ has a group presentation with generators x_1, x_2 and h and relations

$$x_1hx_1^{-1} = h \tag{2.25}$$

$$x_2hx_2^{-1} = h^{-1} \tag{2.26}$$

and

$$x_1^2x_2^2 = h \tag{2.27}$$

where h represents the fibre of the nonorientable Seifert fibration.

Eliminating h from (2.27) and substituting its expression in terms of x_1 and x_2 into (2.25) gives

$$x_1 x_1^2 x_2^2 x_1^{-1} = x_1^2 x_2^2$$

hence

$$x_1 x_2^2 x_1^{-1} x_2^{-2} = 1$$

which is equivalent to relation (2.23) for $x_1 = a^{-1}$ and $x_2 = b$. Substituting (2.27) into (2.26) yields

$$x_2 x_1^2 x_2^2 x_2^{-1} = x_2^{-2} x_1^{-2}$$

hence

$$x_2^3 x_1^2 x_2 x_1^2 = 1$$

which is relation (2.24) for $x_1 = a^{-1}$ and $x_2 = b$. This proves the group isomorphisms

$$\pi_1(M_5) \cong \pi_1(M_5^{'}) \cong Pna2_1.$$

The manifolds M_5 and $M_5^{'}$ are not lens spaces as the integral first homology group is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}$. Furthermore, M_5 and $M_5^{'}$ are prime because the centre of $Pna2_1$ is not trivial. In fact, it contains x^2 from (2.20). Of course, M_5 and $M_5^{'}$ are closed, connected nonorientable Seifert manifolds. Thus [14, Theorem 4] implies that M_5 and $M_5^{'}$ are homeomorphic.

Remark 1. The above methods and the theory of spines also permit to recognize the topological structure of all the manifolds presented by minimal fundamental tetrahedra in [9, §5]. For example, the fundamental group of the manifold with Schlegel diagram in [9, Figure 10.f, p.455] has a geometric presentation with generators x and y and relations $x^2 y^{-1} x y^{-1} = 1$ and $x^2 y x y = 1$. Setting $a = xy$ and $b = x$ (with inverse relations $x = b$ and $y = b^{-1}a$), the above relations become $a^{-6}b^3 = 1$ and $a^2 b = 1$, respectively. This presentation also corresponds to a spine of the considered manifold. By [15], we see that the represented manifold is homeomorphic to the lens space $L(12, 1)$, having spherical geometry.

3 Proof of Theorem 2

(3.1) *The euclidean space form* $E^3/P1$. Set $M_6 = E^3/P1$. A Schlegel diagram of M_6 is depicted in [8, Figure 4, p.32], which induces a group presentation for $P1$ with generators p_i, $i = 1, 2, 3$, and relations $p_i p_j p_i^{-1} p_j^{-1} = 1$, for $i, j = 1, 2, 3$, $i \neq j$. Then $P1$ is isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Thus M_6 is homeomorphic to $T \times \mathbb{S}^1$ by Theorem 4.

(3.2) *The euclidean space form* $E^3/Pca2_1$. Set $M_7 = E^3/Pca2_1$. A Schlegel diagram for M_7 is drawn in [8, Figure 12, p.41], which induces a group presentation for $Pca2_1$ with generators a, p, c and relations

$$p\,a\,p\,a^{-1} = 1 \tag{3.1}$$

$$p\,c\,p^{-1}\,c^{-1} = 1 \tag{3.2}$$

and

$$a\,c\,a\,c^{-1} = 1. \tag{3.3}$$

Setting $x_1 = c^{-1}$, $x_2 = ac$ and $h = p$ (with inverse relations $a = x_2x_1$, $c = x_1^{-1}$ and $p = h$), relation (3.2) becomes

$$hx_1^{-1}h^{-1}x_1 = 1$$

hence

$$x_1hx_1^{-1} = h. \tag{3.4}$$

Relation (3.1) becomes

$$hx_2x_1hx_1^{-1}x_2^{-1} = 1$$

hence

$$hx_2hx_2^{-1} = 1$$

by using (3.4), or, equivalently,

$$x_2hx_2^{-1} = h^{-1}. \tag{3.5}$$

Relation (3.3) becomes

$$x_2x_1x_1^{-1}x_2x_1x_1 = 1$$

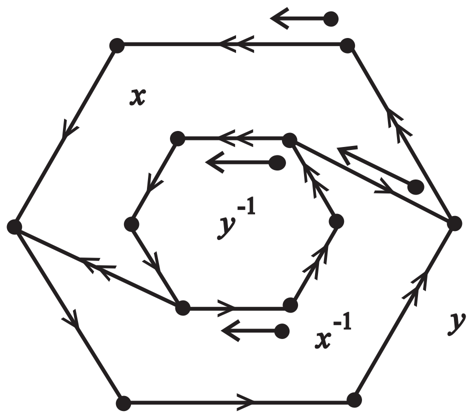
hence

$$x_1^2x_2^2 = 1. \tag{3.6}$$

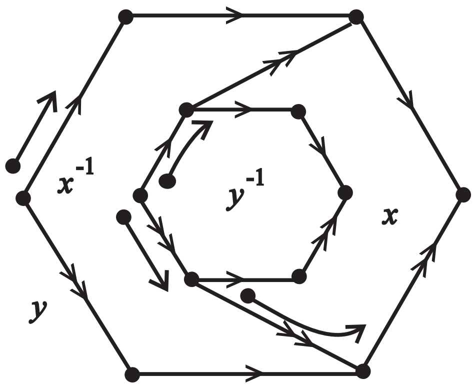
Let $M_7^{'}$ denote the Klein bottle bundle with characteristic map $x_1 \to x_1^{-1}$ and $x_2 \to x_2^{-1}$, that is,

$$M_7^{'} = K_{\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)}.$$

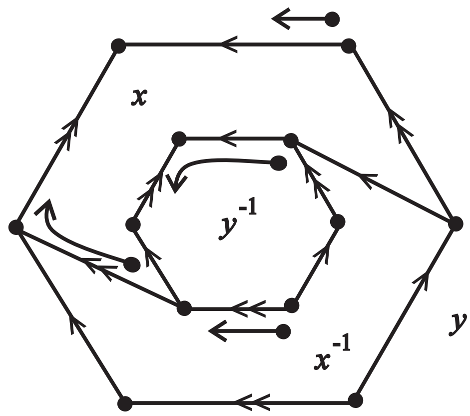
See [14, Theorem 4]. By [14, p.91] a presentation for $\pi_1(M_7^{'})$ has generators x_1, x_2, h and relations (3.4), (3.5) and (3.6). Then $\pi_1(M_7) \cong \pi_1(M_7^{'}) \cong Pca2_1$. The manifolds M_7 and $M_7^{'}$ are prime as $Pca2_1$ is a semidirect product (see [8, p.43]). Furthermore, M_7 and $M_7^{'}$ are not lens spaces as the integral first homology group is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$. It follows from [7, §9] that M_7 and $M_7^{'}$ are homeomorphic.



1.a. $M_1 = E^3/P3_1$

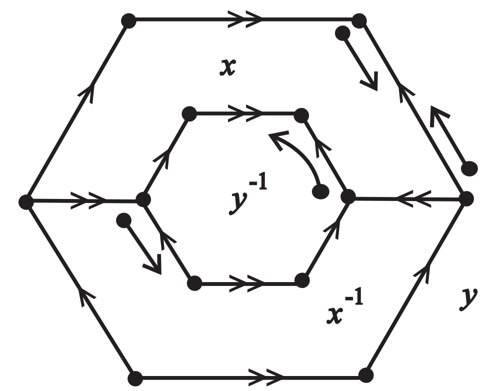


1.b. $M_2 = E^3/P4_1$

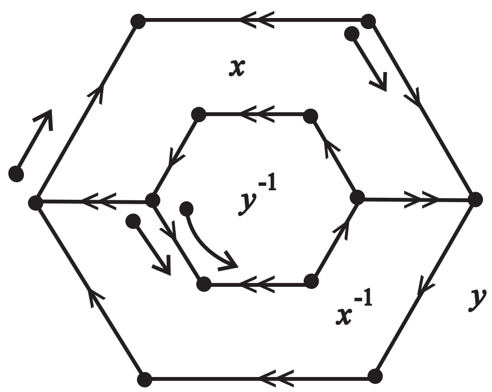


1.c. $M_3 = E^3/P6_1$

Figure 1: The Euclidean space forms presented by special tetrahedra, constructed by Molnár [9]



2.a. $M_4 = E^3/Bb$



2.b. $M_5 = E^3/Pna2_1$

Figure 2: The Euclidean space forms presented by special tetrahedra, constructed by Molnár [9] (continued)

(3.3) *The euclidean space form* $E^3/P2_1$. Set $M_8 = E^3/P2_1$. Schlegel diagrams for M_8 are depicted in [8, §5, Figures 5 and 6, p.33]. A group presentation for $P2_1$ from [8, Formula 4, p.34] has three generators s_i, $i = 0, 1, 2$, and relations

$$s_0^{-2}\, s_1^2 \;=\; 1 \tag{3.7}$$

$$s_0^{-2}\, s_2^2 \;=\; 1 \tag{3.8}$$

and

$$s_0^{-1}\, s_1\, s_2^{-1}\, s_0\, s_1^{-1}\, s_2 \;=\; 1. \tag{3.9}$$

Let $M_8^{'}$ denote the Seifert manifold defined by the invariants

$$(\mathbb{S}^2 \,|\, -2\ (2,1)\ (2,1)\ (2,1)\ (2,1)).$$

We prove that M_8 and $M_8^{'}$ have isomorphic fundamental groups. A presentation for $\pi_1(M_8^{'})$ can be derived from [14, p.91]. More precisely, it has a presentation with generators q_i, $i = 1, \ldots, 4$, and h and relations

$$q_j\, h \;=\; h\, q_j \qquad\qquad j = 1, \ldots, 4 \tag{3.10}$$

$$q_j^2\, h \;=\; 1 \qquad\qquad j = 1, \ldots, 4 \tag{3.11}$$

and

$$q_1\, q_2\, q_3\, q_4 \;=\; h^{-2}. \tag{3.12}$$

Eliminating $h = q_4^{-2}$ from (3.11) and $q_3 = q_2^{-1} q_1^{-1} h^{-2} q_4^{-1} = q_2^{-1} q_1^{-1} q_4^4 q_4^{-1} = q_2^{-1} q_1^{-1} q_4^3$ from (3.12) yields a presentation with generators q_1, q_2, q_4 and relations

$$q_1^2 \;=\; q_2^2 \tag{3.13}$$

$$q_1^2 \;=\; q_4^2 \tag{3.14}$$

and

$$(q_2^{-1}\, q_1^{-1}\, q_4^3)^2 \;=\; q_4^2. \tag{3.15}$$

Using (3.14), relation (3.15) becomes

$$(q_2^{-1}\, q_1^{-1}\, q_1^2\, q_4)^2 \;=\; q_4^2$$

hence

$$(q_2^{-1}\, q_1\, q_4)^2 \;=\; q_4^2$$

or, equivalently,

$$q_2^{-1}\, q_1\, q_4\, q_2^{-1}\, q_1 \;=\; q_4. \tag{3.16}$$

Substituting $q_2^{-1}\, q_1 \;=\; q_2\, q_1^{-1}$ from (3.13) into (3.16) gives

$$q_2\, q_1^{-1}\, q_4\, q_2^{-1}\, q_1\, q_4^{-1} \;=\; 1. \tag{3.17}$$

Setting $q_1 = s_0$, $q_2 = s_2$ and $q_4 = s_1$, it follows that relations (3.7), (3.8) and (3.9) are equivalent to (3.14), (3.13) and (3.17), respectively. This proves that $\pi_1(M_8) \cong \pi_1(M_8^{'}) \cong P2_1$. The manifolds M_8 and $M_8^{'}$ are not lens spaces as the integral first homology group is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$. Furthermore, they are prime since the centre of $P2_1$ is isomorphic to $\mathbb{Z}$ and generated by the fibre h. Then Theorem 4 implies that M_8 and $M_8^{'}$ are homeomorphic.

(3.4) *The euclidean space form* E^3/Pb. Set $M_9 = E^3/Pb$. A Schlegel diagram for M_9 is drawn in [8, Figure 7, p.35], which induces a group presentation for Pb with generators p_1, p_3, b and relations

$$p_1\, p_3\, p_1^{-1}\, p_3^{-1} \;=\; 1 \tag{3.18}$$

$$p_1\, b\, p_1^{-1}\, b^{-1} \;=\; 1 \tag{3.19}$$

and

$$p_3\, b\, p_3\, b^{-1} \;=\; 1. \tag{3.20}$$

Let $M_9^{'}$ be the torus bundle with monodromy given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We prove that $\pi_1(M_9) \cong \pi_1(M_9^{'})$. In fact, $\pi_1(M_9^{'})$ has a presentation with generators x_1, x_2, x_3 and relations

$$x_1\, x_2\, x_1^{-1}\, x_2^{-1} \;=\; 1 \tag{3.21}$$

$$x_3^{-1}\, x_1\, x_3 \;=\; x_1 \tag{3.22}$$

and

$$x_3^{-1}\, x_2\, x_3 \;=\; x_2^{-1}. \tag{3.23}$$

Setting $x_1 = p_1$, $x_2 = p_2$ and $x_3 = b$. relations (3.21), (3.22) and (3.23) are equivalent to (3.18), (3.19) and (3.20), respectively. Then M_9 and $M_9^{'}$ have isomorphic fundamental groups. They are not lens spaces as the abelianization of Pb is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$. Such manifolds are prime since the centre of Pb is not trivial. It follows from [7, §9] that M_9 is homeomorphic to $M_9^{'}$. Furthermore, $T_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)}$ is homeomorphic to the topological product $K \times \mathbb{S}^1$, whose fundamental group has a presentation with generators x_1, x_2, h and relations $x_1 h x_1^{-1} = h$, $x_2 h x_2^{-1} = h$ and $x_1^2 x_2^2 = 1$.

(3.5) *The euclidean space form* $E^3/P2_12_12_1$. Set $M_{10} = E^3/P2_12_12_1$. A Schlegel diagram for M_{10} is depicted in [8, Figure 11, p.35], which induces a group presentation for $\pi_1(M_{10}) \cong P2_12_12_1$ with generators s_1 and s_3 and relations

$$s_1\, s_3^2\, s_1^{-1}\, s_3^2 \;=\; 1 \tag{3.24}$$

and

$$s_3\, s_1^2\, s_3^{-1}\, s_1^2 \;=\; 1. \tag{3.25}$$

Let $M_{10}^{'}$ denote the Seifert manifold defined by the invariants

$$(\,\mathbb{R}P^2 \,|\; 0\ (2,1)\ (2,-1)).$$

A presentation for $\pi_1(M_{10}^{'})$ can be derived from [14, p.91]. More precisely, if $\pi_1(\mathbb{R}P^2) =<\ v\ :\ v^2\ =\ 1\ >$, then $\pi_1(M_{10}^{'})$ has a group presentation with generators q_1, q_2, h, v and relations

$$q_1\, h\, q_1^{-1} = h \qquad\qquad q_2\, h\, q_2^{-1} = h \tag{3.26}$$

$$v\, h\, v^{-1} \;=\; h^{-1} \tag{3.27}$$

$$q_1^2\, h \;=\; 1 \qquad\qquad q_2^2\, h^{-1} \;=\; 1 \tag{3.28}$$

and

$$q_1\, q_2\, v^2 \;= h^0 = 1. \tag{3.29}$$

Eliminating $h = q_1^{-2}$ from (3.28) and $q_2 = q_1^{-1}\, v^{-2}$ from (3.29), we get a presentation for $\pi_1(M_{10}^{'})$ with generators q_1 and v and relations

$$q_1^2 \;=\; (v^2\, q_1)^2 \tag{3.30}$$

and

$$v\, q_1^{-2}\, v^{-1} \;=\; q_1^2. \tag{3.31}$$

Setting $q_1 = s_1$ and $v = s_3$, relations (3.30) and (3.31) are equivalent to (3.24) and (3.25), respectively. Then $\pi_1(M_{10}) \cong \pi_1(M_{10}^{'})$. The considered manifolds are not lens spaces as the integral first homology group is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$. Such manifolds are prime since the centre of $P2_12_12_1$ is not trivial. Thus M_{10} and $M_{10}^{'}$ are homeomorphic by Theorem 4. Finally, the Seifert space $M_{10}^{'}$ can also be represented by the invariants

$$(\,\mathbb{R}P^2 \,|\; -1\ (2,1)\ (2,1))$$

by using the surgery instructions described in [13, p.147].

Remark 2. Recently, Mednykh with his coauthors Chelnokov and Deryagina have written a series of important papers [2-6] on closed euclidean manifolds, by using different notations. They denote the six orientable euclidean 3-manifolds by $\mathcal{G}_i$, $i = 1, \dots, 6$, and the four nonorientable ones by $\mathcal{B}_j$, $j = 1, \dots, 4$. The correspondence between the crystallographic notations from Molnár [8-12], used here, and the above notations is as follows: $\mathcal{G}_1 = E^3/P1$, $\mathcal{G}_2 = E^3/P2_1$, $\mathcal{G}_3 = E^3/P3_1$, $\mathcal{G}_4 = E^3/P4_1$, $\mathcal{G}_5 = E^3/P6_1$, $\mathcal{G}_6 = E^3/P2_12_12_1$, $\mathcal{B}_1 = E^3/Pb$, $\mathcal{B}_2 = E^3/Bb$, $\mathcal{B}_3 = E^3/Pca2_1$ and $\mathcal{B}_4 = E^3/Pna2_1$, where $\mathcal{G}_6$ is also known as the *Hantzsche-Wendt manifold* [5]. The cited authors describe all types of n-fold coverings over $\mathcal{G}_i$, $i = 2, \dots, 6$, and calculate the numbers of non-equivalent coverings of each type. Then they classify the subgroups in the fundamental groups $\pi_1(\mathcal{G}_i)$, up to isomorphism, and compute the numbers of conjugated classes of each type of subgroups of index n. See [3-5]. The same problems have been solved for $\mathcal{B}_j$, $j = 1, \dots, 4$, in [2] and [6].

Remark 3. As pointed out by the referee, only "minimal coverings" would have a special geometric interest for the space form problem, in general. This is related to a fixed point free group (as fundamental group of a space form) and to its maximal fixed point free supergroup(s) (as covering group of the covered space form), up to affine equivalence (conjugacy). As a consequence of the results proved in [2-6], closed euclidean space forms satisfy the following finite-sheeted covering properties: $E^3/P1$ has itself and any other closed euclidean 3-manifold as covered forms; $E^3/P2_1$ has itself, $E^3/P4_1$, $E^3/P6_1$, $E^3/P2_12_12_1$, $E^3/Pca2_1$ and $E^3/Pna2_1$ as covered forms; $E^3/P3_1$ (resp. $E^3/Pna2_1$) has only itself and $E^3/P6_1$ (resp. $E^3/Pca2_1$) as covered forms; $E^3/P4_1$ (resp. $E^3/P6_1$, $E^3/P2_12_12_1$ and $E^3/Pca2_1$) has only itself as covered form; and E^3/Pb (resp. E^3/Bb) has itself, E^3/Bb (resp. E^3/Pb), $E^3/Pca2_1$ and $E^3/Pna2_1$ as covered forms.

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