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Bounds for the derivative of a certain class of rational functions

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Abstract. In this paper, we shall obtain the bounds for the derivative of a rational function in the supremum norm on the unit circle in both the directions by involving the moduli of all its zeros. The obtained results strengthen some recently proved results.

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1 Introduction

Let P_n denote the class of complex polynomials $p(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n and $p'(z)$ is the derivative of $p(z)$. For any positive real number k, we denote $T_k = \{z : |z| = k > 0\}$. Let D_k^- represents the set of all points inside T_k and D_k^+ represents the set of all points outside T_k. For $a_j \in \mathbb{C}$ with $j = 1, 2, \ldots, n$, let

$$w(z) := \prod_{j=1}^{n} (z - a_j), \quad B(z) := \frac{w^*(z)}{w(z)} = \prod_{j=1}^{n} \left(\frac{1-\bar{a}_j z}{z-a_j}\right),$$

where $w^*(z) = z^n \overline{w(1/\bar{z})}$, and

$$\Re_n = \Re_n(a_1, a_2, \ldots, a_n) := \left\{\frac{p(z)}{w(z)} : p \in P_n\right\}.$$

The product $B(z)$ is known as Blaschke product and one can easily verify that $|B(z)| = 1$ and $\frac{zB'(z)}{B(z)} = |B'(z)|$ for $z \in T_1$. Then $\Re_n$ is the set of all rational

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functions with at most n poles $a_1, a_2, \ldots, a_n$ and with finite limit at infinity. We observe that $B(z) \in \Re_n$. For f defined on T_1 in the complex plane, we denote $\|f\| = \sup_{z \in T_1} |f(z)|$, the Chebyshev norm of f on T_1. Throughout this paper, we always assume that all poles $a_1, a_2, \ldots, a_n$ are in D_1^+.

For $p \in P_n$, the following result known as Bernstein inequality (for reference see [3]) is well known:

$$\left\|p'\right\| \leq n \left\|p\right\|.$$

For the class of polynomials $p \in P_n$ having all zeros in $T_1 \cup D_1^+$, the following result was conjectured by Erdös and later verified by Lax [5]:

$$\left\|p'\right\| \leq \frac{n}{2} \left\|p\left(z\right)\right\|.$$

In 1995, Li, Mohapatra and Rodriguez [6] have proved Bernstein-type inequalities for rational function $r(z) \in \Re_n$ with prescribed poles where they replaced z^n by Blaschke product $B\left(z\right)$ and established the following results.

Theorem 1.1. If $r \in \Re_n$, and all zeros of r lie in $T_1 \cup D_1^+$, then for $z \in T_1$, we have

$$\left|r'(z)\right| \leq \frac{1}{2} \left|B'(z)\right| \left\|r\right\|. \tag{1.1}$$

Equality holds for $r(z) = aB(z) + b$ with $|a| = |b| = 1$.

Theorem 1.2. Let $r \in \Re_n$, where r has exactly n poles at $a_1, a_2, \ldots, a_n$ and all its zeros lie in $T_1 \cup D_1^-$. Then for $z \in T_1$,

$$\left|r'(z)\right| \geq \frac{1}{2} \left[\left|B'(z)\right| - (n-t)\right] \left|r(z)\right|, \tag{1.2}$$

where t is the number of zeros of r with counting multiplicity. The above result is best possible and equality holds for $r(z) = aB(z) + b$ with $|a| = |b| = 1$.

Remark 1.1. In particular, if r has exactly n zeros in $T_1 \cup D_1^-$, then the inequality (1.2) yields Bernstein-type inequality, namely for $z \in T_1$,

$$\left|r'(z)\right| \geq \frac{1}{2} \left|B'(z)\right| \left|r(z)\right|. \tag{1.3}$$

Aziz and Shah [2] proved the following theorems which improves upon the inequalities (1.2) and (1.3) by introducing $m = \min_{z \in T_1} \left|r(z)\right|$.

Theorem 1.3. If $r \in \Re_n$, with all its zeros lie in $T_1 \cup D_1^+$, then for $z \in T_1$, we have

$$\left|r'(z)\right| \leq \frac{1}{2} \left|B'(z)\right| \left(\left\|r\right\| - m\right), \tag{1.4}$$

where $m = \min_{z\in T_1} |r(z)|$. The result is best possible and equality attains for $r(z) = B(z) + he^{i\alpha}$ with $h \geq 1$ and α real.

Theorem 1.4. Let $r \in \Re_n$, where r has exactly n poles at $a_1, a_2, \dots, a_n$ and all its zeros lie in $T_1 \cup D_1^-$. Then for $z \in T_1$,

$$\left|r'(z)\right| \geq \frac{1}{2}\left|B'(z)\right| \left[\left|r(z)\right| + m\right], \tag{1.5}$$

where $m = \min_{z\in T_1} |r(z)|$. Equality attains for $r(z) = B(z) + he^{i\alpha}$ with $h \leq 1$ and α real.

Recently, Arunrat and Nakprasit [1] proved the following results, which not only improve upon the inequalities (1.4) and (1.5), but also generalize them.

Theorem 1.5. Let $r \in \Re_n$, where r has exactly n poles at $a_1, a_2, \dots, a_n$ and all its zeros lie in $T_k \cup D_k^+$, $k \geq 1$. Then for $z \in T_1$,

$$\left|r'(z)\right| \leq \frac{1}{2}\left[\left|B'(z)\right| - \frac{(n(1+k) - 2t)\left(|r(z)| - m\right)^2}{(1+k)\left(\|r\| - m\right)^2}\right]\left(\|r\| - m\right), \tag{1.6}$$

where t is the number of zeros of r with counting multiplicity and $m = \min_{z\in T_k} |r(z)|$.

Theorem 1.6. Let $r \in \Re_n$, where r has exactly n poles $a_1, a_2, \dots, a_n$ and all its zeros lie in $T_k \cup D_k^-$, $k \leq 1$. Then for $z \in T_1$,

$$\left|r'(z)\right| \geq \frac{1}{2}\left[\left|B'(z)\right| + \frac{2t - n(1+k)}{1+k}\right]\left(|r(z)| + m\right), \tag{1.7}$$

where t is the number of zeros of r with counting multiplicity and $m = \min_{z\in T_k} |r(z)|$.

2 Main results

In this paper, we shall obtain bounds for the derivative of rational functions by involving the moduli of all its zeros. More precisely, we have the following:

Theorem 2.1. Let $r(z) = \frac{p(z)}{w(z)} \in \Re_n$ and $b_1, b_2, \dots, b_t$ are the zeros of $r(z)$ all lying in $T_k \cup D_k^+$, $k \geq 1$. Then for $z \in T_1$,

$$\left|r'(z)\right| \leq \frac{1}{2}\left[\left|B'(z)\right| - \frac{2\left(\frac{n}{2} - \sum_{j=1}^{t}\left(\frac{1}{1+|b_j|}\right)\right)\left(|r(z)| - m\right)^2}{\left(\|r\| - m\right)^2}\right]\left(\|r\| - m\right),$$

where $m = \min_{z\in T_k} |r(z)|$.

If $r(z)$ has exactly n zeros all lying in $T_k \cup D_k^+$, where $k \geq 1$, we obtain the following result from Theorem 2.1.

Corollary 2.1. Let $r(z) = \frac{p(z)}{w(z)} \in \Re_n$ and $b_1, b_2, \dots, b_n$ are the n zeros of $r(z)$ all lying in $T_k \cup D_k^+$, $k \geq 1$. Then for $z \in T_1$,

$$\left|r'(z)\right| \leq \frac{1}{2}\left[\left|B'(z)\right| - \frac{2\left(\frac{n}{2} - \sum_{j=1}^{n}\left(\frac{1}{1+|b_j|}\right)\right)\left(|r(z)| - m\right)^2}{\left(\|r\| - m\right)^2}\right]\left(\|r\| - m\right), \tag{2.1}$$

where $m = \min_{z \in T_k} |r(z)|$.

Remark 2.1. As we have

$$\frac{1}{1+|b_j|} \leq \frac{1}{k+1}, \tag{2.2}$$

for $|b_j| \geq k \geq 1$. Using (2.2) in (3.3), we observe that Corollary 2.1 reduces to the following result:

$$\left|r'(z)\right| \leq \frac{1}{2}\left[\left|B'(z)\right| - \frac{n(k-1)\left(|r(z)| - m\right)^2}{(1+k)\left(\|r\| - m\right)^2}\right]\left(\|r\| - m\right).$$

For $k = 1$, we get inequality (1.4) from the above inequality.

Theorem 2.2. Let $r(z) = \frac{p(z)}{w(z)} \in \Re_n$ and $b_1, b_2, \dots, b_t$ are the zeros of $r(z)$ all lying in $T_k \cup D_k^-$, $k \leq 1$. Then for $z \in T_1$,

$$\left|r'(z)\right| \geq \frac{1}{2}\left[\left|B'(z)\right| + 2\left(\sum_{j=1}^{t}\left(\frac{1}{1+|b_j|}\right) - \frac{n}{2}\right)\right]\left(|r(z)| + m\right),$$

where $m = \min_{z \in T_k} |r(z)|$.

If $r(z)$ has exactly n zeros all lying in $T_k \cup D_k^-$, where $k \leq 1$, we get the following result from Theorem 2.2.

Corollary 2.2. Let $r(z) = \frac{p(z)}{w(z)} \in \Re_n$ and $b_1, b_2, \dots, b_n$ are the zeros of $r(z)$ lie in $T_k \cup D_k^-$, $k \leq 1$. Then for $z \in T_1$,

$$\left|r'(z)\right| \geq \frac{1}{2}\left[\left|B'(z)\right| + 2\left(\sum_{j=1}^{n}\left(\frac{1}{1+|b_j|}\right) - \frac{n}{2}\right)\right]\left(|r(z)| + m\right), \tag{2.3}$$

where $m = \min_{z \in T_k} |r(z)|$.

Remark 2.2. As
$$\frac{1}{1+|b_j|} \geq \frac{1}{1+k}, \tag{2.4}$$
where $|b_j| \leq k \leq 1$. Using (2.4) in (3.1), we see that Corollary 2.2 reduces to the following result due to Mir [7], which improves upon a result recently proved by Tripathi, Hans, and Tyagi [8]:

$$\left|r'(z)\right| \geq \frac{1}{2}\left[\left|B'(z)\right| + \frac{n(1-k)}{k+1}\right](|r(z)| + m).$$

Also, for $k = 1$, we get inequality (1.5) from the above inequality.

3 Lemmas

For the proof of these theorems, we need the following lemmas. The first lemma is due to Li, Mohapatra and Rodriguez [6].

Lemma 3.1. If $r \in \Re_n$ and $r^*(z) = B(z)\overline{r\left(\frac{1}{\bar{z}}\right)}$, then for $z \in T_1$,
$$\left|(r^*(z))'\right| + \left|r'(z)\right| \leq \left|B'(z)\right| \|r\|.$$

This next lemma is due to Bidkham and Shahmansouri [4].

Lemma 3.2. If $z \in T_1$, then
$$Re\left(\frac{zw'(z)}{w(z)}\right) = \frac{n - |B'(z)|}{2}.$$

4 Proofs of Theorems

Proof of Theorem 2.1. Assume that $r \in \Re_n$ has no zero in $|z| < k$, where $k \geq 1$. Let $m = \min_{|z|=k}|r(z)|$. If $r(z)$ has a zero on $|z| = k$, then $m = 0$ and hence for every α with $|\alpha| < 1$, we get $r(z) - \alpha m = r(z)$. In case $r(z)$ has no zero on $|z| = k$, we have for every α with $|\alpha| < 1$ that $|-\alpha m| = |\alpha| m \leq |r(z)|$ for $|z| = k$. By Rouché's theorem $R(z) = r(z) - \alpha m$ and $r(z)$ have the same number of zeros in with $|z| < k$, that is, for every α with $|\alpha| < 1$, $R(z)$ has no zeros in $|z| < k$. Let $b_1, b_2, \ldots, b_t$ are zeros of $R(z)$, $t \leq n$, then $|b_j| \geq k \geq 1$, we have
$$\frac{zR'(z)}{R(z)} = \frac{zr'(z)}{r(z)} = \frac{zp'(z)}{p(z)} - \frac{zw'(z)}{w(z)}$$
$$= \sum_{j=1}^{t} \frac{z}{z-b_j} - \frac{zw'(z)}{w(z)}.$$

On using Lemma 3.2, we have

$$Re\left(\frac{zR'(z)}{R(z)}\right) = Re\left(\sum_{j=1}^{t}\frac{z}{z-b_j}\right) - \left(\frac{n-|B'(z)|}{2}\right).$$

This implies

$$\begin{aligned} Re\left(\frac{zR'(z)}{R(z)}\right) &\le \sum_{j=1}^{t}\frac{1}{1+|b_j|} - \left(\frac{n-|B'(z)|}{2}\right) \\ &= \frac{|B'(z)|}{2} + \left(\sum_{j=1}^{t}\frac{1}{1+|b_j|} - \frac{n}{2}\right). \end{aligned} \tag{4.1}$$

Note that $R^*(z) = B(z)\overline{R\left(\frac{1}{\bar{z}}\right)} = B(z)\bar{R}\left(\frac{1}{z}\right)$. Then

$$\begin{aligned} (R^*(z))' &= B'(z)\bar{R}\left(\frac{1}{z}\right) + B(z)\left(\bar{R}\left(\frac{1}{z}\right)'\right) \\ &= B'(z)\bar{R}\left(\frac{1}{z}\right) + B(z)\left(\bar{R}'\left(\frac{1}{z}\right)\right)\left(-\frac{1}{z^2}\right) \\ &= B'(z)\bar{R}\left(\frac{1}{z}\right) - \frac{B(z)}{z^2}\left(\bar{R}'\left(\frac{1}{z}\right)\right), \end{aligned}$$

which further implies

$$z(R^*(z))' = zB'(z)\bar{R}\left(\frac{1}{z}\right) - \frac{B(z)}{z}\left(\bar{R}'\left(\frac{1}{z}\right)\right).$$

Since $z \in T_1$, we have $\bar{z} = \frac{1}{z}$, $|B(z)| = 1$, $\frac{zB'(z)}{B(z)} = |B'(z)|$, hence

$$\begin{aligned} \left|z(R^*(z))'\right| &= \left|zB'(z)\overline{R(z)} - B(z)\overline{zR'(z)}\right| \\ &= \left|\frac{zB'(z)}{B(z)}\overline{R(z)} - \overline{zR'(z)}\right| \\ &= \left|\left|B'(z)\right|\overline{R(z)} - \overline{zR'(z)}\right|. \end{aligned}$$

Also, $|B'(z)|$ is real, we get

$\left|z(R^*(z))'\right| = ||B'(z)|R(z) - zR'(z)|$. Then by inequality (4.1), we obtain

$$\begin{aligned}\left|\frac{z(R^*(z))'}{R(z)}\right|^2 &= \left|\left|B'(z)\right| - \frac{zR'(z)}{R(z)}\right|^2 \\ &= \left|B'(z)\right|^2 + \left|\frac{zR'(z)}{R(z)}\right|^2 - 2\left|B'(z)\right| Re\left(\frac{zR'(z)}{R(z)}\right) \\ &\geq \left|B'(z)\right|^2 + \left|\frac{zR'(z)}{R(z)}\right|^2 \\ &\quad - 2\left|B'(z)\right|\left[\frac{|B'(z)|}{2} + \left(\sum_{j=1}^{t}\frac{1}{1+|b_j|} - \frac{n}{2}\right)\right] \\ &= \left|\frac{zR'(z)}{R(z)}\right|^2 + 2\left[\frac{n}{2} - \left(\sum_{j=1}^{t}\frac{1}{1+|b_j|}\right)\right]\left|B'(z)\right|.\end{aligned}$$

This implies that for $z \in T_1$,

$$\left[\left|R'(z)\right|^2 + 2\left[\frac{n}{2} - \left(\sum_{j=1}^{t}\frac{1}{1+|b_j|}\right)\right]\left|B'(z)\right|\left|R(z)\right|^2\right]^{\frac{1}{2}} \leq \left|(R^*(z))'\right|, \quad (4.2)$$

where $R^*(z) = B(z)\overline{R\left(\frac{1}{\bar{z}}\right)} = r^*(z) - \bar{\alpha}mB(z)$. Moreover, $(R^*(z))' = (r^*(z))' - \bar{\alpha}mB'(z)$ and $R'(z) = r'(z) = (r(z) - \alpha m)'$. On applying these relations into (4.2), we obtain

$$\left[\left|r'(z)\right|^2 + 2\left[\frac{n}{2} - \left(\sum_{j=1}^{t}\frac{1}{1+|b_j|}\right)\right]\left|B'(z)\right|\left|r(z) - \alpha m\right|^2\right]^{\frac{1}{2}}$$

$$\leq \left|(r^*(z))' - \bar{\alpha}mB'(z)\right|, \quad (4.3)$$

for $z \in T_1$ and for α with $|\alpha| < 1$. Choose the argument of α such that $\left|r^*(z)' - \bar{\alpha}mB'(z)\right| = \left|r^*(z)'\right| - m|\alpha|\left|B'(z)\right|$, for $z \in T_1$. Since $|r(z) - m\alpha| \geq ||r(z)| - m|\alpha||$. Note that $||r(z)| - m|\alpha||^2 = (|r(z)| - m|\alpha|)^2$,
which implies that $|r(z) - m\alpha|^2 \geq (|r(z)| - m|\alpha|)^2$,
which on using in (4.3), gives

$$\left[\left|r'(z)\right|^2 + 2\left[\frac{n}{2} - \left(\sum_{j=1}^{t}\frac{1}{1+|b_j|}\right)\right]\left|B'(z)\right|(|r(z)| - m|\alpha|)^2\right]^{\frac{1}{2}}$$

$$\leq \left|(r^*(z))'\right| - m|\alpha|\left|B'(z)\right|.$$

Letting $|\alpha| \to 1$, we get

$$\left[\left|r'(z)\right|^2 + 2\left[\frac{n}{2} - \left(\sum_{j=1}^{t}\frac{1}{1+|b_j|}\right)\right]\left|B'(z)\right|(|r(z)| - m)^2\right]^{\frac{1}{2}}$$
$$\leq \left|(r^*(z))'\right| - m\left|B'(z)\right|.$$

By lemma 3.1, implies that

$$\left[\left|r'(z)\right|^2 + 2\left[\frac{n}{2} - \left(\sum_{j=1}^{t}\frac{1}{1+|b_j|}\right)\right]\left|B'(z)\right|(|r(z)| - m)^2\right]^{\frac{1}{2}}$$
$$\leq \left|B'(z)\right|\|r\| - \left|r'(z)\right| - m\left|B'(z)\right|.$$

Further simplifying and squaring both sides, gives us

$$\left|r'(z)\right|^2 + 2\left[\frac{n}{2} - \left(\sum_{j=1}^{t}\frac{1}{1+|b_j|}\right)\right]\left|B'(z)\right|(|r(z)| - m)^2$$
$$\leq (\|r\| - m)^2\left|B'(z)\right|^2 + \left|r'(z)\right|^2 - 2(\|r\| - m)\left|B'(z)\right|\left|r'(z)\right|.$$

This implies that

$$\left|r'(z)\right| \leq \frac{(\|r\| - m)^2|B'(z)|^2}{2(\|r\| - m)|B'(z)|} - \frac{2\left[\frac{n}{2} - \left(\sum_{j=1}^{t}\frac{1}{1+|b_j|}\right)\right]|B'(z)|(|r(z)| - m)^2}{2(\|r\| - m)|B'(z)|}.$$

Thus

$$\left|r'(z)\right| \leq \frac{1}{2}\left[\left|B'(z)\right| - \frac{2\left[\frac{n}{2} - \left(\sum_{j=1}^{t}\frac{1}{1+|b_j|}\right)\right](|r(z)| - m)^2}{(\|r\| - m)^2}\right](\|r\| - m).$$

This proves inequality for $R(z) \neq 0$. In case $R(z) = 0$, we obtain that $r'(z) = 0$. This implies that the above inequality is trivially true. Therefore, inequality holds for all $z \in T_1$. QED

Proof of Theorem 2.2. Assume that $r \in \Re_n$ has no zeros in D_k^+ where $k \leq 1$. Let $m = \min_{|z|=k}|r(z)|$, then $m \leq |r(z)|$ for $z \in T_k$. If $r(z)$ has a zero on $|z| = k$, then $m = 0$, hence for every α with $|\alpha| < 1$ we get $r(z) + \alpha m = r(z)$. In case $r(z)$ has no zeros on $|z| = k$, we have for every α with $|\alpha| < 1$ that $|\alpha m| < |r(z)|$

for $|z| = k$. It follows by Rouché's theorem that $R(z) = r(z) + \alpha m$ and $r(z)$ have same number of zeros in D_k^-, that is, for every α with $|\alpha| < 1$, $R(z)$ has no zero in D_k^+. If $b_1, b_2, \ldots, b_t$ are zeros of $R(z)$, $t \leq n$ and $|b_j| \leq k \leq 1$, we have

$$\begin{aligned}\frac{zR'(z)}{R(z)} = \frac{zr'(z)}{r(z)} &= \frac{zp'(z)}{p(z)} - \frac{zw'(z)}{w(z)} \\ &= \sum_{j=1}^{t} \frac{z}{z-b_j} - \frac{zw'(z)}{w(z)}.\end{aligned}$$

For $z \in T_1$, gives with the help of Lemma 3.2, that

$$\begin{aligned}Re\left(\frac{zR'(z)}{R(z)}\right) &= Re\left(\sum_{j=1}^{t} \frac{z}{z-b_j}\right) - \left(\frac{n-|B'(z)|}{2}\right) \\ &\geq \sum_{j=1}^{t} \frac{1}{1+|b_j|} - \left(\frac{n-|B'(z)|}{2}\right) \\ &= \frac{|B'(z)|}{2} + \left(\sum_{j=1}^{t} \frac{1}{1+|b_j|} - \frac{n}{2}\right),\end{aligned}$$

where $R(z) \neq 0$. Then

$$\left|\frac{R'(z)}{R(z)}\right| = \left|\frac{zR'(z)}{R(z)}\right| \geq Re\left(\frac{zR'(z)}{R(z)}\right) \geq \frac{|B'(z)|}{2} + \left(\sum_{j=1}^{t} \frac{1}{1+|b_j|} - \frac{n}{2}\right).$$

This implies that

$$\left|R'(z)\right| \geq \left[\frac{|B'(z)|}{2} + \left(\sum_{j=1}^{t} \frac{1}{1+|b_j|} - \frac{n}{2}\right)\right] |R(z)|, \text{ for } z \in T_1.$$

As $R(z) = r(z) + \alpha m$, therefore, we get

$$\left|r'(z)\right| \geq \left[\frac{|B'(z)|}{2} + \left(\sum_{j=1}^{t} \frac{1}{1+|b_j|} - \frac{n}{2}\right)\right] |r(z) + \alpha m|, \text{ for } z \in T_1.$$

Note that this inequality is trivially true for $R(z) = 0$. Therefore, this inequality holds for all $z \in T_1$. Choosing the argument of α suitably in the right side of the above inequality and noting that the left side is independent of α, we get that

$$\left|r'(z)\right| \geq \left[\frac{|B'(z)|}{2} + \left(\sum_{j=1}^{t} \frac{1}{1+|b_j|} - \frac{n}{2}\right)\right] \left(|r(z)| + |\alpha|\, m\right),$$

for $z \in T_1$. Letting $|\alpha| \to 1$, we get for $z \in T_1$, that

$$\begin{aligned} \left|r'(z)\right| &\geq \left[\frac{|B'(z)|}{2} + \left(\sum_{j=1}^{t} \frac{1}{1+|b_j|} - \frac{n}{2}\right)\right] \left(|r(z)| + m\right) \\ &= \frac{1}{2}\left[\left|B'(z)\right| + 2\left(\sum_{j=1}^{t} \frac{1}{1+|b_j|} - \frac{n}{2}\right)\right] \left(|r(z)| + m\right), \end{aligned}$$

which proves the Theorem 2.2. *QED*

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