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On an autoregressive process driven by a sequence of Gaussian cylindrical random variables

Felix Che Shu[i]
The University of Bamenda
shufche@gmail.com

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Abstract. Let $\{Z_n\}_{n\in\mathbb{Z}}$ be a sequence of identically distributed, weakly independent and weakly Gaussian cylindrical random variables in a separable Banach space U. We consider the cylindrical difference equation, $X_n = AX_{n-1} + Z_n,\ n \in \mathbb{Z}$, in U and determine a cylindrical process $\{Y_n\}_{n\in\mathbb{Z}}$ which solves the equation. The cylindrical distribution of Y_n is shown to be weakly Gaussian and independent of n. It is also shown to be strongly Gaussian if the cylindrical distribution of Z_1 is strongly Gaussian. We determine the characteristic functional of Y_n and give conditions under which $\{Y_n\}_{n\in\mathbb{Z}}$ is unique.

Keywords: Autoregressive process, Cylindrical process, Cylindrical measure, Cylindrical random variable, Stationary process

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1 Introduction

The stochastic sequence $\{X_n\}_{n\geq 1}$, defined recursively by the stochastic difference equation

$$X_n = A_n X_{n-1} + B_n, \tag{1.1}$$

where $(A_n, B_n),\ n \geq 1$ are random pairs in $\mathbb{R}^2$ and X_0 is given, is used in stochastic modeling of phenomena in various disciplines. In most of the applications, X_n represents the quantity of some stock at time n, B_n an amount added to the stock just before time n and A_n is the rate of decay of the stock between the times $n-1$ and n. The wide range of applicability of this model is one of the major reasons for the attention paid to it over the last several decades. In [20], the author investigates among other questions, conditions under which the sequence $\{X_n\}_{n\geq 1}$ in (1.1) converges in distribution, when $\{(A_n, B_n)\}_{n\geq 1}$ is an i.i.d. sequence in $\mathbb{R}^2$ and studies the limit distribution. He shows that under

[i]

certain conditions, this convergence is equivalent to the almost sure convergence of $\sum_{k=1}^{\infty} A_1 A_2 \cdots A_{k-1} B_k$ and to the existence of a solution to the equation $X \overset{d}{=} AX + B$, in which X and (A, B) are independent and A and B are generic elements of the sequences $\{A_n\}_{n \geq 1}$ and $\{B_n\}_{n \geq 1}$ respectively. These results are some of the most fundamental in the i.i.d. setting, in one dimension.

In [11], Grincevicius shows that if $A \neq 0$ with probability 1, and $X \overset{d}{=} AX + B$ has a solution X, then the distribution of X is absolutely continuous or singular and continuous or degenerate. In particular, non degenerate discrete distributions do not occur as distributions of solutions of this equation whenever $A \neq 0$ with probability 1.

The results in [20] are obtained, assuming that $\{(A_n, B_n)\}_{n \geq 1}$ is a sequence of i.i.d. random vectors in $\mathbb{R}^2$. Given that this assumption is restrictive in view of applications, Brandt [9] extends these results to the case of stationary and ergodic sequences in $\mathbb{R}^2$. He determines conditions under which a uniquely determined stationary solution of the equation

$$X_{n+1} = A_n X_n + B_n, \tag{1.2}$$

$n \in \mathbb{Z}$, exists. He shows that if $\{(A_n, B_n)\}_{n \in \mathbb{Z}}$ is a stationary and ergodic sequence in $\mathbb{R}^2$ for which $-\infty \leq \mathbb{E} \ln |A_0| < 0$ and $\mathbb{E} \ln^+ |B_1| < \infty$ or $\mathbb{P}(A_0 = 0) > 0$ then $\{X_n\}_{n \in \mathbb{Z}}$, where $X_n := \sum_{j=0}^{\infty} A_{n-1} \cdots A_{n-j} B_{n-j-1}$, (we define $A_{n-1} A_n = 1$), is the only proper stationary solution of (1.2) for the given sequence $\{(A_n, B_n)\}_{n \in \mathbb{Z}}$. In this case, $\sum_{j=0}^{\infty} A_{n-1} \cdots A_{n-j} B_{n-j-1}$ converges absolutely almost surely for all $n \in \mathbb{Z}$ and $\mathbb{P}(\lim_{n \to \infty} |y_n(Y, \Psi) - y_n(\Psi)| = 0) = 1$, where

$y_n(Y, \Psi) := \sum_{j=0}^{n-1} A_{n-1} \cdots A_{n-j} B_{n-j-1} + A_{n-1} \cdots A_0 Y$,

$y_n(\Psi) := \sum_{j=0}^{\infty} A_{n-1} \cdots A_{n-j} B_{n-j-1}$ and Y is an arbitrary random variable defined on the same space as the vectors of the sequence $\Psi := \{(A_n, B_n)\}_{n \in \mathbb{Z}}$. In particular, under these conditions, $\sum_{j=0}^{n-1} A_{n-1} \cdots A_{n-j} B_{n-j-1} + A_{n-1} \cdots A_0 Y$ converges to $\sum_{k=0}^{\infty} A_{n-1} \cdots A_{n-k} B_{n-k-1}$ in distribution as n tends to infinity. The main differences in the studies by Vervaat [20] and Brandt [9] are that, while [20] considers sequences $\{(A_n\ B_n)\}_{n \geq 1}$ which are i.i.d., [9] considers sequences $\{(A_n\ B_n)\}_{n \in \mathbb{Z}}$ which are stationary and ergodic and in addition, while the series $y_n(\Psi)$ in [9] uses a backward construction, the corresponding series Y_n^* in [20] uses a forward construction.

Results in the one dimensional setting for (1.2), when $\{(A_n, B_n)\}_{n\in\mathbb{N}}$ is non stationary are found in Horst [12], where he gives conditions on the sequence $\{(A_n, B_n)\}_{n\in\mathbb{N}}$ under which the finite dimensional distributions of $\{X_n\}_{n\in\mathbb{N}}$ converge weakly to the finite dimensional distributions of a unique stationary solution of (1.2) under some measure $\mathbb{P}^*$ on the same space $(\Omega, \mathcal{F}, \mathbb{P})$ on which $\{(A_n, B_n)\}_{n\in\mathbb{N}}$ is defined. The results of [12] generalize those of Brandt [9] and Borovkov [4] from the stationary to the non stationary case.

For real multidimensional autoregressive processes, we mention the works of Bougerol and Picard [8] and Kesten [13]. In [8], Bougerol and Picard consider the equation (1.1) where $\{(A_n, B_n)\}_{n\in\mathbb{Z}}$ is a strictly stationary and ergodic process, X_n and B_n are random vectors in $\mathbb{R}^d$ and A_n is a $d \times d$ random matrix for each n. Assuming that $\{(A_n, B_n)\}_{n\in\mathbb{Z}}$ is i.i.d., they give conditions under which a non anticipative strictly stationary solution $\{X_n\}_{n\in\mathbb{Z}}$ of (1.1), if it exists, satisfies $X_n = \sum_{k=0}^{\infty} A_n A_{n-1} \cdots A_{n-k+1} B_{n-k}$ almost surely (the series converging almost surely) and is the unique strictly stationary solution of (1.1). They also show that if $\{(A_n, B_n)\}_{n\in\mathbb{Z}}$ is i.i.d., (1.1) is irreducible, $\mathbb{E}\ln^+ \|A_0\| < \infty$, and $\mathbb{E}\ln^+ \|B_0\| < \infty$, then (1.1) has a non anticipative strictly stationary solution if and only if $\inf\{\frac{1}{n+1}\mathbb{E}\ln \|A_0 \cdots A_n\| : n \in \mathbb{N}\} < 0$. In essence, the results presented in [8] are an extension of the results in [9] to the multidimensional case, when $\{(A_n, B_n)\}_{n\in\mathbb{Z}}$ is i.i.d.

Considerations on autoregressive processes in Hilbert and Banach spaces appear in Bosq [5], [6], [7]. In [6], Bosq considers the autoregressive Hilbertian process $\{X_n\}_{n\in\mathbb{Z}}$ in a Hilbert space H-

$$X_n - \mu = \rho\,(X_{n-1} - \mu) + \varepsilon_n,\ n \in \mathbb{Z}, \tag{1.3}$$

where $\mu \in H$, ρ is a bounded linear operator, $\rho : H \to H$ and $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ is a H-white noise process, i.e. a sequence of H-valued random variables for which $0 \le \mathbb{E}\|\varepsilon_n\|^2 < \infty$, $\mathbb{E}\varepsilon_n = 0$ for all $n \in \mathbb{Z}$, the covariance operator C_{ε_n} of ε_n does not depend on n and the random variables in the sequence $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ are orthogonal, i.e. if $n \neq m$, then $\mathbb{E}\langle\varepsilon_n\, x\rangle\langle\varepsilon_m\, y\rangle = 0$ for all $x, y \in H$. He shows that if for some integer $j_0 \ge 1$, $\|\rho^{j_0}\| < 1$, then (1.3) has a unique stationary solution given by $X_n := \mu + \sum_{j=0}^{\infty} \rho^j \varepsilon_{n-j}$, $n \in \mathbb{Z}$, where the series converges in L_2 and almost surely. In addition, $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ is the innovation process of $\{(X_n - \mu)\}_{n\in\mathbb{Z}}$. He also gives a necessary and sufficient condition for the existence of a stationary solution of (1.3) when $\mu = 0$ and ρ is a symmetric and compact operator. For future purposes, we point out that by Lemma 3.1 in [6], the condition $\|\rho^{j_0}\| < 1$ for some $j_0 \ge 1$ implies that $\sum_{k=0}^{\infty} \|\rho^j\| < \infty$.

Results in [5] on the existence of stationary solutions to Banach space valued autoregressive processes are generalizations of the studies in Hilbert spaces which in turn generalize the case in finite dimensional Euclidean space.

In what follows, we consider the difference equation

$$X_n = AX_{n-1} + Z_n,\ n \in \mathbb{Z}, \tag{1.4}$$

in a separable Banach space U, where A is a bounded linear operator on U and $\{Z_n\}_{n\in\mathbb{Z}}$ is a sequence of identically distributed, weakly independent, weakly Gaussian cylindrical random variables in U, and determine a cylindrical process $\{Y_n\}_{n\in\mathbb{Z}}$ in U which solves the equation. The cylindrical distribution of Y_n is shown to be weakly Gaussian and independent of n. Conditions are also given for the cylindrical distribution of Y_n to be strongly Gaussian. We determine the characteristic functional of Y_n and give conditions under which $\{Y_n\}_{n\in\mathbb{Z}}$ is unique. We note that in the equations (1.1) and (1.2), A_n is random while the operator A in the case we study is non random and does not depend on n. Thus the equation we study is more similar to the case of Hilbert and Banach space valued autoregressive processes as presented in [5] and [6]. The difference equation (1.4) can be considered as a discrete analogue of the stochastic differential equation

$$dY(t) = AY(t) + CdM(t), \tag{1.5}$$

studied in [14], [2] (see [21] for a related comparison), in which M is a cylindrical Lévy process in a Banach space U, A is the infinitesimal generator of a strongly continuous semi-group of linear operators and C is a linear bounded operator, these operators being defined on appropriate spaces. Applications of the concepts of cylindrical stochastic differential equations and cylindrical stochastic processes can be found in Da Prato et al. [10], Peszat et al. [17] etc. In comparison with the results in Banach and Hilbert spaces, our results are an extension and generalization of the corresponding results for Hilbert and Banach space valued autoregressive processes to the case of cylindrical random variables. The rest of this paper is organized as follows: In the next section we introduce definitions and cite the theorems we need. Our main theorem is presented in section 3. We then end this note with a conclusion.

2 Prerequisites

Throughout the note, U is a fixed separable Banach space with dual U^* and dual pairing $\langle\cdot,\cdot\rangle$. We also fix a probability space $(\Omega,\mathcal{F},\mathbb{P})$. For $\Gamma \subseteq U^*$, let $\mathcal{Z}(\Gamma) := \{\pi^{-1}_{a_1,\dots,a_n}(B) \mid n \in \mathbb{N}, a_1,\dots,a_n \in \Gamma, B \in \mathcal{B}(\mathbb{R}^n)\}$, where for $a_1,\dots,a_n \in U^*$, we define $\pi_{a_1,\dots,a_n} : U \to \mathbb{R}^n$ by,

$\pi_{a_1,\dots,a_n}(u) := (\langle u, a_1\rangle, \dots, \langle u, a_n\rangle)$, $u \in U$, and write $\mathcal{B}(\mathbb{R}^n)$ for the Borel σ-algebra on $\mathbb{R}^n$. $\mathcal{Z}(\Gamma)$ is an algebra. It is not a σ-algebra in general, but if Γ is finite, then it is the smallest σ-algebra relative to which the elements of Γ are measurable.

Definition 1. Let Γ be a subspace of U^*. A map $\mu : \mathcal{Z}(\Gamma) \to \mathbb{R}^+$ is called a cylindrical measure on $\mathcal{Z}(\Gamma)$ if $\mu(U) = 1$ and for each finite set $\triangle \subset \Gamma$, $\mu|_{\mathcal{Z}(\triangle)}$ is a measure.

Definition 2. Let V be a vector space. A function $\chi : V \to \mathbb{C}$ is said to be pseudo-continuous if its restriction to any finite-dimensional subspace of V is continuous.

If Γ is a subspace of U^* and μ is a cylindrical measure on $\mathcal{Z}(\Gamma)$, then the functional $\hat{\mu} : \Gamma \to \mathbb{C}$ defined by the formula $\hat{\mu}(a) := \int\limits_U e^{i\langle x,a\rangle}\mu(dx)$, where the integral is the usual Lebesgue integral on the measure space $(U, \mathcal{Z}(\{a\}), \mu)$, is called the characteristic functional of the cylindrical measure μ. The following Proposition is taken from Vakhania [19].

Proposition 1. *Let V be a set and Γ be a vector space of real valued functions defined on V. For a cylindrical measure μ on $\mathcal{Z}(\Gamma)$, let $\hat{\mu}$ denote its characteristic functional, then*

(a) The map $\mu \mapsto \hat{\mu}$ establishes a one-to-one correspondence between the collection of all cylindrical measures defined on $\mathcal{Z}(\Gamma)$ and the collection of their characteristic functionals.

(b) The characteristic functional $\hat{\mu} : \Gamma \to \mathbb{C}$ of an arbitrary cylindrical measure on $\mathcal{Z}(\Gamma)$ is positive definite, pseudo-continuous and $\hat{\mu}(0) = 1$.

(c) If V is a linear space and Γ consists of linear functionals, then for any positive-definite pseudocontinuous functional $\varphi : \Gamma \to \mathbb{C}$ with $\varphi(0) = 1$, there exists a cylindrical measure μ on $\mathcal{Z}(\Gamma)$ such that $\hat{\mu} = \varphi$.

Proof. See [19] Proposition VI 3.2. QED

A cylindrical random variable X in U is a linear map $X : U^* \to L_0(\Omega, \mathcal{F}, \mathbb{P})$. If X is a cylindrical random variable in U, then the functional $\varphi_X : U^* \to \mathbb{C}$ defined by the formula $\varphi_X(a) := \boldsymbol{E}(e^{iXa}),\ a \in U^*$, is called the characteristic functional of X.

Remark 1. (i) The characteristic functional of a cylindrical random variable X is positive definite pseudo-continuous and $\varphi_X(0) = 1$. By Proposition 1 c, there exists on $\mathcal{Z}(U^*)$ a cylindrical measure μ, such that $\hat{\mu} = \varphi_X$.

μ is called the cylindrical distribution of X. If μ is the cylindrical distribution of X, then for $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in U^*$ we have that $\mu_{\pi_{a_1,\ldots,a_n}} = \mathbb{P}_{Xa_1,\ldots,Xa_n}$. Here $\mu_{\pi_{a_1,\ldots,a_n}}$ is defined by $\mu_{\pi_{a_1,\ldots,a_n}}(B) := \mu(\{ u \in U \mid \pi_{a_1,\ldots,a_n}(u) \in B \}), B \in \mathcal{B}(\mathbb{R}^n)$. For more on this, see [19] section VI 3.2.

(ii) In the definition of a cylindrical random variable, it is not assumed to be continuous. However, to avoid technicalities, we shall require that cylindrical random variables be continuous.

Definition 3. We shall say that two cylindrical random variables X and Y in U are identically distributed if they have the same cylindrical distribution.

Remark 2. We shall henceforth write $X \stackrel{d}{=} Y$ to mean that the cylindrical random variables X and Y are identically distributed according to Definition 3, while $X = Y$ means that for each fixed $a \in U^*$, $Xa = Ya$, i.e. $Xa(\omega) = Ya(\omega)$ for all $\omega \in \Omega$, i.e. Xa and Ya are equal as functions defined on Ω. We write $Xa = Ya$ almost surely to mean that the equality holds on a set of measure 1.

Remark 3. By Proposition 1 (a) and Remark 1 (i), cylindrical random variables in U are identically distributed if and only if they have the same characteristic functional.

We write $\mathfrak{C}_2$ for the set of all cylindrical random variables X in U with $X(U^*) \subseteq G$ (where G is a fixed, closed, separable subspace of $L_2(\Omega, \mathcal{F}, \mathbb{P})$), with the property that X is continuous with respect to the norm $\|\cdot\|_{\mathfrak{C}_2}$ in $\mathfrak{C}_2$, where $\|X\|_{\mathfrak{C}_2} := \sup\{ \|Xa\|_{L_2} : a \in U^*, \|a\| \leq 1 \}$, $X \in \mathfrak{C}_2$. We endow $\mathfrak{C}_2$ with the topology generated by the norm, then $(\mathfrak{C}_2, \|\cdot\|_{\mathfrak{C}_2})$ is a Banach space (see Mamporia [15] page 602).

The following Definition is taken from Riedle [18]:

Definition 4. A cylindrical measure μ on $\mathcal{Z}(U^*)$ is said to be a weakly Gaussian cylindrical measure if its one dimensional projections μ_{π_a}, $a \in U^*$ are one dimensional Gaussian measures on $\mathcal{B}(\mathbb{R})$.

If μ is a weakly Gaussian cylindrical measure on $\mathcal{Z}(U^*)$ and $(U^*)'$ is the algebraic dual of U^*, then the operator $Q : U^* \to (U^*)'$ defined by $Q(a)(b) := \int_U \langle x, a\rangle\langle x, b\rangle \mu(dx) - \int_U \langle x, a\rangle \mu(dx) \int_U \langle x, b\rangle \mu(dx)$, is called the covariance operator of μ.

Part (i) of the following Definition is also taken from Riedle [18]:

Definition 5. (i) A centered weakly Gaussian cylindrical measure μ on $\mathcal{Z}(U^*)$ is said to be strongly Gaussian, if its covariance operator is U-valued.

(ii) We say that a cylindrical random variable $X : U^* \to L_0(\Omega, \mathcal{F}, \mathbb{P})$ is a weakly Gaussian cylindrical random variable, if its cylindrical distribution is a weakly Gaussian cylindrical measure on $\mathcal{Z}(U^*)$ and that it is a strongly Gaussian cylindrical random variable, if its cylindrical distribution is a strongly Gaussian cylindrical measure on $\mathcal{Z}(U^*)$.

It therefore follows that X is a weakly Gaussian cylindrical random variable if and only if for each $a \in U^*$, Xa is a one dimensional real Gaussian random variable.

We shall use the following Theorem from [18].

Theorem 1. *Let U be a separable Banach space with dual pairing $\langle \cdot, \cdot \rangle$. If μ is a weakly Gaussian cylindrical measure on $\mathcal{Z}(U^*)$, then its characteristic functional is given by* $\hat{\mu} : U^* \to \mathbb{C}$, $\hat{\mu}(a) := \int\limits_U e^{i\langle x,a\rangle} d\mu(x) = \exp\{\, im(a) - \frac{1}{2}\sigma(a)\,\}$, *where* $m(a) = \int\limits_U \langle x, a\rangle \mu(dx)$ *and* $\sigma(a) = \int\limits_U \langle x, a\rangle^2 \mu(dx) - (m(a))^2$.

Conversely, if μ is a cylindrical measure on $\mathcal{Z}(U^*)$ with characteristic functional of the form $\hat{\mu}(a) := \exp\{\, im(a) - \frac{1}{2}\sigma(a)\,\}$, *where $m : U^* \to \mathbb{R}$ is a linear functional and $\sigma : U^* \to \mathbb{R}^+$ is a quadratic form on U^*, then μ is a weakly Gaussian cylindrical measure.*

Proof. See [18] Theorem 2 page 194. QED

Remark 4. It follows that if X is a weakly Gaussian cylindrical random variable, then its characteristic functional has the form

$$\varphi_X(a) := \boldsymbol{E}(e^{iXa}) = \exp\left\{ im_X(a) - \frac{1}{2}\sigma_X(a) \right\}, \tag{2.6}$$

where $m_X : U^* \to \mathbb{R}$ is a linear functional and $\sigma_X : U^* \to \mathbb{R}^+$ is a quadratic form on U^* and if the characteristic functional of a cylindrical random variable is of the form above , then it is a weakly Gaussian cylindrical random variable. We also note that $m_X(a)$ is the expectation of Xa and $\sigma_X(a)$ is the variance of Xa. Since X is continuous, there exists a constant M such that $\boldsymbol{E}|Xa|^2 \leq M\|a\|^2$ for all $a \in U^*$. We can check that if μ is the cylindrical distribution of X, then $\int\limits_U \langle x, a\rangle^2 \mu(dx) = \int\limits_{\mathbb{R}} z^2 \mathbb{P}_{Xa}(dz) = \boldsymbol{E}|Xa|^2 \leq M \parallel a \parallel^2$ and $|m_X(a)| \leq \sqrt{M}\|a\|$. Therefore σ_X and m_X in Theorem, 1 are continuous. For a cylindrical random variable X in U and $a \in U^*$, we shall write Xa or $X(a)$, whichever is more convenient and define $\Lambda(a) := \begin{cases} 0 & : \quad a = 0 \\ \frac{a}{\|a\|} & : \quad a \neq 0 \end{cases}$. Note that $\|\Lambda(a)\| \leq 1$ for all

$a \in U^*$. We shall use indices to differentiate between the different norms we use when there is danger of confusion.

Suppose that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of cylindrical random variables in U which converges in $\mathfrak{C}_2$ to a cylindrical random variable X in U and $a \in U^*$, then from the inequality $\|X_n a - Xa\|_{L_2} \leq \|a\|\|X_n - X\|_{\mathfrak{C}_2}$, it follows that $Xa = (\, \mathfrak{C}_2\text{-}\lim_{n\to\infty} X_n \,)a = L_2\text{-}\lim_{n\to\infty}(X_n a)$. In particular, if the series $\sum_{k=0}^{\infty} X_k$ of cylindrical random variables in U converges in $\mathfrak{C}_2$ and $a \in U^*$, then $\left(\sum_{k=0}^{\infty} X_k\right) a = \sum_{k=0}^{\infty}(X_k a)$, where the convergence on the right of the equality is understood to be in L_2 and the convergence of the series on the left of the equality is understood to be in $\mathfrak{C}_2$.

Definition 6. We say that a stochastic process $\{X_t\}_{t\in T}$, $T = \mathbb{N}$ *or* $\mathbb{Z}$ with values in a Banach space E is strictly stationary, if for all $m \geq 1$, $t_1 < \cdots < t_m \in T$ and $h \geq 0$, $(\, X_{t_1}, \ldots, X_{t_m} \,) \stackrel{d}{=} (\, X_{t_1+h}, \ldots, X_{t_m+h} \,)$. The process is said to be weakly stationary, if the following hold:

i. $\mathbb{E}\|X_t\|^2 < \infty$ for all $t \in T$.

ii. $\mathbb{E}(X_t) = \nu$ for all $t \in T$, where $\nu \in E$ is a constant.

iii. For all $s,\ t,\ h \in T$, $h \geq 0$ and $x^*, y^* \in E^*$, $\mathbb{E}(\, \langle X_{s+h} - \mu, x^*\rangle\langle X_{t+h} - \mu, y^*\rangle \,) = \mathbb{E}(\, \langle X_s - \mu, x^*\rangle\langle X_t - \mu, y^*\rangle \,)$.

Let X be a cylindrical random variable in U and $a_1, \ldots, a_n \in U^*$, then following [2], we shall write $X(a_1, \ldots, a_n)$ for the $\mathbb{R}^n$ -valued random vector $(\, Xa_1, \ldots, Xa_n \,)$.

Definition 7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X_t : U^* \to L^2(\Omega, \mathcal{F}, \mathbb{P})$, $t \in T$, $T = \mathbb{N}$ or $\mathbb{Z}$ be a cylindrical process in U. We say that $\{X_t\}_{t\in T}$ is a cylindrical strictly stationary process in U if for all $m \geq 1$ and $a_1, \ldots, a_m \in U^*$, the process $\{X_t(a_1, \ldots, a_n)\}_{t\in T}$ is strictly stationary in $\mathbb{R}^m$. We say that $\{X_t\}_{t\in T}$ is a cylindrical weakly stationary process in U if for all $m \geq 1$ and $a_1, \ldots, a_m \in U^*$, $\{X_t(a_1 \cdots a_m)\}_{t\in T}$ is weakly stationary in $\mathbb{R}^m$.

3 THE AUTOREGRESSIVE PROCESS

Definition 8. We say that a sequence $\{Z_n\}_{n\in\mathbb{Z}}$ of cylindrical random variables in U is weakly independent if for all $n \in \mathbb{N}$, distinct indices $i_1, \ldots, i_n \in \mathbb{Z}$ and $a_{i_1}, \ldots, a_{i_n} \in U^*$, the random variables $Z_{i_1}a_{i_1}, \ldots, Z_{i_n}a_{i_n}$ are independent.

Remark 5. This definition of weak independence in Definition 8 is essentially Definition 2.4 in [3] and the corresponding definition in [2] page 705. If

$X = \{X_k\}_{k\in T}$, $T = \mathbb{N}$ or $\mathbb{Z}$, is a sequence of weakly Gaussian cylindrical random variables with characteristic functionals $\boldsymbol{E}(e^{iX_k a}) = \exp\{\, i m_{X_k}(a) - \frac{1}{2}\sigma_{X_k}(a)\,\}$, then we shall write $\sigma_X(a)$ when $\sigma_{X_k}(a)$ does not depend on k. Same applies to m_{X_k}.

Our main Theorem uses the following Lemmas:

Lemma 1. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be a sequence of cylindrical random variables in U, $X_n : U^* \to G$, which satisfies*

$$X_n = AX_{n-1} + Z_n,\ n \in \mathbb{Z}, \tag{3.7}$$

where $\{Z_n\}_{n\in\mathbb{Z}}$ is a given sequence of identically distributed, weakly independent, weakly Gaussian cylindrical random variables in U, $Z_n : U^* \to G$ and A be a bounded linear operator on U. Suppose that $X_0 \in \mathfrak{C}_2$, then:

(i) For each $n \geq 1$, there exist weakly independent, cylindrical random variables ξ_n and η_n in U, such that $X_n = \xi_n + \eta_n$ and η_n is a weakly Gaussian cylindrical random variable in U.

(ii) $\xi_n \in \mathfrak{C}_2$, $\eta_n \in \mathfrak{C}_2$ and $X_n \in \mathfrak{C}_2$ for all $n \geq 1$.

(iii) If for each $n \in \mathbb{Z}$ and $l \geq 1$ we define $\overline{\eta}_{nl} := \sum_{k=0}^{l-1} Z_{n-k}((A^*)^k(\cdot))$, then $\eta_l \overset{d}{=} \overline{\eta}_{nl}$. In addition, for all $l \geq 1$, we have that $\eta_l = \overline{\eta}_{ll}$.

Proof. For a linear operator A on the Banach space U and a cylindrical random variable X in U, the cylindrical random variable AX is defined by $(AX)a := X(A^*a)$, $a \in U^*$ (see [2] page 720), hence for each $n \geq 1$ and $a \in U^*$, X_n in (3.7) satisfies $X_n a = X_0((A^*)^n a) + Z_1((A^*)^{n-1}a) + \cdots + Z_{n-1}(A^*a) + Z_n a$. Define ξ_n by $\xi_n(\cdot) := X_0((A^*)^n(\cdot))$ and $\eta_n(\cdot) := Z_1((A^*)^{n-1}(\cdot)) + \cdots + Z_{n-1}(A^*(\cdot)) + Z_n(\cdot)$. That ξ_n is a cylindrical random variable in U, follows from the fact that for all $k \geq 0$, $k \in \mathbb{Z}$, $(A^*)^k : U^* \to U^*$ is linear, $X_0 : U^* \to G$ is linear and since X_0 and A are continuous, ξ_n is a linear and continuous map from U^* into G.

We now show that η_n is a weakly Gaussian cylindrical random variable in U. For $a \in U^*$ and $k \in \mathbb{Z}$, $Z_k(a)$ is a real Gaussian random variable. Since the sequence $\{Z_n\}_{n\in\mathbb{Z}}$ is a sequence of weakly independent cylindrical random variables, $\eta_n(a)$ is a sum of independent real Gaussian random variables and hence is real Gaussian. η_n is a finite sum of linear maps and is thus linear. Also $\eta_n : U^* \to G$ since it is a finite sum of maps which map U^* into G. η_n is also continuous as a finite sum of continuous maps. If μ is the cylindrical distribution of η_n, then for all $a \in U^*$, $\mu_{\pi_a} = \mathbb{P}_{\eta_n a}$ and hence μ_{π_a} is one dimensional real Gaussian. From these considerations, it follows that η_n is a weakly Gaussian cylindrical random variable. It is obvious that $\{\xi_n, \eta_n\}$ is weakly independent, since, if $\{X_n\}_{n\in\mathbb{Z}}$ satisfies (3.7), then X_0 depends on $\{Z_k : k \le 0\}$, hence $\{X_0\} \cup \{Z_n\}_{n\ge1}$ is weakly independent.

(ii) From the proof of *(i)*, $\xi_n : U^* \to G$, $\eta_n : U^* \to G$. We have that for any $a \in U^*$, $\|\xi_n a\|_{L_2}^2 = \boldsymbol{E}|X_0((A^*)^n a)|^2 = \|(A^*)^n a\|^2 \boldsymbol{E}\big|X_0(\Lambda((A^*)^n a))\big|^2$, thus $\| \xi_n \|_{\mathfrak{C}_2}^2 \le \|(A^*)^n\|^2 \|X_0\|_{\mathfrak{C}_2}^2 < \infty$, since A is bounded and $X_0 \in \mathfrak{C}_2$. Therefore $\xi_n \in \mathfrak{C}_2$. Further, $\|\eta_n a\|_{L_2}^2 \le 2^n \sum_{k=1}^{n} \|Z_k((A^*)^{n-k}a)\|_{L_2}^2$. Since for each k, Z_k is a weakly Gaussian cylindrical random variable, it follows that for $k \in \{1, \ldots, n\}$, and by Remark 1 *(ii)* and Remark 4, $\|Z_k((A^*)^{n-k}a)\|_{L_2}^2 = \sigma_Z((A^*)^{n-k}a) + (m_Z((A^*)^{n-k}a))^2 \le 2M\|(A^*)^{n-k}a\|^2$, where $M < \infty$ is a constant, $m_Z : U^* \to \mathbb{R}$ is linear and $\sigma_Z : U^* \to \mathbb{R}^+$ is a quadratic form.
Thus $\|\eta_n a\|_{L_2}^2 \le 2^{n+1} M \sum_{k=1}^{n} \|(A^*)^{n-k}\|^2 \|a\|^2$. Therefore

$\|\eta_n\|_{\mathfrak{C}_2}^2 \le 2^{n+1} M \sum_{k=1}^{n} \|(A^*)^{n-k}\|^2 < \infty$. Since η_n is a cylindrical random variable with values in G, it follows that $\eta_n \in \mathfrak{C}_2$. Since $\|X_n\|_{\mathfrak{C}_2}^2 \le 2\|\xi_n\|_{\mathfrak{C}_2}^2 + 2\|\eta_n\|_{\mathfrak{C}_2}^2 < \infty$ and by the assumptions of the Lemma, X_n is a cylindrical random variable with values in G, we have that $X_n \in \mathfrak{C}_2$.

(iii) That for each $n \in \mathbb{Z}$ and $l \ge 1$, $\eta_l \overset{d}{=} \overline{\eta}_{nl}$ follows by computing the characteristic functionals. It is also clear that $\eta_l = \overline{\eta}_{ll}$ for all $l \ge 1$. QED

Lemma 2. *Let $\{X_t\}_{t\in T}$, $T = \mathbb{N}$ or $\mathbb{Z}$ be a sequence of weakly independent and identically distributed cylindrical random variables in U, then for all $m \ge 1$ and $a_1, \ldots, a_m \in U^*$, the sequence $\{X_t(a_1, \ldots, a_m)\}_{t\in T}$ is a sequence of i.i.d. $\mathbb{R}^m$-valued random vectors.*

Proof. Let $m \ge 1$, $a_1, \ldots, a_m \in U^*$, $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, then
$\boldsymbol{E}(e^{i\langle X_t(a_1,\ldots,a_m),\alpha\rangle}) = \boldsymbol{E}\exp\{ i \sum_{k=1}^{m} \alpha_k X_t a_k \} = \boldsymbol{E}\exp\{ iX_t(\sum_{k=1}^{m} \alpha_k a_k) \}$
$= \boldsymbol{E}\exp\{ iX_1(\sum_{k=1}^{m} \alpha_k a_k) \}$. Therefore the joint distribution of $X_t(a_1, \ldots, a_m)$ is independent of t and hence $\{X_t(a_1, \ldots, a_m)\}_{t\in T}$ is identically distributed.

Let now $n \ge 1$, $m \ge 1$ and $t_1 < t_2 < \cdots < t_n \in T$, $a_1, \ldots, a_m \in U^*$ and $\alpha_1, \ldots, \alpha_n$ be arbitrary $\mathbb{R}^m$- vectors, $\alpha_j := (\alpha_{j1}, \ldots, \alpha_{jm})$, then

$$\boldsymbol{E}\exp\left\{ i \sum_{j=1}^{n} \langle X_{t_j}(a_1, \ldots, a_m), \alpha_j \rangle \right\} = \boldsymbol{E}\exp\left\{ i \sum_{j=1}^{n} \sum_{k=1}^{m} X_{t_j} a_k \alpha_{jk} \right\} =$$

$$\boldsymbol{E}\exp\left\{ i \sum_{j=1}^{n} X_{t_j} \left(\sum_{k=1}^{m} a_k \alpha_{jk} \right) \right\} = \prod_{j=1}^{n} \boldsymbol{E}\exp\left\{ iX_{t_j} \left(\sum_{k=1}^{m} a_k \alpha_{jk} \right) \right\} =$$

$\prod_{j=1}^{n} \boldsymbol{E} \exp\left\{i\langle X_{t_j}(a_1, \ldots, a_m), \alpha_j\rangle\right\}$. Since the α_j, s were arbitrary $\mathbb{R}^m$-vectors, the assertion follows from Kac's theorem (see [1] Theorem 2.1). QED

Theorem 2. *(i) Let the assumptions of Lemma 1 hold,* $\sum_{k=0}^{\infty} \|(A^*)^k\| < \infty$ *and* Z_1 *be centered with* $\|Z_1\|_{\mathfrak{C}_2} < \infty$*, then for each* $n \in \mathbb{Z}$*,* $\{\overline{\eta}_{nl}\}_{l\geq 1}$ *converges in* $(\mathfrak{C}_2, \|\cdot\|_{\mathfrak{C}_2})$ *as* l *tends to infinity, to a weakly Gaussian cylindrical random variable* $\overline{\eta}_n$ *in* U*, whose cylindrical distribution is independent of* n*. Further,* $\overline{\eta}_n$ *has characteristic functional*

$\varphi_\eta(a) := \exp\left\{-\frac{1}{2}\sum_{k=0}^{\infty}\sigma_Z((A^*)^k a)\right\}$*, where* $a \mapsto \exp\left\{-\frac{1}{2}\sigma_Z(a)\right\}$ *is the characteristic functional of* Z_1*.*

(ii) If for each $n \in \mathbb{Z}$*, we define* $Y_n := \lim_{l\to\infty} \overline{\eta}_{nl}$*, where the limit is taken in* $(\mathfrak{C}_2, \|\cdot\|_{\mathfrak{C}_2})$*, then*

(1) Y_n *is well defined for all* $n \in \mathbb{Z}$*,* $Y_n \in \mathfrak{C}_2$ *and for all* $a \in U^*$*,* $Y_n a = (AY_{n-1})a + Z_n a$ *almost surely.*

(2) If $\{X_n\}_{n\in\mathbb{Z}}$ *satisfies (3.7) and* $X_0 \overset{d}{=} Y_0$*, then* $Y_n \overset{d}{=} X_n$ *for all* $n \in \mathbb{N}$*.*

(3) If $\{X_n\}_{n\in\mathbb{Z}}$ *satisfies (3.7) and* $X_0 \in \mathfrak{C}_2$*, then* $\lim_{n\to\infty} \|X_n - Y_n\|_{\mathfrak{C}_2} = 0$*.*

(iii) The process $\{Y_n\}_{n\in\mathbb{Z}}$ *is cylindrical weakly stationary.*

(iv) If the cylindrical distribution of Z_1 *is strongly Gaussian, then the cylindrical distribution of* Y_n *is strongly Gaussian for all* $n \in \mathbb{Z}$*.*

(v) $\{Y_n\}_{n\in\mathbb{Z}}$ *is the unique cylindrical weakly stationary, centered, weakly Gaussian cylindrical process,* $Y_n \in \mathfrak{C}_2$ *for all* n*, for which, for all* $a \in U^*$*,* $Y_n a = (AY_{n-1})a + Z_n a$ *almost surely, for all* $n \in \mathbb{Z}$*, i.e. if* $\{W_n\}_{n\in\mathbb{Z}}$ *is any other cylindrical weakly stationary, centered, weakly Gaussian cylindrical process,* $W_n \in \mathfrak{C}_2$ *for all* n*, for which for all* $a \in U^*$*,* $W_n a = (AW_{n-1})a + Z_n a$ *almost surely for all* $n \in \mathbb{Z}$*, then for all* $a \in U^*$*,* $W_n a = Y_n a$ *almost surely, for all* $n \in \mathbb{Z}$*.*

Proof. (i) We show that $\{\overline{\eta}_{nl}\}_{l\geq 1}$ converges in $(\mathfrak{C}_2, \|\cdot\|_{\mathfrak{C}_2})$ as l tends to infinity. From Lemma 1 (iii), for each fixed $n \in \mathbb{Z}$ and all $l \geq 1$, $\eta_l \overset{d}{=} \overline{\eta}_{nl}$. Therefore by Lemma 1 (ii), $\overline{\eta}_{nl} \in \mathfrak{C}_2$ for each fixed $n \in \mathbb{Z}$ and all $l \geq 1$. We will now show that $\{\overline{\eta}_{nl}\}_{l\geq 1}$ is a Cauchy sequence in $(\mathfrak{C}_2, \|\cdot\|_{\mathfrak{C}_2})$. Since Z_1 is centered, i.e. $\boldsymbol{E}(Z_1 a) = 0$ for all $a \in U^*$ and the cylindrical random variables Z_k are weakly independent and identically distributed, we have that for $p \geq l \geq 1$ and $a \in U^*$,

$\|\overline{\eta}_{np}a-\overline{\eta}_{nl}a\|^2_{L_2} = \left\|\sum_{k=l}^{p-1} Z_{n-k}((A^*)^k a)\right\|^2 = \left\|\sum_{k=l}^{p-1} \|(A^*)^k a\| Z_{n-k}(\Lambda((A^*)^k a))\right\|^2 =$
$\boldsymbol{E}\left(\sum_{k=l}^{p-1} \|(A^*)^k a\| Z_{n-k}(\Lambda((A^*)^k a))\right)^2 =$
$\sum_{k=l}^{p-1} \|(A^*)^k a\|^2 \boldsymbol{E}\left(Z_{n-k}(\Lambda((A^*)^k a))\right)^2 \leq \|Z_1\|^2_{\mathfrak{C}_2} \sum_{k=l}^{p-1} \|(A^*)^k a\|^2$. Therefore

$$\|\overline{\eta}_{np}a-\overline{\eta}_{nl}a\|^2_{L_2} \leq \|Z_1\|^2_{\mathfrak{C}_2} \sum_{k=l}^{p-1} \|(A^*)^k a\|^2 \leq \|Z_1\|^2_{\mathfrak{C}_2}\|a\|^2 \sum_{k=l}^{p-1} \|(A^*)^k\|^2. \tag{3.8}$$

From this, it follows that $\|\overline{\eta}_{np}-\overline{\eta}_{nl}\|^2_{\mathfrak{C}_2} \leq \|Z_1\|^2_{\mathfrak{C}_2} \sum_{k=l}^{p-1} \|(A^*)^k\|^2$. Now, from the assumption, we have that $\sum_{k=0}^{\infty} \|(A^*)^k\|^2 < \infty$ and hence $\{\overline{\eta}_{nl}\}_{l\geq 1}$ is a Cauchy sequence in $(\mathfrak{C}_2, \|\cdot\|_{\mathfrak{C}_2})$. Since $(\mathfrak{C}_2, \|\cdot\|_{\mathfrak{C}_2})$ is complete, it follows that $\{\overline{\eta}_{nl}\}_{l\geq 1}$ converges in $(\mathfrak{C}_2, \|\cdot\|_{\mathfrak{C}_2})$ to some cylindrical random variable $\overline{\eta}_n \in \mathfrak{C}_2$ as l tends to infinity.

We now show that $\overline{\eta}_n$ is a weakly Gaussian cylindrical random variable. For $a \in U^*$ and $l \geq 1$, we have that $\boldsymbol{E}|\overline{\eta}_{nl}a-\overline{\eta}_n a|^2 \leq \|a\|^2\|\overline{\eta}_{nl}-\overline{\eta}_n\|^2_{\mathfrak{C}_2}$. Therefore $\{\overline{\eta}_{nl}a\}_{l\geq 1}$ converges to $\overline{\eta}_n a$ in L_2 as l tends to infinity. Since by Lemma 1 (i) and (iii) $\{\overline{\eta}_{nl}a\}_{l\geq 1}$ is a sequence of real Gaussian random variables, it follows that $\overline{\eta}_n a$ is a real Gaussian random variable. Since $\overline{\eta}_n$ is the limit of a sequence of elements of $\mathfrak{C}_2$ which is a Banach space, we have that it is an element of $\mathfrak{C}_2$, hence it is linear, continuous and $\sup\{\|\overline{\eta}_n a\| : \|a\| \leq 1\} < \infty$. We therefore have that $\overline{\eta}_n$ is linear, continuous and $\overline{\eta}_n a$ is real Gaussian for all $a \in U^*$. It thus follows that $\overline{\eta}_n$ is a weakly Gaussian cylindrical random variable. It remains to show that the cylindrical distribution of $\overline{\eta}_n$ is independent of n. To do this, we compute the characteristic functional of $\overline{\eta}_n$.

For the characteristic functional $\varphi_{\overline{\eta}_{nl}}$ of $\overline{\eta}_{nl}$, we have that for $a \in U^*$, $\varphi_{\overline{\eta}_{nl}}(a) = \varphi_{\overline{\eta}_{nl}a}(1) = \boldsymbol{E}\left(\exp\{i\overline{\eta}_{nl}a\}\right) = \boldsymbol{E}\exp\left\{\sum_{k=0}^{l-1} iZ_{n-k}((A^*)^k a)\right\}$
$= \prod_{k=0}^{l-1} \boldsymbol{E}\exp\{iZ_{n-k}((A^*)^k a)\} = \exp\left\{-\frac{1}{2}\sum_{k=0}^{l-1} \sigma_Z((A^*)^k a)\right\}$.

Since $\{\overline{\eta}_{nl}a\}$ converges to $\overline{\eta}_n a$ in L_2, it follows that $\varphi_{\overline{\eta}_{nl}a}(1)$ converges to $\varphi_{\overline{\eta}_n a}(1) = \varphi_{\overline{\eta}_n}(a)$ as l tends to infinity. Therefore $\varphi_{\overline{\eta}_n}(a) = \exp\{-\frac{1}{2}\sum_{k=0}^{\infty} \sigma_Z((A^*)^k a)\}$. If we show that the map $\varphi_{\overline{\eta}_n}(\cdot)$ is indeed the characteristic functional of a weakly Gaussian cylindrical random variable, then it will follow from Proposition 1 (a), that it is the characteristic functional of $\overline{\eta}_n$. Further, it will also follow that the cylindrical distribution of $\overline{\eta}_n$ is independent of n and by Theorem 1, that $\overline{\eta}_n$ is a weakly Gaussian cylindrical random variable. Note that the series $\sum_{k=0}^{\infty} \sigma_Z((A^*)^k a)$ is convergent for all $a \in U^*$. This follows from the following argument: From the assumption of the theorem, $\sum_{k=0}^{\infty} \|(A^*)^k\| < \infty$, thus there exists n_0 such that for all $n \geq n_0$, $\|(A^*)^k\| < \delta < 1$ for any fixed $0 < \delta < 1$ chosen arbitrarily and hence for all $n \geq n_0$, $\|(A^*)^k\|^2 \leq \|(A^*)^k\| < \delta$ so that $\sum_{k=n_0}^{\infty} \|(A^*)^k\|^2 \leq \sum_{k=n_0}^{\infty} \|(A^*)^k\| < \infty$. Also, by Remark 4, $|\sigma_Z(a)| \leq M\|a\|^2$ for some constant M and all $a \in U^*$. Therefore $\sum_{k=0}^{\infty} |\sigma_Z((A^*)^k a)| \leq \sum_{k=0}^{\infty} M\|(A^*)^k\|^2\|a\|^2 \leq M \sum_{k=0}^{\infty} \|(A^*)^k\|\|a\|^2 < \infty$.

On U^*, define $m(a) := 0$ and $\sigma(a) := \sum_{k=0}^{\infty} \sigma_Z((A^*)^k a)$, then m is linear. Since σ_Z is a quadratic form, if we define $f(x,y) := \frac{1}{2}[\,\sigma_Z(x) + \sigma_Z(y) - \sigma_Z(x-y)\,]$, then f will be bilinear, so that for all $k \geq 0$, $(x,y) \mapsto f((A^*)^k x, (A^*)^k y)$ is bilinear and $(x,y) \mapsto \sum_{k=0}^{l-1} f((A^*)^k x, (A^*)^k y)$ is bilinear for all $l \geq 1$. Therefore for all $l \geq 1$, $h_l(x,y) := \displaystyle\sum_{k=0}^{l-1} \frac{1}{2}[\,\sigma_Z((A^*)^k x) + \sigma_Z((A^*)^k y) - \sigma_Z((A^*)^k (x-y))\,]$ is a bilinear function of (x,y). Since $\sum_{k=0}^{\infty} \sigma_Z((A^*)^k a) < \infty$ for all $a \in U^*$, it follows that $h(x,y) := \lim_{l\to\infty} h_l(x,y)$ is well defined for all $(x,y) \in U^* \times U^*$ and h defines a bilinear form. Thus $\sigma(a) = h(a,a)$ defines a quadratic form; i.e. $\sigma(a) := \sum_{k=0}^{\infty} \sigma_Z((A^*)^k a)$ defines a quadratic form on U^*. Therefore $\varphi_\eta(a) := \exp\{\,-\frac{1}{2} \sum_{k=0}^{\infty} \sigma_Z((A^*)^k a)\,\}$ is the characteristic functional of a weakly Gaussian cylindrical measure.

(ii) (1) From (i), we have that for each $n \in \mathbb{Z}$, Y_n is well defined as a weakly Gaussian cylindrical random variable which is the $\mathfrak{C}_2$-limit of $\{\overline{\eta}_{nl}\}_{l\geq 1}$ as l tends to infinity. By (3.8), for each $a \in U^*$, $\sum_{k=0}^{\infty} Z_{n-k}((A^*)^k a)$ converges in L_2. It holds that $\boldsymbol{E}Z_{n-k}((A^*)^k a) = 0$ for each $a \in U^*$. By Remark 4 and the fact that $\{Z_k\}_{k\in\mathbb{Z}}$ is i.i.d., we also have that $\sum_{k=0}^{\infty} \boldsymbol{E}[Z_{n-k}((A^*)^k a)]^2 = \sum_{k=0}^{\infty} \sigma_Z((A^*)^k a)) <$

∞. It therefore follows that $\sum_{k=0}^{\infty} Z_{n-k}((A^*)^k a)$ converges almost surely for each $a \in U^*$. We then have that for each $a \in U^*$, $Y_n a = \sum_{k=0}^{\infty} Z_{n-k}((A^*)^k a) = \sum_{k=1}^{\infty} Z_{n-k}((A^*)^k a) + Z_n(a) = \sum_{k=0}^{\infty} Z_{n-k-1}((A^*)^{k+1} a) + Z_n a$ almost surely. We note that Y_{n-1} is a cylindrical random variable and $(AY_{n-1})a = \sum_{k=0}^{\infty} Z_{n-1-k}((A^*)^{k+1} a)$ almost surely. Therefore $Y_n a = (AY_{n-1})a + Z_n a$ almost surely for all $a \in U^*$.

(ii) (2) We note first that for each $a \in U^*$ and $k \in \mathbb{N}$, the series $Y_0((A^*)^k a)$ converges almost surely. By assumption, $Y_0 \overset{d}{=} X_0$. Since $\eta_n = \overline{\eta}_{nn}$ for all $n \geq 1$, X_0 and η_n are weakly independent, and Y_0 and $\overline{\eta}_{nn}$ are weakly independent, we have that for each $a \in U^*$ and $n \in \mathbb{N}$, $X_0((A^*)^n a) + \eta_n a \overset{d}{=} Y_0((A^*)^n a) + \overline{\eta}_{nn} a$. By Lemma 1 (i), we have that for $n \geq 1$, $X_n a = X_0((A^*)^n a) + \eta_n a$ and hence for $n \geq 1$, $X_n a \overset{d}{=} Y_0((A^*)^n a) + \overline{\eta}_{nn} a$. Further, we have that for all $a \in U^*$,

$$\begin{aligned} Y_0((A^*)^n a) + \overline{\eta}_{nn} a &= \sum_{k=0}^{\infty} Z_{-k}((A^*)^{n+k} a) + \sum_{k=0}^{n-1} Z_{n-k}((A^*)^k a) \\ &= \sum_{k=n}^{\infty} Z_{n-k}((A^*)^k a) + \sum_{k=0}^{n-1} Z_{n-k}((A^*)^k a) \\ &= \sum_{k=0}^{\infty} Z_{n-k}((A^*)^k a) = Y_n a, \end{aligned}$$

almost surely. It thus holds that X_n and Y_n have the same characteristic functional for each n and hence $X_n \overset{d}{=} Y_n$ for all $n \in \mathbb{N}$.

(ii) (3) Suppose that $X_0 \in \mathfrak{C}_2$. By (ii) (1), for all $a \in U^*$, $Y_n a = Y_0((A^*)^n a) + \eta_n a$ almost surely. Also $X_n a = X_0((A^*)^n a) + \eta_n a$. It follows that on a set of measure 1 depending on a, $|X_n a - Y_n a|^2 = |X_0((A^*)^n a) + \eta_n - (Y_0((A^*)^n a) - \eta_n|^2 = |X_0((A^*)^n a) - Y_0((A^*)^n a)|^2$. Therefore

$$\begin{aligned} \|X_n a - Y_n a\|^2 &= \boldsymbol{E}|X_n a - Y_n a|^2 = \boldsymbol{E}|X_0((A^*)^n a) - Y_0((A^*)^n a)|^2 \\ &= \|X_0((A^*)^n a) - Y_0((A^*)^n a)\|^2 \leq \|(A^*)^n\|^2 \|X_0 - Y_0\|_{\mathfrak{C}_2}^2 \|a\|^2. \end{aligned}$$

Hence $\|X_n - Y_n\|_{\mathfrak{C}_2} \leq \|(A^*)^n\| \|X_0 - Y_0\|_{\mathfrak{C}_2}$. Since $\sum_{k=0}^{\infty} \|(A^*)^k\| < \infty$, the assertion follows since we also have that $Y_0 \in \mathfrak{C}_2$.

(iii) Let $n \in \mathbb{Z}$. For $m \geq 1$, let $a_1, \ldots, a_m \in U^*$, then since $\sum_{k=0}^{\infty} Z_{n-k}((A^*)^k a)$ converges in L_2 for all $a \in U^*$, we have that for some constant $M < \infty$, $\boldsymbol{E}\|Y_n(a_1, \ldots, a_m)\|^2 = \sum_{j=1}^{m} \boldsymbol{E}|Y_n a_j|^2 = \sum_{j=1}^{m} \boldsymbol{E} \sum_{kl=0}^{\infty} Z_{n-k}((A^*)^k a_j) Z_{n-l}((A^*)^l a_j) = \sum_{j=1}^{m} \sum_{kl=0}^{\infty} \boldsymbol{E} Z_{n-k}((A^*)^k a_j) Z_{n-l}((A^*)^l a_j) = \sum_{j=1}^{m} \sum_{k=0}^{\infty} \boldsymbol{E} Z_{n-k}((A^*)^k a_j)^2 = \sum_{j=1}^{m} \sum_{k=0}^{\infty} \sigma_Z((A^*)^k a_j) \leq M \sum_{j=1}^{m} \|a_j\|^2 \sum_{k=0}^{\infty} \|(A^*)^k\| < \infty$. Also, for each $n \in \mathbb{Z}$, $\boldsymbol{E} Y_n(a_1, \cdots, a_m) = (\boldsymbol{E} Y_n a_1, \ldots, \boldsymbol{E} Y_n a_m) = 0$ and lastly, if $m \geq 1$, $a_1, \ldots, a_m \in U^*$, s, t, $h \in \mathbb{Z}$, $h \geq 0$, α, $\beta \in \mathbb{R}^m$, $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_m)$, then $\boldsymbol{E}\langle Y_{s+h}(a_1, \ldots, a_m), \alpha\rangle \langle Y_{t+h}(a_1, \ldots, a_m), \beta\rangle = \boldsymbol{E}[\sum_{j=1}^{m} \alpha_j Y_{s+h} a_j][\sum_{l=1}^{m} \beta_l Y_{t+h} a_l] = \sum_{j,l=1}^{m} \alpha_j \beta_l \boldsymbol{E}(Y_{s+h} a_j)(Y_{t+h} a_l) = \sum_{j,l=1}^{m} \alpha_j \beta_l \sum_{n,k=0}^{\infty} \boldsymbol{E} Z_{s+h-k}((A^*)^k a_j) Z_{t+h-n}((A^*)^n a_l)$

$$= \sum_{j,l=1}^{m} \alpha_j \beta_l \sum_{\{k,n|s-k=t-n\}}^{\infty} \boldsymbol{E} Z_{s+h-k}((A^*)^k a_j) Z_{t+h-n}((A^*)^n a_l).$$

A similar argument shows that, $\boldsymbol{E}\langle Y_s(a_1, \ldots, a_m), \alpha\rangle \langle Y_t(a_1, \ldots, a_m), \beta\rangle = \sum_{j,l=1}^{m} \alpha_j \beta_l \sum_{\{k,n|s-k=t-n\}}^{\infty} \boldsymbol{E} Z_{s-k}((A^*)^k a_j) Z_{t-n}((A^*)^n a_l)$. By Lemma 2, for a, $b \in U^*$, the joint distribution of $Z_m(a, b) := (Z_m a, Z_m b)$ is independent of m. It therefore follows that $\boldsymbol{E}\left[Z_m((A^*)^k a_j)(Z_m((A^*)^n a_l)\right]$ is independent of m and the proof is complete.

(iv) Suppose that the cylindrical distribution of Z_1 is strongly Gaussian with covariance operator Q, then $\sigma_Z(a) = \langle Qa, a\rangle$, where Q is a positive symmetric Operator, $Q : U^* \to U$(also see [18] page 196).

Define $\tilde{Q}a := \sum_{k=0}^{\infty} A^k Q (A^k)^* a$, $a \in U^*$, then obviously, $\tilde{Q} : U^* \to U$ and $\|\tilde{Q}\| \leq \|Q\| \sum_{k=0}^{\infty} \|A^k\|^2 < \infty$. Also, $\langle \tilde{Q}a, a\rangle = \sum_{k=0}^{\infty} \langle Q(A^k)^* a, (A^k)^* a\rangle \geq 0$ since Q is positive. Finally, if u^*, $v^* \in U^*$, then using the symmetry of Q we get

$$\begin{aligned}
\langle \tilde{Q}u^*, v^*\rangle &= \langle \sum_{k=0}^{\infty} A^k Q (A^k)^* u^*, v^*\rangle = \sum_{k=0}^{\infty} \langle Q(A^*)^k u^*, (A^k)^* v^*\rangle \\
&= \sum_{k=0}^{\infty} \langle Q(A^k)^* v^*, (A^k)^* u^*\rangle = \langle \sum_{k=0}^{\infty} A^k Q (A^*)^k v^*, u^*\rangle = \langle \tilde{Q}v^*, u^*\rangle.
\end{aligned}$$

Therefore $\tilde{Q}$ is positive, symmetric and $\tilde{Q} : U^* \to U$. Also we have that

$$\begin{aligned} \sigma_Y(a) &= \sum_{k=0}^{\infty} \sigma_Z((A^k)^*a) = \sum_{k=0}^{\infty} \langle Q(A^k)^*a, (A^k)^*a\rangle = \langle \sum_{k=0}^{\infty} A^k Q (A^k)^* a, a\rangle \\ &= \langle \tilde{Q}a, a\rangle. \end{aligned}$$

Therefore the cylindrical distribution of Y_n is strongly Gaussian.

(v) Suppose that $\{W_n\}_{n\in\mathbb{Z}}$ is another cylindrical weakly stationary, centered, weakly Gaussian cylindrical process such that $W_n \in \mathfrak{C}_2$ for all n, $W_n a = (AW_{n-1})a + Z_n a$ almost surely for all $a \in U^*$, then $|W_n a - Y_n a| = |(AW_{n-1})a - (AY_{n-1})a| = |((W_{n-1}(A^*)a) - Y_{n-1}((A^*)a)|$, so that by iterating, for all $k \geq 1$, we have that

$$\begin{aligned} |W_n a - Y_n a|^2 &= |W_{n-k}((A^*)^k a) - Y_{n-k}((A^*)^k a)|^2 \\ &\leq \|(A^*)^k a\|^2 |W_{n-k}(\Lambda((A^*)^k a)) - Y_{n-k}(\Lambda((A^*)^k a))|^2 \\ &\leq 2\|(A^*)^k\|^2 \|a\|^2 |W_{n-k}(\Lambda((A^*)^k a))|^2 \\ &+ 2\|(A^*)^k\|^2 \|a\|^2 |Y_{n-k}(\Lambda((A^*)^k a))|^2. \end{aligned}$$

Since $\{W_l\}_{l\in\mathbb{Z}}$ is cylindrical weakly stationary, it follows that for fixed k and each $a \in U^*$, $\{W_l(\Lambda((A^*)^k a))\}_{l\in\mathbb{Z}}$ is weakly stationary in $\mathbb{R}^1$.
Therefore $\boldsymbol{E}|W_l(\Lambda((A^*)^k a))|^2$ is independent of l, hence
$\boldsymbol{E}|W_l(\Lambda((A^*)^k a))|^2 = \boldsymbol{E}|W_1(\Lambda((A^*)^k a))|^2$ for all $l \in \mathbb{Z}$. In particular,
$\boldsymbol{E}|W_{n-k}(\Lambda((A^*)^k a))|^2 = \boldsymbol{E}|W_1(\Lambda((A^*)^k a))|^2$. Since $W_1 \in \mathfrak{C}_2$, $a \mapsto \sigma_W(a)$ is continuous and hence $\boldsymbol{E}|W_1(\Lambda((A^*)^k a))|^2 = \sigma_W(\Lambda((A^*)^k a)) \leq M_W \|\Lambda((A^*)^k a)\|^2 = M_W$ for some constant M_W. For arbitrary $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}(\|(A^*)^k\| |W_{n-k}(\Lambda((A^*)^k a))| > \varepsilon) &\leq \frac{\boldsymbol{E}\|(A^*)^k\|^2 |W_{n-k}(\Lambda((A^*)^k a))|^2}{\varepsilon^2} \\ &\leq \frac{\|(A^*)^k\|^2 M_W}{\varepsilon^2}. \end{aligned}$$

Therefore $\sum_{k=0}^{\infty} \mathbb{P}(\|(A^*)^k\| |W_{n-k}(\Lambda((A^*)^k a))| > \varepsilon) \leq \sum_{k=0}^{\infty} \frac{\|(A^*)^k\|^2 M_W}{\varepsilon^2} < \infty$ and hence by Borel Cantelli's Lemma, $\lim_{k\to\infty} \|(A^*)^k\| |W_{n-k}(\Lambda((A^*)^k a))| = 0$ almost surely. A similar argument applied to $\{Y_n\}_{n\in\mathbb{Z}}$ shows that for all $a \in U^*$, $\lim_{k\to\infty} \|(A^*)^k\| |Y_{n-k}(\Lambda((A^*)^k a))| = 0$ almost surely. This shows that for all $a \in U^*$, $Y_n a = W_n a$ almost surely. QED

Example 1. Let U be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $\{Z_k\}$ be a sequence of weakly independent standard Gaussian cylindrical random variables in U, then the characteristic functional of Z_1 is given by $\varphi_{Z_1}(a) = \varphi_{Z_1 a}(1) = \boldsymbol{E}(e^{iZ_1 a}) = \exp\{-\frac{1}{2}\|a\|^2\}$. Therefore $m_Z(a) := \boldsymbol{E}(Z_1 a) = 0$ and hence $\sigma_Z(a) = \boldsymbol{E}((Z_1 a)^2) = \|a\|^2$. Thus $\|Z_1\|_{\mathfrak{C}_2} = 1 < \infty$, $\|Z_1 a\| = \|a\|$, i.e. Z_1 is continuous. If the operaror $A : U \to U$ is such that $\sum_{k=0}^{\infty} \|(A^*)^k\| < \infty$, for example when $\|A\| < 1$ then Y_n converges in $\mathfrak{C}_2$ for all $n \in \mathbb{Z}$ and the assertions of Theorem 2 hold.

4 Conclusion

We have considered the cylindrical auto-regressive process $X_n = AX_{n-1} + Z_n$, $n \in \mathbb{Z}$ in a seperable Banach space U, $Z_n a \in G$ for all n and $a \in U^*$, where G is a fixed, closed, seperable subspace of $L_2(\Omega, \mathcal{F}, \mathbb{P})$, A is a bounded linear operator on U and $\{Z_n\}_{n\in\mathbb{Z}}$ is a sequence of identically distributed, weakly independent, weakly Gaussian cylindrical random variables in U and determined a cylindrical process $\{Y_n\}_{n\in\mathbb{Z}}$ in U such that for all $a \in U^*$, $Y_n a = (AY_{n-1})a + Z_n a$ almost surely. We have shown that the cylindrical distribution of Y_n is weakly Gaussian and independent of n. Conditions are given for the cylindrical distribution of Y_n to be strongly Gaussian. We have also determined the characteristic functional of Y_n and conditions under which $\{Y_n\}_{n\in\mathbb{Z}}$ is unique are given.

The results above are obtained under the condition that $\sum_{k=0}^{\infty} \|(A^*)^k\| < \infty$ which ensures almost sure convergence of the series $\sum_{k=0}^{\infty} Z_{n-k}((A^*)a)$, $a \in u^*$. This is weaker than the condition, $\lim_{n\to\infty} \frac{1}{n+1} \ln \|A_0 \cdots A_n\| < 0$ almost surely, required for the almost sure convergence of the series $\sum_{k=0}^{\infty} A_n \cdots A_{n-k+1} B_{n-k}$ in [8], since the terms of the series in the latter case are required to tend to 0 at an exponential rate.

Just as the results on Autoregressive processes in Banach spaces are essentially an extension of results for the Euclidean space case, the results we have obtained extend results on the general Banach space situation to the case of cylindrical random variables. In addition it is very interesting that the concepts of stationarity for Banach space random variables can be extended in the spirit of cylindrical processes.

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