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The Ricci soliton equation for homogeneous Siklos spacetimes

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Abstract. We complete the classification of Ricci solitons within all classes of homogeneous Siklos metrics.

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1 Introduction

In global coordinates $(v, u, x, y) = (x_1, x_2, x_3, x_4)$, Siklos metrics are described as the family of Lorentzian metrics of the form

$$g = -\frac{3}{\Lambda x_3^2}\left(2dx_1dx_2 + Hdx_2^2 + dx_3^2 + dx_4^2\right), \tag{1.1}$$

where $H = H(x_2, x_3, x_4)$ is an arbitrary smooth function (see [24],[23]). These metrics are of Petrov type N, and yield exact solutions to Einstein's field equations with an Einstein-Maxwell source, with a cosmological constant $\Lambda < 0$. All of them admit a null non-twisting Killing field. For several subclasses of (1.1), additional Killing vector fields appear. In particular, several homogeneous subclasses of Siklos spacetimes exist [24]. These metrics have been intensively studied. We may refer to [23],[20],[21],[9]-[11],[15] for several results about these spacetimes.

A *Ricci soliton* is a pseudo-Riemannian manifold (M, g), together with some smooth tangent vector field X and a real constant λ, such that

$$\mathcal{L}_X g + \varrho = \lambda g, \tag{1.2}$$

with $\mathcal{L}_X$ and ϱ respectively denoting the Lie derivative in the direction of X and the Ricci tensor. The Ricci soliton is called either *shrinking*, *steady* or

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expanding, depending on whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$. An Einstein manifold (M, g), together with a Killing vector field X, yields a trivial solution to the Ricci soliton equation. References [1]-[14],[22] provide several examples of study of Ricci solitons in the framework of pseudo-Riemannian (and in particular, Lorentzian) geometry.

In the general context of investigation of the Ricci soliton equation, metrics with a higher degree of symmetry, and in particular homogeneous examples, play a special role. It may be readily observed that if (M, g) and X satisfy the Ricci soliton equation (1.2), then the same is true for the same real constant λ substituting X by $X+Y$, where Y denotes any Killing vector field on M. This simple observation gives more freedom in the research for solutions of (1.2) in presence of additional symmetries of (M, g).

Moreover, it is worth to emphasize the fact that homogeneous *Riemannian* Ricci solitons $(M = G/H, g)$ are necessarily *algebraic* ([18],[19]) and so, determined by suitable derivations of the Lie algebra $\mathfrak{g}$ of the transitive Lie group G. On the other hand, algebraic Ricci solitons do not exhaust the homogeneous solutions of (1.2) in pseudo-Riemannian settings (see for example [8],[14]).

With respect to a system of local coordinates, the Ricci soliton equation translates into an overdetermined system of nonlinear second order PDEs, which is in general very difficult to solve. With regard to homogeneous Siklos metrics, we started investigating the solutions to the Ricci soliton equation (1.2) in [9], solving it completely for a well known subclass. Further contributions to the study of homogeneous Siklos metrics giving rise to Ricci solitons, were obtained in [10] and [11]. The aim of this paper is to treat the remaining homogeneous Siklos metrics, determining for them all solutions of (1.2).

We will also show that such solutions are higly nontrivial, because they are not gradient Ricci solitons. A *gradient* Ricci soliton satisfies (1.2) for some vector field $X = \mathrm{grad}_g(f)$, where f is a smooth function. The existence of a gradient Ricci soliton usually imposes some very strong restrictions on the geometry of the manifold.

The paper is organized in the following way. In Section 2 we recall some general information about the curvature of Siklos metrics and the classification of the homogeneous examples. In Section 3 we shall determine when homogeneous Siklos metrics, not investigated in previous works, are Ricci solitons and we prove that the solutions are not of gradient type. Calculations have been checked using the software *Maple 16*©.

2 On the geometry of Siklos metrics

We briefly report the essential information concerning the Levi-Civita connection and curvature of Siklos metrics, as deduced in [23] and [9]. With respect to the global coordinates (x_1, x_2, x_3, x_4) used in (1.1), the Levi-Civita connection ∇ of g is completely determined by the following possibly non-vanishing components:

$$\begin{aligned}
\nabla_{\partial_1}\partial_2 &= \tfrac{1}{x_3}\partial_3, \\
\nabla_{\partial_1}\partial_3 &= -\tfrac{1}{x_3}\partial_1, \\
\nabla_{\partial_2}\partial_2 &= \tfrac{1}{2}(\partial_2 H)\partial_1 + \tfrac{1}{2x_3}(2H - x_3\partial_3 H)\partial_3 - \tfrac{1}{2}(\partial_4 H)\partial_4, \\
\nabla_{\partial_2}\partial_3 &= \tfrac{1}{2}(\partial_3 H)\partial_1 - \tfrac{1}{x_3}\partial_2, \\
\nabla_{\partial_2}\partial_4 &= \tfrac{1}{2}(\partial_4 H)\partial_1, \\
\nabla_{\partial_3}\partial_3 &= -\tfrac{1}{x_3}\partial_3, \\
\nabla_{\partial_3}\partial_4 &= -\tfrac{1}{x_3}\partial_4, \\
\nabla_{\partial_4}\partial_4 &= \tfrac{1}{x_3}\partial_3,
\end{aligned} \tag{2.1}$$

where $\partial_i := \frac{\partial}{\partial_i}$ are coordinate vector fields. We already observed in [9] that Siklos metrics do not admit any parallel vector field. In particular, they are not locally reducible.

The Riemann-Christoffel curvature tensor $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ of g is completely determined by the following possibly non-vanishing components:

$$\begin{aligned}
R_{1212} &= -\tfrac{3}{\Lambda x_3^4}, & R_{1323} &= \tfrac{3}{\Lambda x_3^4}, \\
R_{1424} &= \tfrac{3}{\Lambda x_3^4}, & R_{2323} &= \tfrac{3\left(2H - x_3(\partial_3 H) + x_3^2(\partial_{33}^2 H)\right)}{2\Lambda x_3^4}, \\
R_{2324} &= \tfrac{3\left(\partial_{34}^2 H\right)}{2\Lambda x_3^2}, & R_{2424} &= \tfrac{3\left(2H - x_3(\partial_3 H) + x_3^2(\partial_{44}^2 H)\right)}{2\Lambda x_3^4}, \\
R_{3434} &= -\tfrac{3}{\Lambda x_3^4},
\end{aligned} \tag{2.2}$$

where $R_{ijkl} = g(\partial_i, R(\partial_k, \partial_l)\partial_j)$. In terms of its components with respect to $\{\partial_i\}$, the *Ricci tensor* of g is determined by the matrix

$$\varrho = \begin{pmatrix}
0 & -3x_3^{-2} & 0 & 0 \\
-3x_3^{-2} & -\frac{6H - 2x_3(\partial_3 H) + x_3^2\left(\partial_{33}^2 H + \partial_{44}^2 H\right)}{2x_3^2} & 0 & 0 \\
0 & 0 & -3x_3^{-2} & 0 \\
0 & 0 & 0 & -3x_3^{-2}
\end{pmatrix} \tag{2.3}$$

and the *Ricci operator Ric*, which is defined by $\varrho(X,Y) = g(Ric(X),Y)$, is determined by

$$Ric = \begin{pmatrix} \Lambda & -\frac{1}{6}\Lambda x_3 \left\{2(\partial_3 H) - x_3\left(\partial^2_{33}H + \partial^2_{44}H\right)\right\} & 0 & 0 \\ 0 & \Lambda & 0 & 0 \\ 0 & 0 & \Lambda & 0 \\ 0 & 0 & 0 & \Lambda \end{pmatrix}. \tag{2.4}$$

The *scalar curvature* of a Siklos metric is $\tau = 4\Lambda$. Einstein and locally conformally flat Siklos metrics are characterized in the following propositions.

Proposition 1 ([24],[23],[9])**.** For an arbitrary Siklos metric g, as described in (1.1), the following conditions are equivalent:

(i) g is Einstein. More precisely, $\varrho = \Lambda g$;

(ii) g is Ricci-parallel (that is, $\nabla\varrho = 0$);

(iii) the defining function $H = H(x_2, x_3, x_4)$ satisfies the PDE

$$\frac{2}{x_3}(\partial_3 H) - \partial^2_{33}H - \partial^2_{44}H = 0. \tag{2.5}$$

Proposition 2 ([10])**.** A Siklos metric g, as described in (1.1), is locally conformally flat if and only if the defining function $H = H(x_2, x_3, x_4)$ satisfies the system of PDEs

$$\begin{cases} \partial^2_{33}H - \partial^2_{44}H = 0, \\ \partial^2_{34}H = 0, \end{cases} \tag{2.6}$$

that is, when H is explicitly given by

$$H(x_2, x_3, x_4) = \frac{1}{2}\, T(x_2)\left(x_3^2 + x_4^2\right) + L(x_2)\, x_3 + M(x_2)\, x_4 + N(x_2), \tag{2.7}$$

where T, L, M, N are arbitrary smooth functions.

As we already recalled in the Introduction, Siklos [24] completely classified metrics of the form (1.1) admitting some additional vector fields. Homogeneous Siklos metrics correspond to cases admitting at least four linearly independent Killing vector fields. They form five subclasses I),. . . ,V), and also include as a special case the homogeneous Siklos metrics isometric to the anti-de Sitter spacetime, which, being Einstein, are trivial cases for the actual investigation.

For each of subclasses I),. . . ,V), we report below the special form of the defining function H, using the notation adopted in this paper for the global coordinates and the gravitational constant. Following [24], $A(x_i)$ will denote an arbitrary smooth function of variables x_i, while we shall use $A_\alpha(x_i)$ to denote a homogeneous function of degree α of the specified variables.

I) $H = A_{-2}(x_3, x_4)$.

II) $H = A(x_3)$.

III) $H = A(x_2)x_3^2$.

IV) $H = x_2^{2\beta-2}\, A(x_2^\beta x_3)$.

V) $H = \pm x_3^\alpha$.

Ricci solitons have already been completely determined in previous works for cases from II) to V). In fact, the main result of [9] classified Ricci solitons in the class V).

Theorem 1 ([9])**.** All homogeneous Siklos metrics defined by $H = \pm x_3^\alpha$, are solutions to the Ricci soliton equation. These Ricci solitons are not gradient.

Moreover, the following result holds.

Theorem 2 ([11])**.** Let g denote an arbitrary Siklos metric of the form (1.1) with defining function $H = F(x_2, x_3)+G(x_2, x_4)$, for arbitrary smooth functions F and G. Then, g is a Ricci soliton if and only if H takes one of the following forms:

$$\begin{aligned}(i) \quad H(x_2, x_3, x_4) \quad &= \quad P(x_2)x_3^{\frac{3\Lambda-6\alpha+\sqrt{9\Lambda^2+12\Lambda\alpha+36\alpha^2}}{2\Lambda}} \\ &\quad +Q(x_2)x_3^{\frac{3\Lambda-6\alpha-\sqrt{9\Lambda^2+12\Lambda\alpha+36\alpha^2}}{2\Lambda}} \\ &\quad +\tfrac{1}{2}\, T(x_2)\left(x_3^2 + x_4^2\right) + M(x_2)\, x_4 \\ &\quad +\tfrac{1}{2\alpha}\big(2A_1'(x_2) + M(x_2)A_4(x_2)\big),\end{aligned}$$

where P, Q, T, M, A_1 are some arbitrary smooth functions, $\alpha \neq 0$ is a real constant and A_4 satisfies

$$2A_4''(x_2) - T(x_2)A_4(x_2) + \alpha M(x_2) = 0. \tag{2.8}$$

$$\begin{aligned}(ii) \quad H(x_2, x_3, x_4) \quad &= \quad \tfrac{1}{3}\, S(x_2)x_3^3 + \tfrac{1}{2}\, T(x_2)\left(x_3^2 + x_4^2\right) \\ &\quad +\tfrac{2\ln(x_3)}{\Lambda}\big(2A_1'(x_2) + M(x_2)A_4(x_2)\big) \\ &\quad +M(x_2)\, x_4 + N(x_2),\end{aligned}$$

where S, T, M, N, A_1 are some arbitrary smooth functions and A_4 satisfies equation (2.8) with $\alpha = 0$.

As proved in [11, Proposition 3.2], with the obvious exception of the trivial Einstein cases, the above Ricci solitons are not gradient.

We may observe that homogeneous Siklos metrics within the classes II), III) and IV) are defined by a function H which is a special case of $H = F(x_2, x_3) + G(x_2, x_4)$. Therefore, comparing the results of Theorem 2 with the defining function H of cases II)-IV), we get the following (see also [11, Section 4]).

Corollary 1. A) A homogeneous Siklos metric in class *II)*, i.e., defined by $H = A(x_3)$, is a solution to the Ricci soliton equation if and only if H is of the form either

$$(i) \quad H = A(x_3) \quad = \quad P x_3^{\frac{3\Lambda-6\alpha+\sqrt{9\Lambda^2+12\Lambda\alpha+36\alpha^2}}{2\Lambda}} + Q x_3^{\frac{3\Lambda-6\alpha-\sqrt{9\Lambda^2+12\Lambda\alpha+36\alpha^2}}{2\Lambda}} + \tfrac{1}{\alpha} C,$$

where P, Q, C and $\alpha \neq 0$ are real constants; or

$$(ii) \quad H = A(x_3) = \frac{1}{3} S x_3^3 + \frac{C}{\Lambda} \ln(x_3) + N,$$

where S, C, N are real constants.

B) Homogeneous Siklos metrics in class *III)*, i.e., defined by $H = A(x_2) x_3^2$, are not solutions to the Ricci soliton equation (except in the trivial case $A = 0$).

C) A homogeneous Siklos metric in class *IV)*, i.e., defined by $H = x_2^{2\beta-2} A(x_2^\beta x_3)$, is a solution to the Ricci soliton equation if and only if $\beta = 0$ and H is of the form either

$$\begin{aligned} (i)\ H(x_2, x_3, x_4) \quad &= \quad P x_2^{-2} x_3^{\frac{3\Lambda-6\alpha+\sqrt{9\Lambda^2+12\Lambda\alpha+36\alpha^2}}{2\Lambda}} \\ &\quad + Q x_2^{-2} x_3^{\frac{3\Lambda-6\alpha-\sqrt{9\Lambda^2+12\Lambda\alpha+36\alpha^2}}{2\Lambda}} + \tfrac{1}{\alpha} C x_2^{-2}, \end{aligned}$$

where P, Q, C are real constants and $\alpha = -\frac{1}{3}\Lambda$; or

$$(ii) \quad H(x_2, x_3, x_4) = \frac{1}{3} S x_2^{-2} x_3^3 + \frac{C}{\Lambda} x_2^{-2} \ln(x_3) + N x_2^{-2},$$

where S, C, N are some real constants.

In the next section we shall consider the remaining case I), so achieving the complete classification of homogeneous Siklos metrics which are Ricci solitons.

3 Ricci soliton Siklos metrics

We again refer to the system of global coordinates (x_1, x_2, x_3, x_4) used in (1.1) to describe the whole class of Siklos metrics. Let $X = X_i \partial_i$ be an arbitrary vector field, where $X_i = X_i(x_1, x_2, x_3, x_4)$, $i = 1, \ldots, 4$, are smooth functions. The Lie derivative $\mathcal{L}_X g$ is completely determined by the components $(\mathcal{L}_X g)_{ij} = (\mathcal{L}_X g)(\partial_i, \partial_j), i \leq j = 1, \ldots, 4$, and can be calculated starting from (2.1) (see also [9]). Explicitly, we have:

$$
\left\{
\begin{array}{l}
(\mathcal{L}_X g)_{11} = -\frac{6}{\Lambda x_3^2}\,\partial_1 X_2, \\
(\mathcal{L}_X g)_{12} = -\frac{3}{\Lambda x_3^3}\,\{x_3\partial_1 X_1 + x_3 H \partial_1 X_2 + x_3 \partial_2 X_2 - 2X_3\}, \\
(\mathcal{L}_X g)_{13} = -\frac{3}{\Lambda x_3^2}\,\{\partial_3 X_2 + \partial_1 X_3\}, \\
(\mathcal{L}_X g)_{14} = -\frac{3}{\Lambda x_3^2}\,\{\partial_4 X_2 + \partial_1 X_4\}, \\
(\mathcal{L}_X g)_{22} = -\frac{3}{\Lambda x_3^3}\,\{2x_3\partial_2 X_1 + x_3 \partial_2 H\, X_2 + 2x_3 H\,\partial_2 X_2 - 2H\,X_3 \\
\qquad\qquad\qquad + x_3\partial_3 H\,X_3 + x_3 \partial_4 H\,X_4\}, \\
(\mathcal{L}_X g)_{23} = -\frac{3}{\Lambda x_3^2}\,\{\partial_3 X_1 + H\,\partial_3 X_2 + \partial_2 X_3\}, \\
(\mathcal{L}_X g)_{24} = -\frac{3}{\Lambda x_3^2}\,\{\partial_4 X_1 + H\,\partial_4 X_2 + \partial_2 X_4\}, \\
(\mathcal{L}_X g)_{33} = -\frac{6}{\Lambda x_3^3}\,\{x_3\partial_3 X_3 - X_3\}, \\
(\mathcal{L}_X g)_{34} = -\frac{3}{\Lambda x_3^2}\,\{\partial_4 X_3 + \partial_3 X_4\}, \\
(\mathcal{L}_X g)_{44} = -\frac{6}{\Lambda x_3^3}\,\{x_3\partial_4 X_4 - X_3\}.
\end{array}
\right.
\tag{3.1}
$$

Using the components of $\mathcal{L}_X g$ and the ones of the metric tensor g and the Ricci tensor ϱ, the Ricci soliton equation (1.2) is expressed by the following system of ten PDEs for the components X_i of vector field X:

$$
\begin{cases}
\partial_1 X_2 = 0, \\
x_3\partial_1 X_1 + x_3 H\partial_1 X_2 + x_3\partial_2 X_2 - 2X_3 + (\Lambda - \lambda)x_3 = 0, \\
\partial_3 X_2 + \partial_1 X_3 = 0, \\
\partial_4 X_2 + \partial_1 X_4 = 0, \\
12x_3\partial_2 X_1 + 6x_3\partial_2 H\, X_2 + 12x_3 H\,\partial_2 X_2 - 12H\, X_3 + 6x_3\partial_3 H\, X_3 \\
+6x_3\partial_4 H\, X_4 - 2\Lambda x_3^2\partial_3 H + \Lambda x_3^3(\partial^2_{33}H + \partial^2_{44}H) + 6(\Lambda - \lambda)x_3 H = 0, \\
\partial_3 X_1 + H\,\partial_3 X_2 + \partial_2 X_3 = 0, \\
\partial_4 X_1 + H\,\partial_4 X_2 + \partial_2 X_4 = 0, \\
2x_3\partial_3 X_3 - 2X_3 + (\Lambda - \lambda)x_3 = 0, \\
\partial_4 X_3 + \partial_3 X_4 = 0, \\
2x_3\partial_4 X_4 - 2X_3 + (\Lambda - \lambda)x_3 = 0.
\end{cases}
\tag{3.2}
$$

One then proceeds integrating equations in (3.2) one by one, starting from the simpler ones. As already observed in [11], it is possible to integrate all but two equations (namely, the fifth and the seventh) in the above system (3.2) in full generality, without any restriction on the defining function H, obtaining that necessarily $\lambda = \Lambda < 0$ (whence, a Ricci soliton is necessarily expanding) and the general form of the components X_i, in order to satisfy all equations in (3.2) but the fifth and the seventh, is given by:

$$
\begin{cases}
X_1 &= -2b_2x_1^2 + x_1\left(2B_3(x_2) - A_2'(x_2) - D_2'(x_2)\right) + x_1x_4C_2'(x_2) \\
& \quad -\frac{1}{2}x_3^2x_4C_2'''(x_2) - \frac{1}{2}x_3^2B_3'(x_2) - 2b_2\int (x_3H)\,dx_3 + G_1(x_2,x_4), \\
X_2 &= b_2(x_3^2 + x_4^2) + x_4C_2(x_2) + A_2(x_2) + D_2(x_2), \\
X_3 &= -2b_2x_1x_3 + x_3x_4C_2'(x_2) + x_3B_3(x_2), \\
X_4 &= -2b_2x_1x_4 - x_1C_2(x_2) - \frac{1}{2}x_3^2C_2'(x_2) + \frac{1}{2}x_4^2C_2'(x_2) \\
& \quad +x_4B_3(x_2) + A_4(x_2),
\end{cases}
\tag{3.3}
$$

where $G_1, A_2, C_2, D_2, B_3, A_4$ are arbitrary smooth functions and b_2 is a real constant.

We shall now specialize our study to the remaining homogeneous case I). Thus, we assume that $H = A_{-2}(x_3, x_4)$, that is, H is a homogeneous function of degree -2 of variables x_3, x_4. Explicitly, we have

$$H = k_1 x_3^{-2} + k_2 x_3^{-1} x_4^{-1} + k_3 x_4^{-2}, \tag{3.4}$$

for some real constants k_1, k_2, k_3. We substitute from (3.3) and (3.4) into the seventh equation of (3.2) and differentiate it three times by x_3, obtaining

$$2C_2''(x_2)x_3x_4^3 - b_2(k_2x_4 + 4k_3x_3) = 0,$$

which must hold for all values of x_3, x_4. Therefore, the above equation necessarily yields $C(x_2) = u_2x_2 + v_2$ for some real constants u_2, v_2, and either $b_2 = 0$, or $k_2 = k_3 = 0$. We checked both possibilities: it turns out that assuming $k_2 = k_3 = 0$ we also necessarily get $b_2 = 0$ from the remaining equations, and conversely. We report below the details for the more general case for the defining function H, that is, we assume here $b_2 = 0$ without restrictions on H, the other case (setting $k_2 = k_3 = 0$) leads exactly to the same result.

Substituting $C(x_2) = u_2x_2 + v_2$ and $b_2 = 0$, the seventh equation of (3.2) is now equivalent to the following equation, written down as a polynomial in the variable x_3:

$$\begin{aligned}&\big(\partial_4 G_1(x_2,x_4)x_4^2 + B_3'(x_2)x_4^3 + A_4'(x_2)x_4^2 + k_3u_2x_2 + k_3v_2\big)x_3^2\\ &+k_2\big(u_2x_2x_4 + v_2x_4\big)x_3 + k_1\big(u_2x_2x_4^2 + v_2x_4^2\big) = 0.\end{aligned}$$

The above equation must hold for all values of x_3. So, it yields either $k_1 = k_2 = 0$, or $u_2 = v_2 = 0$. Straigthforward calculations show that the first case eventually leads to the trivial solution $H = 0$. Thus, we continue with the case $u_2 = v_2 = 0$, whence the above equation reduces to

$$\partial_4 G_1(x_2,x_4) + x_4B_3'(x_2) + A_4'(x_2) = 0,$$

whose general integral is given by

$$G_1(x_2,x_4) = -\frac{1}{2}x_4^2B_3'(x_2) - x_4A_4'(x_2) + A_1(x_2), \tag{3.5}$$

for an arbitrary smooth function A_1. We are now left with the fifth equation of (3.2), which we write down as a polynomial equation in the variable x_1, of the form

$$\tfrac{6}{\Lambda x_3^2}\big(A_2''(x_2) + D_2''(x_2) - 2B_3'(x_2)\big)x_1 + R(x_2,x_3,x_4) = 0,$$

for some suitable smooth function R independent of x_1. As the above equation must hold for all values of x_1, from the vanishing of the coefficient of x_1, namely,

$$2B_3'(x_2) - A_2''(x_2) - B_2''(x_2) = 0,$$

by integration we get

$$B_3\,(x_2) = \frac{1}{2}A_2'(x_2) + \frac{1}{2}D_2'(x_2) + u_3, \tag{3.6}$$

where u_3 is a real constant. The fifth equation of (3.2) now reduces to the remaining condition $R(x_2, x_3, x_4) = 0$, which we simplify taking into account of (3.6) and rewrite as a polynomial in the variable x_3, with coefficients independent of x_3. Explicitly, we have:

$$\begin{aligned} &\tfrac{3}{2\Lambda x_4^4}\big((A_2'''(x_2) + D_2'''(x_2))x_4^4 - 6\Lambda k_3\big)x_3^4 - \tfrac{k_2}{x_4^3}x_3^3 \\ &+\tfrac{3}{2\Lambda x_4^3}\big((A_2'''(x_2) + D_2'''(x_2))x_5^4 + 4A_4''(x_2)x_4^4 \\ &\qquad -4A_1'(x_2)x_4^3 + 8u_3k_3x_4 + 4k_3A_4(x_2)\big)x_3^2 \\ &+\tfrac{k_2}{\Lambda x_4^2}\big(12u_3x_4 - 2\Lambda x_4 + 3A_4(x_2)\big)x_3 + \tfrac{k_1}{\Lambda}\big(12u_3 - 5\Lambda\big) = 0, \end{aligned} \tag{3.7}$$

for all values of x_3, so that all coefficients must vanish. In particular, setting equal to zero the coefficients of x_3^3 and x_3^4 in (3.7), we get $k_2 = 0$ and

$$\big(A_2'''(x_2) + D_2'''(x_2)\big)x_4^4 - 6\Lambda k_3 = 0,$$

which must vanish for all values of x_4, so that $k_3 = 0$ and $A_2''' + D_2''' = 0$, whence,

$$D_2(x_2) = -A_2(x_2) + \frac{1}{2}r_2x_2^2 + s_2x_2 + t_2,$$

for some real constants r_2, s_2, t_2. Observe that now necessarily $k_1 \neq 0$, otherwise $H = 0$ and we get a trivial solution corresponding to the anti-de Sitter space. Hence, again by (3.7), since $k_1(12u_3 - 5\Lambda) = 0$, we necessarily get $u_3 = \frac{5}{12}\Lambda$. Finally, (3.7) now reduces to

$$-A_4''(x_2)x_4 + A_1'(x_2) = 0,$$

for all values of x_4, and by integration we get

$$A_1(x_2) = a_1, \qquad A_4(x_2) = a_4x_2 + b_4,$$

for some real constants a_1, a_4, b_4.

All equations of system (3.2) are now satisfied. We get the explicit description of the components X_i of vector field X with respect to $\{\partial_i\}$, simply substituting all the previous formulas into (3.3). Explicitly, we find

$$\begin{cases} X_1 = \frac{5}{6}\Lambda x_1 - \frac{1}{4}r_2(x_3^2 + x_4^2) - a_4x_4 + a_1, \\ X_2 = \frac{1}{2}r_2x_2^2 + s_2x_2 + t_2, \\ X_3 = \frac{1}{2}(r_2x_2 + s_2)x_3 + \frac{5}{12}\Lambda x_3, \\ X_4 = a_4x_2 + \frac{1}{2}(r_2x_2 + s_2)x_4 + \frac{5}{12}\Lambda x_4 + b_4. \end{cases} \tag{3.8}$$

In order to check the above conclusions, we computed $(\mathcal{L}_X g)_{ij}$ and $\Lambda g_{ij} - \varrho_{ij}$ for all indices $i, j = 1, \dots, 4$. Using (1.1), (2.3), (3.1), (3.4) and (3.8) we obtain

$$\begin{aligned}
&(\mathcal{L}_X g)_{22} = 5k_1 x_3^{-4} = \Lambda g_{22} - \varrho_{22}, \\
&(\mathcal{L}_X g)_{ij} = 0 = \Lambda g_{ij} - \varrho_{ij} \qquad\qquad \text{in the other cases,}
\end{aligned}$$

so that equation (1.2) holds with $\lambda = \Lambda$. It is easy to check from Propositions 1 and 2 that, as $k_1 \neq 0$, these examples are neither Einstein nor conformally flat.

We shall now prove that the ones described above are not gradient Ricci solitons. In fact, suppose that this Ricci soliton is gradient. Then, there exists some smooth function $f = f(x_1, x_2, x_3, x_4)$, such that $X = \mathrm{grad}_g(f)$.

We use (1.1) to determine the inverse matrix $g^{-1} = (g^{ij})$ of the matrix describing the metric tensor g in coordinates (x_1, x_2, x_3, x_4). We then use it to compute $\mathrm{grad}_g(f) = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x_i} \partial_i$. We obtain that $X = X_i \partial_i = \mathrm{grad}_g(f)$, where X_i are given by (3.8), if and only if f is a solution of the following system of 4 PDEs:

$$\begin{cases}
\frac{1}{3}\Lambda k_1 \partial_1 f - \frac{1}{3}\Lambda x_3^2 \partial_2 f = \frac{5}{6}\Lambda x_1 - \frac{1}{4} r_2 (x_3^2 + x_4^2) - a_4 x_4 + a_1, \\
-\frac{1}{3}\Lambda x_3^2 \partial_1 f = \frac{1}{2} r_2 x_2^2 + s_2 x_2 + t_2, \\
-\frac{1}{3}\Lambda x_3^2 \partial_3 f = \frac{1}{2}(r_2 x_2 + s_2) x_3 + \frac{5}{12}\Lambda x_3, \\
-\frac{1}{3}\Lambda x_3^2 \partial_4 f = a_4 x_2 + \frac{1}{2}(r_2 x_2 + s_2) x_4 + \frac{5}{12}\Lambda x_4 + b_4.
\end{cases} \tag{3.9}$$

Integrating the last equation in (3.9) we get

$$f = -\tfrac{1}{\Lambda x_3^2}\left(\tfrac{5}{8}\Lambda x_4^2 + \tfrac{3}{4}(r_2 x_2 + s_2) x_4^2 + 3(a_4 x_2 + b_4) x_4\right) + p(x_1, x_2, x_3), \tag{3.10}$$

for some smooth function p. We substitute from (3.10) into the second equation of (3.9) and integrate. We find

$$p(x_1, x_2, x_3) = -\frac{3x_1}{2\Lambda x_3^2}\left((r_2 x_2 + 2s_2) x_2 + 2t_2\right) + q(x_2, x_3),$$

where q is a smooth function. Next, we substitute the above expressions into the third equation of (3.9). Writing it as a polynomial equation in the variable x_4, we have

$$\begin{aligned}
&(6r_2 x_2 + 6s_2 + 5\Lambda) x_4^2 + 24(a_4 x_2 + b_4) x_4 \\
&+\left(4\Lambda x_3^3 \partial_3 q(x_2, x_3) + 12 x_1 x_2 (r_2 x_2 + 2s_2) + 24 t_2 x_1 \right. \\
&\quad \left. + 6x_3^2 (r_2 x_2 + s_2) + 5\Lambda x_3^2\right) = 0,
\end{aligned}$$

for all values of x_4 and x_2, whence, $r_2 = 0$, $s_2 = -\frac{5}{6}\Lambda$, $a_4 = b_4 = 0$ and the remaining equation reads

$$\Lambda x_3^3 \partial_3 q(x_2, x_3) - 5\Lambda x_1 x_2 + 6 t_2 x_1 = 0,$$

for all values of x_1. But this contradicts $\Lambda \neq 0$. Therefore, no gradient Ricci solitons occur in this case. The above results lead to the following.

Theorem 3. Let g be a Siklos metric, as described by (1.1), having a defining function of the form $H = A_{-2}(x_3, x_4)$, that is, as explicitly given in (3.4). Then, g is a (nontrivial) Ricci soliton if and only if $k_2 = k_3 = 0$. In this case, equation (1.2) holds with $\lambda = \Lambda < 0$ and $X = X_i \partial_i$ described by (3.8). This Ricci soliton is not gradient.

We summarize the complete classification of homogeneous Siklos metrics which are Ricci solitons in the following Table I. For each class of homogeneous Siklos spacetimes, we list the type, the defining function H and the cases where H gives rise to solutions to the Ricci soliton equation. The checkmark "✓" means that for such defining function H, all homogeneous Siklos metrics are Ricci solitons, while "NO" means that homogeneous Siklos metrics corresponding to such H are never Ricci solitons, except in the trivial case where $H = 0$.

Type	Defining H	Ricci soliton cases
I)	$\frac{k_1}{x_3^2} + \frac{k_2}{x_3 x_4} + \frac{k_3}{x_4^2}$	$k_2 = k_3 = 0$
II)	$A(x_3)$	$P x_3^{\frac{3\Lambda - 6\alpha + \sqrt{9\Lambda^2 + 12\Lambda\alpha + 36\alpha^2}}{2\Lambda}} + Q x_3^{\frac{3\Lambda - 6\alpha - \sqrt{9\Lambda^2 + 12\Lambda\alpha + 36\alpha^2}}{2\Lambda}} + \frac{1}{\alpha} C$ or $\frac{1}{3} S x_3^3 + \frac{C}{\Lambda} \ln(x_3) + N$
III)	$A(x_2) x_3^2$	NO
IV)	$x_2^{2\beta - 2} A(x_3 x_2^\beta)$	$P x_2^{-2} x_3^{\frac{3\Lambda - 6\alpha + \sqrt{9\Lambda^2 + 12\Lambda\alpha + 36\alpha^2}}{2\Lambda}} + Q x_2^{-2} x_3^{\frac{3\Lambda - 6\alpha - \sqrt{9\Lambda^2 + 12\Lambda\alpha + 36\alpha^2}}{2\Lambda}} + \frac{1}{\alpha} C x_2^{-2}$ or $\frac{1}{3} S x_2^{-2} x_3^3 + \frac{C}{\Lambda} x_2^{-2} \ln(x_3) + N x_2^{-2}$
V)	$\pm x_3^\alpha$	✓

Table I: Homogeneous Ricci soliton Siklos metrics

As reported in the above Table I, very different behaviours occur for the different classes of homogeneous Siklos spacetimes in reference to the Ricci soliton equation (1.2). In some cases, all metrics in the considered subclass are Ricci solitons. On the other hand, in some other classes there are some specific solutions to the Ricci soliton equation, while in other cases no Ricci solitons occur. Observe that the Ricci soliton equation, applied to a four-dimensional Lorentzian metric, is itself a special case of Einstein's field equations. Thus, Siklos spacetimes satisfying the Ricci soliton equation provide solutions to Einstein's field equations in more than one sense.

Finally, we mention that Ricci solitons also highlight the existence of special kinds of symmetries. In fact, a smooth vector field X appearing in the Ricci soliton equation (1.2) is an *infinitesimal harmonic transformation*, that is, satisfies $\operatorname{tr}(\mathcal{L}_X \nabla) = 0$. (Infinitesimal harmonic transformations are also known as 1-*harmonic vector fields*, because this harmonicity property is equivalent to the vanishing of the linear part of the tension field of the local one-parameter group of infinitesimal point transformations.) We shall investigate in detail the symmetries of homogeneous Siklos metrics in future work.

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