Note di Matematica
*Note Mat.* **41** (2021) no. 1, 19–29.
ISSN 1123-2536, e-ISSN 1590-0932
doi:10.1285/i15900932v41n1p19

# Open $X$-ranks with respect to Segre and Veronese varieties

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Received: 2.11.2020; accepted: 27.2.2021.

**Abstract.** Let $X \subset \mathbb{P}^N$ be an integral and non-degenerate variety. Recall (A. Białynicki-Birula, A. Schinzel, J. Jelisiejew and others) that for any $q \in \mathbb{P}^N$ the open rank $or_X(q)$ is the minimal positive integer such that for each closed set $B \subsetneq X$ there is a set $S \subset X \setminus B$ with $\#S \le or_X(q)$ and $q \in \langle S\rangle$, where $\langle \ \rangle$ denotes the linear span. For an arbitrary $X$ we give an upper bound for $or_X(q)$ in terms of the upper bound for $or_X(q')$ when $q'$ is a point in the maximal proper secant variety of $X$ and a similar result using only points $q'$ with submaximal border rank. We study $or_X(q)$ when $X$ is a Segre variety (points with $X$-rank 1 and 2) and when $X$ is a Veronese variety (points with $X$-rank $\le 3$ or with border rank 2).

**Keywords:** open rank, open $X$-rank, Segre variety, Veronese variety, secant variety;,border rank

**MSC 2020 classification:** primary 14N07, secondary 14N07,15A69

## Introduction

Let $X \subset \mathbb{P}^N$ be an integral and non-degenerate projective variety. We recall the following definition ([5, 7, 8, 11]). The papers [7, 8, 11] study Veronese varieties, i.e. homogeneous polynomials, but [7, 8] also consider the case of non-homogeneous polynomials, which is harder.

**Definition 1.** For any $q \in \mathbb{P}^N$ the *open rank* or *open $X$-rank* $or_X(q)$ of $q$ is the minimal integer with the following property: for any closed set $B \subsetneq X$ there exists $S \subset X \setminus B$ such that $\#S \le or_X(q)$ and $q \in \langle S\rangle$, where $\langle \ \rangle$ denotes the linear span.

We recall that the $X$-rank $r_X(q)$ of $q$ is the minimal integer such that there is $S \subset X$ with $\#S = r_X(q)$ and $q \in \langle S\rangle$ ([13]).

Since $X$ is non-degenerate, for any closed set $B \subsetneq X$, $X \setminus B$ spans $\mathbb{P}^N$. Thus the integer $or_X(q)$ is a well-defined positive integer $\le N+1$. Obviously $or_X(q) \ge r_X(q)$. In general it is not easy to compute $or_X(q)$. For instance there is no $q \in \mathbb{P}^N$ such that $or_X(q) = 1$ (Remark 3).

[i]This work is partially supported by MIUR and GNSAGA (INdAM)

We recall that for each integer $t > 0$ the $t$-secant variety $\sigma_t(X) \subsetneq \mathbb{P}^N$ is the closure in $\mathbb{P}^N$ of the union of all linear spaces $\langle S \rangle$ for some $S \subset X$ with $\#S = t$ ([1, 13]). Each $\sigma_t(X)$ is irreducible, $\sigma_1(X) = X$ and either $\sigma_t(X) = \mathbb{P}^N$ or $\sigma_t(X) \subsetneq \sigma_{t+1}(X)$ ([1, Observation 1.2]). The *border rank* $b_X(q)$ of $q \in \mathbb{P}^N$ is the first positive integer $t$ such that $q \in \sigma_t(X)$. Let $g$ be the generic $X$-rank, i.e. the minimal positive integer such that $\sigma_g(X) = \mathbb{P}^N$. For each integer $k \in \{1, \dots, g\}$ let $\gamma_k$ denote the maximal integer $or_X(q)$ for some $q \in \sigma_k(X)$. Hence $\gamma_g$ is the maximal open $X$-rank of some $q \in \mathbb{P}^N$. Let $\mu_1$ be the minimal integer $or_X(q)$ for some $q \in X$. In general the integer $\mu_1$ is not the minimal integer $or_X(q)$ for some $q \in \mathbb{P}^N$ (Examples 1 and 2). Obviously $\mu_1 = \gamma_1$ if $X \subset \mathbb{P}^N$ is a homogeneous embedding of a homogeneous variety. For $i = 2, \dots, g$ let $\tilde{\gamma}_i$ be the maximum integer $or_X(q)$ with $q \in \sigma_i(X) \setminus \sigma_{i-1}(X)$. Note that all $q \in \sigma_i(X) \setminus \sigma_{i-1}(X)$ have $r_X(q) = i$, but that $\sigma_{i-1}(X)$ may contain points with $X$-rank $i$. Set $\tilde{\gamma}_1 := \gamma_1$. Obviously $\gamma_i = \max_{1 \le j \le i} \tilde{\gamma}_j$. In particular $\gamma_i \ge \gamma_{i-1}$ for all $i = 2, \dots, g$.

In section 1 we give a few remarks on the open $X$-rank and prove the following result.

**Theorem 1.** *Set $e := N - \dim \sigma_{g-1}(X)$. Then $\gamma_g \le \gamma_{g-1} + e$ and $\tilde{\gamma}_g \le \tilde{\gamma}_{g-1} + e$.*

We ask the following Question.

For the rational normal curve this sequence is strictly decreasing (Remark 6), but there are many examples of $X$ and $i$ such that $\tilde{\gamma}_i < \tilde{\gamma}_{i+1}$, e.g. the case $n \ge 2$, $d \ge 2$ and $i = 1$ for the order $d$ Veronese embedding of $\mathbb{P}^n$ (Theorem 2).

In section 2 $X$ is a Veronese variety, i.e. each $q \in \mathbb{P}^N$ is an equivalence class (up to a non-zero multiplicative constant) $[f]$ of a homogeneous polynomial $f$ and $r_X(q)$ is the minimal number of addenda needed to write $f$ as a sum of powers of linear forms. In section 3 $X$ is a Segre variety, i.e., each $q \in \mathbb{P}^N$ is an equivalence class $q = [T]$ (up to a non-zero multiplicative constant) of a tensor $T \ne 0$ and $r_X(q)$ is the tensor rank of $T$. For Veronese varieties we study the case in which $r_X(q) = 1$ (Example 1), $r_X(q) = 2$ (Theorem 2) and $r_X(q) = 3$ and the polynomial associated to $q$ effectively depends on more than 2 variables (Theorem 3). We describe the open ranks of all $q \in \sigma_2(X)$ (Theorem 2). For the Segre variety we study the case $r_X(q) = 1$ (Theorem 4) and the case $r_X(q) = 2$ when the tensor depends on all factors of $Y$ (Theorem 5).

We work over an algebraically closed field $\mathbb{K}$.

# 1 General remarks and proof of Theorem 1

Let $X \subset \mathbb{P}^N$ be an integral and non-degenerate projective variety. For any $q \in \mathbb{P}^N$ let $\mathcal{S}(X, q)$ denote the set of all $A \subset X$ such that $\#A = r_X(q)$ and $q \in \langle A \rangle$.

**Remark 1.** Fix $q \in \mathbb{P}^N$ with $\mathcal{S}(X,q)$ finite, say $\mathcal{S}(X,q) = \{S_1,\dots,S_c\}$. Thus $B := S_1 \cup \cdots \cup S_c$ is a proper closed subset of $X$. Every set $A \subset X$ such that $\#A = r_X(q)$ and $q \in \langle A \rangle$ is contained in $B$. The definition of open $X$-rank gives $or_X(q) > r_X(q)$. Since all $q \in \sigma_1(X) = X$ have $\#\mathcal{S}(X,q) = 1$, it follows that $r_X(q) > 1$ for all $q \in X$. Thus $\gamma_1 \ge \mu_1 > 1$. The same proof shows that $or_X(q) > r_X(q)$ for all $q \in \mathbb{P}^N$ such that $\cup_{A \in \mathcal{S}(X,q)} A$ is not Zariski dense in $X$.

**Remark 2.** Take any $q \in \mathbb{P}^N$, any closed $B \subsetneq X$ and any $A \subset X$ such that $\#A = r_X(q)$ and $q \in \langle A \rangle$. By the definition of open $X$-rank for each $a \in A$ there is $S_a \subset X \setminus B$ such that $\#S_a := or_X(a)$ and $q \in \langle S_a \rangle$. Set $S := \cup_{a \in A} S_a$. Since $S \subset X \setminus B$ and $\#S \le \gamma_1 r_X(q)$, we get $or_X(q) \le \gamma_1 r_X(q)$ for all $q \in \mathbb{P}^N$.

**Remark 3.** Since $or_X(q) \ge r_X(q)$ and $r_X(q) = 1$ if and only if $q \in X$, Remark 1 gives $or_X(q) > 1$ for all $q \in \mathbb{P}^N$.

**Remark 4.** Let $\rho$ be the maximal positive integer such that each $S \subset X$ with $\#S \le \rho$ is linearly independent. For any $q \in \mathbb{P}^N$ with $r_X(q) \le \lfloor \rho/2 \rfloor$, there is a unique set $A \subset X$ such that $\#A \le \lfloor \rho/2 \rfloor$, $q \in \langle A \rangle$ and $q \notin \langle A' \rangle$ for any $A' \subsetneq A$. Thus $or_X(q) > \lfloor \rho/2 \rfloor$ for all $q \in \mathbb{P}^N$. Since each set with cardinality $\le \rho$ is linearly independent and $\mathcal{S}(X,o) = \{o\}$ for all $o \in X$, $\mu_1 \ge \rho$.

**Example 1.** Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$, $N = \binom{n+d}{n} - 1$, be the order $d$ Veronese embedding of $\mathbb{P}^n$. Set $X := \nu_d(\mathbb{P}^n)$. The last part of Remark 3 gives $\mu_1 \ge d+1$. Fix a closed set $B \subsetneq X$ and set $B' := \nu_d^{-1}(B)$. Let $L \subseteq \mathbb{P}^n$ be a line containing $o$ and containing at least one point of $\mathbb{P}^n \setminus B'$. Thus $L \cap B'$ is finite. Take any $A \subset L \setminus L \cap B'$ such that $\#A = d+1$. Since $\nu_d(L)$ is a degree $d$ rational normal curve in its linear span, we have $q \in \langle \nu_d(L) \rangle = \langle \nu_d(A) \rangle$. Hence $or_X(q) = d+1$.

**Example 2.** Take $X = \mathbb{P}^N$. This is the case $d = 1$ of Example 1. Thus $or_X(q) = 2$ for all $q \in \mathbb{P}^N$. In this case all $q \in \mathbb{P}^N$ have $or_X(q) > r_X(q)$.

**Example 3.** Let $X \subset \mathbb{P}^N$ be a hypersurface of degree $d > 1$. A point $o \in \mathbb{P}^N$ is said to be a *strange point* of $X$ if for each smooth point $a$ of $X$ the tangent space $T_aX$ of $X$ contains $o$ ([10, 12]). Fix $q \in \mathbb{P}^N \setminus X$. Remark 3 gives $or_X(q) > 1$. Note that $or_X(q) = 2$ if and only if a general line $L \subset \mathbb{P}^N$ containing $q$ contains at least 2 points of $X$, i.e. if and only if the separable degree of the morphism $X \to \mathbb{P}^{n-1}$ induced by the linear projection from $q$ is at least 2. Take an arbitrary $q' \in \mathbb{P}^N$. If $X$ is a cone with vertex $q'$ (and hence $q' \in X$), then a general line through $q'$ contained in $X$ shows that $or_X(q') \le 2$. Remark 3 gives $or_X(q') = 2$. Now assume that $X$ is not a cone with vertex containing $q'$. Fix a closed set $B \subsetneq X$. Fix a general $(p_1,p_2) \in X^2$. Since $X$ is non-degenerate, $L := \langle \{p_1,p_2\} \rangle$ is a line not contained in $X$ and $q' \notin L$. Thus $E := \langle \{q',p_1,p_2\} \rangle$ is a plane. The scheme $X \cap E$ is a plane curve, possible with multiple components. Let $Y \subset E$ be the reduction of $X \cap E$. Since $L \nsubseteq X$, $Y$ is not a line. Thus $\langle Y \rangle = E$ (even if $Y$ is reducible). Since $p_1,p_2$ are general,

$p_i \notin B$, $i = 1, 2$. Thus $Y$ has either at least two irreducible components or an irreducible component not contained in $B$. Thus there is $p_3 \in Y \setminus Y \cap B$ such that $E = \langle \{p_1, p_2, p_3\} \rangle$. Thus $or_X(q') \leq 3$. Now assume $q' \in X$ with $X$ not a cone with vertex containing $q'$. Let $a$ be the multiplicity of $X$ at $q'$. We see that $or_X(q') = 2$ if and only of if the general line containing $q'$ meets $X$ in at least 2 other points, i.e. (since $X$ is not a cone with vertex containing $q'$) if and only the morphism $f : X \setminus \{q'\} \to \mathbb{P}^{N-1}$ induced by the linear projection from $q'$ has separable degree at least 2. This is never the case if $a = d - 1$ and in particular this is never the case if $d = 2$. Now assume $d \geq a + 2$. Thus under the assumption $d \geq a + 2$ $or_X(q') = 2$ if either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > d - a$. In summary, $or_X(q) \in \{2, 3\}$ for all $q \in \mathbb{P}^N$ and we gave a geometric description of the points $q$ with $or_X(q) = 3$.

**Remark 5.** Set $n := \dim X$ and assume $t := (N + 1)/(n + 1) \in \mathbb{N}$ and $\sigma_t(X) = \mathbb{P}^N$. For a general $q \in \mathbb{P}^N$ we have $r_X(q) = t$ and $\mathcal{S}(X, q)$ is finite. Thus $or_X(q) > t$ for a general $q \in \mathbb{P}^N$. Fix $q \in \mathbb{P}^N$ such that $r_X(q) = t$ and $\mathcal{S}(X, q)$ infinite. If there is at least one $o \in X$ such that no $A \in \mathcal{S}(X, q)$ contains $o$, then $or_X(q) > t$. Now assume that $N$ is odd and that $X$ is a curve. In this case $\sigma_t(X) = \mathbb{P}^N$ ([1, Remark 1.6]).

The following lemma is a variation of the proof of [14, Proposition 5.1].

**Lemma 1.** *Assume* $\mathrm{char}(\mathbb{K}) = 0$. *Set* $n := \dim X$. *Then* $or_X(q) \leq N+1-n$ *for all* $q \in \mathbb{P}^N \setminus X$.

*Proof.* Fix $q \in \mathbb{P}^N \setminus X$ and a closed set $B \subsetneq X$. Let $V \subset \mathbb{P}^N$ be a general linear subspace of codimension $n$ containing $q$. By the uniform position lemma, the set $V \cap X$ is formed by $\deg(X)$ points, any $N + 1 - n$ of them spanning $V$ ([9, Lemma 3.4]). Since $V \cap B = \emptyset$ for a general $V$, $or_X(q) \leq N + 1 - n$. *QED*

Let $X$ be an projective variety, $D$ an effective Cartier divisor of $X$ and $Z \subset X$ a zero-dimensional scheme. The *residual scheme* $\mathrm{Res}_D(Z)$ *of* $Z$ *with respect to* $D$ is the closed subscheme of $X$ with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. We have $\mathrm{Res}_D(Z) \subseteq Z$ and $\deg(Z) = \deg(Z \cap D) + \deg(\mathrm{Res}_D(Z))$. If $Z_1, \ldots, Z_a$ are the connected components of $Z$, then $\mathrm{Res}_D(Z) = \mathrm{Res}_D(Z_1) \cup \cdots \cup \mathrm{Res}_D(Z_a)$. If $Z$ is reduced, them $\mathrm{Res}_D(Z) = Z \setminus D$. For any line bundle $Ll$ on $X$ the following sequence, often called the *residual sequence of* $D$,

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{L}(-D) \to \mathcal{I}_Z \otimes \mathcal{L} \to \mathcal{I}_{Z \cap D, D} \otimes \mathcal{L}_{|D} \to 0$$

is exact.

The following lemma is just [3, Lemma 5.1] (see [4, Lemmas 2.4, 2.5] for similar statements).

**Lemma 2.** *Let $X \subset \mathbb{P}^N$ be a linearly normal projective variety and $D$ an effective Cartier divisor of $X$. Assume $h^1(\mathcal{O}_X(1)) = h^1(\mathcal{O}_X(1)(-D)) = 0$. Fix $q \in \mathbb{P}^N$ zero-dimensional schemes $A, B \subset X$ such that $A \neq B$, $q \in \langle A\rangle \cap \langle B\rangle$, $q \notin \langle A'\rangle$ for any $A' \subseteq A$ and $q \notin \langle B'\rangle$ for any $B' \subset B$. Set $Z := B \cup A$. Assume $h^1(X, \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{O}_X(1)(-D)) = 0$ and that one of the following conditions is satisfied:*

*(a)* $\mathrm{Res}_D(A) \cap \mathrm{Res}_D(B) = \emptyset$.

*(b) At least one among $A$ and $B$ is reduced.*

*Then* $\mathrm{Res}_D(A) = \mathrm{Res}_D(B)$.

*Proof of Theorem 1:* We first prove the inequality $\gamma_g \le \gamma_{g-1} + e$. Fix $q \in \mathbb{P}^N$. If $q \in \sigma_{g-1}(X)$, then $or_X(q) \le \gamma_{g-1}$ by the definition of $\gamma_{g-1}$. Thus we may assume $q \in \mathbb{P}^N \setminus \sigma_{g-1}(X)$. Fix a closed set $B \subsetneq X$ and take a general $(p_1, \dots, p_e) \in (X \setminus B)^e$. Set $V := \langle \{q, p_1, \dots, p_e\}\rangle$. Since $X$ is non-degenerate and $e \le N$, $\dim V = e$. Hence $V \cap \sigma_{g-1}(X) \neq \emptyset$. Fix $q' \in \sigma_{g-1}(X) \cap V$. Since $q \notin \sigma_{g-1}(X)$ and $\{p_1, \dots, p_e\}$ is general, $\langle \{q\} \cup E\rangle \cap \sigma_{g-1}(X) = \emptyset$ for all $E \subsetneq \{p_1, \dots, p_e\}$ and $\langle \{p_1, \dots, p_e\}\rangle \cap \sigma_{g-1}(X) = \emptyset$. Thus $q \in \langle \{q', p_1, \dots, p_e\}\rangle$. By the definition of open $X$-rank and the inequality $or_X(q) \le \gamma_{g-1}$ there is $A \subset X \setminus B$ such that $\#A \le \gamma_{g-1}$ and $q' \in \langle A\rangle$. Set $S := A \cup \{p_1, \dots, p_e\}$. Since $S \subset X \setminus B$ and $q \in \langle \{q', p_1, \dots, p_e\}\rangle \subseteq \langle S\rangle$, $or_X(q) \le \#S \le \gamma_{g-1} + e$.

Now we modify the proof just given to prove that $\tilde{\gamma}_g \le \tilde{\gamma}_{g-1} + e$. Since $\tilde{\gamma}_g \le \gamma_g$ and $\tilde{\gamma}_1 = \gamma_1$, we may assume $g \ge 3$. By the definition of $\tilde{\gamma}_g$ we start with $q \notin \sigma_{g-1}(X)$. Note that $\sigma_{g-2}(X)$ has codimension $> e$ ([1, Observation 1.2]). By the generality of $\{p_1, \dots, p_e\}$ we have $V \cap \sigma_{g-2}(X) = \emptyset$. Thus $q' \in \sigma_{g-1}(X) \setminus \sigma_{g-2}(X)$ and we may repeat the proof of the first inequality. QED

## 2 Veronese varieties

Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$, $N = \binom{n+d}{n} - 1$, be the order $d$ Veronese embedding of $\mathbb{P}^n$. Set $X = X_{n,d} = \nu_d(\mathbb{P}^n)$.

**Remark 6.** Let $X \subset \mathbb{P}^d$ be a rational normal curve, i.e. take $X = X_{1,d}$. By [5, Proposition 3.1] $or_X(q) = d + 2 - b_X(q)$ for all $q$.

**Remark 7.** Take $X$ as in Remark 6. All $q \notin X$ have $or_X(q) < \mu_1$.

The following result is (in a weak form) the opposite of concision for the open rank of symmetric tensors.

**Proposition 1.** *Let $M \subsetneq \mathbb{P}^n$, $n \ge 2$, be a positive dimensional linear space. Take any $q \in \langle \nu_d(M)\rangle$. Then $or_{X_{n,d}}(q) \ge or_{\nu_d(M)}(q)$.*

*Proof.* Using induction on the codimension of $M$ we reduce to the case $\dim M = n - 1$. Set $a := or_{X_{n,d}}(q)$. Fix a closed subset $B' \subsetneq M$. Take any $S \subset \mathbb{P}^n \setminus M$

such that $b := \#S \le a$ and $q \in \langle \nu_d(S)\rangle$. It is easy to check that for a general $o \in \mathbb{P}^n$ we have $o \notin S$ and $\ell(S) \cap B' = \emptyset$, where $\ell : \mathbb{P}^n \setminus \{o\} \to M$ denotes the linear projection from $o$. Since $\#\ell(S) \le a$ and $\ell(S) \cap B' = \emptyset$, to prove that $or_X(q) \le a$ it is sufficient to prove that $q \in \langle \nu_d(\ell(S))\rangle$. Fix homogeneous coordinates $x_0, \dots, x_n$ such that $M = \{x_0 = 0\}$. Take homogeneous polynomials $f(x_0, \dots, x_n)$ representing $q$ and $f_i(x_0, \dots, x_n)$, $1 \le i \le b$, representing the points of $S$. By assumptions there are constants $c_1, \dots, c_b$ such that $f = c_1 f_1 + \cdots + c_b f_b$. For any $[f_i] \in S$, $[f_i(0, x_1, \dots, x_n)]$ represents $\ell([f_i])$ . Since $q \in \langle \nu_d(M)\rangle$, $f$ does not depend on $x_0$. Thus $f = \sum_{i=1}^{b} c_i f_i(0, x_1, \dots, x_n)$. QED

**Theorem 2.** *Take $X_{n,d}$. Fix $q \in \mathbb{P}^N$ such that $b_X(q) = 2$.*

*(1) If $n = 1$, then $or_X(q) = d$.*

*(2) If $n \ge 2$, then $or_X(q) = 2d$.*

*Proof.* Until step (e) we assume $r_X(q) = 2$. By Remark 6 we may assume $n \ge 2$. Fix $A \subset \mathbb{P}^n$ such that $\nu(A) \in \mathcal{S}(X, q)$. Let $L \subset \mathbb{P}^n$ be the line spanned by $A$. Fix any closed $B \subsetneq X$ containing $\nu_d(L)$ and set $B' := \nu_d^{-1}(B)$. Take a general $u \in \mathbb{P}^n \setminus B'$ and call $M$ the plane spanned by $L$ and $u$. Let $D \subset M$ be a smooth conic containing $\{u\} \cup A$. Since $u \notin B$, $D \cap B$ is a finite set. Since $D$ is a projectively normal curve, $\dim\langle \nu_d(D)\rangle = 2d$. Since $or_{\nu_d(D)}(q) = 2d$ (Remark 2), $or_X(q) \le 2d$. Assume $or_X(q) \le 2d - 1$ and take $S \subset \mathbb{P}^n \setminus B'$ such that $\#S \le 2d - 1$ and $q \in \langle \nu_d(S)\rangle$. Note that $h^1(\mathcal{I}_{S\cup A}(d)) > 0$. Since $B' \supset L$, $S \cap A = \emptyset$. Applying case (b) of Lemma 2 with as Cartier divisor a general hyperplane $H \supseteq L$, we obtain $h^1(\mathcal{I}_S(d - 1)) > 0$. Since $\#S \le 2(d - 1) + 1$, [6, Lemma 34] gives the existence of a line $R \subset \mathbb{P}^n$ such that $\#(R \cap S) \ge d + 1$. Since $S \cap L = \emptyset$, $R \ne L$.

(a) Assume $n = 2$. Applying case (b) of Lemma 2 taking as the Cartier divisor the conic $L \cup R$ we get that either $h^1(\mathcal{I}_{S\setminus S\cap R}(d-2)) > 0$ or $S \subset R$. Since $\#(S \setminus S \cap R) \le d - 2$, $h^1(\mathcal{I}_{S\setminus S\cap R}(d - 2)) = 0$. Thus $S \subset R$. Since $q \in \langle \nu_d(S)\rangle$, we get $q \in \langle \nu_d(R)\rangle$. Concision gives $A \subset R$ ([13, Ex. 3.2.2.2]). Thus $R = L$, a contradiction.

(b) Assume $n = 3$ and $L \cap R \ne \emptyset$. Set $H := \langle L \cup R\rangle$. Applying any of the two cases of Lemma 2 with respect to the Cartier divisor $H$ we get that either $h^1(\mathcal{I}_{S\setminus S\cap H}(d - 1)) > 0$ or $S \subset H$. Since $\#(S \setminus S \cap H) \le \#(S \setminus S \cap R) \le d - 2$, we get $S \subset H$. Since $q \in \langle \nu_d(H)\rangle$ and $S \subset H$, step (a) gives a contradiction.

(c) Assume $n = 3$ and $L \cap R = \emptyset$. Since $\mathcal{I}_{L\cup R}(2)$ is globally generated and $S$ is a finite set, there is $Q \in |\mathcal{I}_{L\cup R}(2)|$ such that $S \cap Q = S \cap R$. Applying part (b) of Lemma 2 to the Cartier divisor $Q$ we get $S \subset R$.

(d) Assume $n \ge 4$. There is a hyperplane $H \subset \mathbb{P}^n$ containing $R \cup L$. As in step (b) we get a contradiction using induction on $n$.

(e) Assume $r_X(q) > 2$. There is a degree 2 connected zero-dimensional scheme $v \subset \mathbb{P}^n$ such that $q \in \langle \nu_d(v) \rangle$. We repeat the proof of the previous steps using $v$ instead of $A$. In all cases we take $B'$ containing the reduction of $v$ and hence $v \cap S = \emptyset$ in all steps. Thus we may apply any of the two cases of Lemma 2. QED

**Remark 8.** Let $X \subsetneq \mathbb{P}^N$ be a Veronese variety. Since $X$ is homogeneous and the embedding is homogeneous, $\mu_1 = \gamma_1$. Example 1 and Theorem 2 show that when $n = 1$ there are points $q$ with $or_X(q) < \mu_1$.

**Theorem 3.** *Take $X = X_{n,d}$, $n \geq 2$, $d \geq 4$ and $N = \binom{n+d}{n}$. Take $q \in \mathbb{P}^N$ such that $r_X(q) = 3$ and there is no line $L \subset \mathbb{P}^n$ such that $q \in \langle \nu_d(L) \rangle$.*

*(1) If $n = 2$, then $or_X(q) = 2d - 1$.*

*(2) If $n > 2$, then $or_X(q) = 3d - 1$.*

*Proof.* Fix $A \subset \mathbb{P}^n$ such that $\nu_d(A) \in \mathcal{S}(X, q)$. Since there is no line $L \subset \mathbb{P}^n$ such that $q \in \langle \nu_d(L) \rangle$, concision gives $\dim\langle A \rangle = 2$ ([13, Ex. 3.2.2.2]). Take a closed set $B' \subsetneq \mathbb{P}^n$. If $n = 2$ we assume that $B'$ contains the 3 lines spanned by 2 of the points of $A$. If $n > 2$ we assume $B' \supseteq \langle A \rangle$.

(a) Assume $n = 2$. Fix a general $u \in \mathbb{P}^2 \setminus B'$. Since $A \cup \{u\}$ is contained in a smooth conic, the case $n \geq 2$ of the proof of Theorem 2 gives $or_X(q) = 2d - 1$. Assume $or_X(q) \leq 2d - 2$ and take $E \subset \mathbb{P}^2 \setminus B'$ such that $\#E \leq 2d - 2$ and $q \in \langle \nu_d(E) \rangle$. Since $or_X(q) \geq r_X(q)$, $\#E \geq 3$. Since $E \cap B' = \emptyset$, we have $E \cap A = \emptyset$. Since $q \in \langle \nu_d(E) \rangle \cap \langle \nu_d(A) \rangle$, $h^1(\mathcal{I}_{E \cup A}(d)) > 0$. Take a line $L \subset \mathbb{P}^2$ spanned by 2 of the points of $A$, say $A = (A \cap L) \cup \{o\}$. By the choice of $B'$, $L \cap E = \emptyset$. Since $E \neq \{o\}$, part (b) of Lemma 2 gives $h^1(\mathcal{I}_{E \cup \{o\}}(d-1)) > 0$. Since $\#(E \cup \{o\}) \leq 2d - 1 = 2(d-1) + 1$, [6, Lemma 34] gives the existence of a line $R$ such that $\#(R \cap (E \cup \{o\}) \geq d + 1$. Note that $\#(R \cap A) \leq 1$. Part (b) of Lemma 2 gives $h^1(\mathcal{I}_{(E \cup A) \setminus (E \cup A) \cap R}(d-1)) > 0$. The inequality $\#((E \cup A) \setminus (E \cup A) \cap R) \leq d$ contradicts [6, Lemma 34].

(b) Assume $n \geq 3$. Take a general $u \in \mathbb{P}^n \setminus B'$ and set $M := \langle A \cup \{u\} \rangle$. We have $\dim M = 3$ and there is a degree 3 rational normal curve $G \subset M$ containing $A \cup \{u\}$. Thus $G \cap B'$ is a finite set containing $A$. Since $G$ is projectively normal, the restriction map $H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_G(d))$ is surjective. Thus $\dim\langle \nu_d(G) \rangle = 3d$ and $\nu_d(G)$ is a rational normal curve of $\langle \nu_d(G) \rangle$. Remark 6 gives the existence of $S \subset G \setminus G \cap B'$ such that $\#S = 3d - 1$ and $q \in \langle \nu_d(S) \rangle$. Thus $or_X(q) \leq 3d - 1$.

Assume $or_X(q) \leq 3d - 2$ and take $E \subset \mathbb{P}^n \setminus B'$ such that $\#E \leq 3d - 2$. Recall that $B' \supseteq \langle A \rangle$ and hence $E \cap A = \emptyset$. Set $S := E \cup A$. Since $q \in \langle \nu_d(E) \rangle \cap \langle \nu_d(A) \rangle$, $h^1(\mathcal{I}_S(d)) > 0$. Since $\#S \leq 3d + 1$, by [2, Theorem 1] one of the following cases occurs:

(1) there is a line $L \subset \mathbb{P}^n$ such that $\#(L \cap S) \geq d+2$;

(2) there is a reduced conic $D$ such that $\#(D \cap S) \geq 2d+2$;

(3) there is a reduced plane cubic $T$ and $S' \subseteq S$ such that $\#S' = 3d$ and $S' \in |\mathcal{O}_T(d-1)|$;

(4) $\#S = 3d+1$ and there is a reduced plane cubic $F \subset \mathbb{P}^n$ such that $S \subset F$.

(b1) Case (4) is excluded, because it would force $E \subset \langle A \rangle$, contradicting our choice of $B'$.

(b2) For the same reason in case (3) we have $\#S = 3d+1$ and $S \setminus S'$ is a point, $o$, of $A$. Consider the plane $\langle T \rangle$ and call $H \subset \mathbb{P}^n$ a general hyperplane containing $\langle T \rangle$ (hence $H = \langle T \rangle$ if $n = 3$). Since $S \setminus S \cap H = \{o\}$ and $h^1(\mathcal{I}_o(d-1)) = 0$, case (a) of Lemma 2 gives a contradiction.

(b3) Assume the existence of a reduced conic $D$ such that $\#(D \cap S) \geq 2d+2$. Since $E \cap \langle A \rangle = \emptyset$, we have $\#(\langle D \rangle \cap A) \leq 2$. Let $H$ be a general hyperplane containing $\langle D \rangle$. Thus $H \cap S = \langle D \rangle \cap S$ and $1 \leq \#(S \setminus S \cap H) \leq d-1$. Thus $h^1(\mathcal{I}_{S \setminus S \cap H}(d-1)) = 0$, contradicting part (a) of Lemma 2.

(b4) Assume the existence of a line $L \subset \mathbb{P}^n$ such that $\#(L \cap S) \geq d+2$. Since $E \cap \langle A \rangle = \emptyset$, we have $\#(L \cap A) \leq 1$. Take a hyperplane $H \subset \mathbb{P}^n$ such that $H \supset L$ and $A \nsubseteq H$. Part (b) of Lemma 2 gives $h^1(\mathcal{I}_{S \setminus S \cap H}(d-1)) > 0$. Since $\#(S \setminus S \cap H) \leq 2d-1 = 2(d-1)+1$, there is a line $R \subset \mathbb{P}^n$ such that $\#(R \cap (S \setminus S \cap H)) \geq d+1$ ([6, Lemma 34]). Note that $R \neq L$ and hence $\#(L \cap R) \leq 1$.

(b4.1) Assume either $n > 3$ or $R \cap L \neq \emptyset$. These assumptions are equivalent to the existence of a hyperplane $U \supset L \cup R$. Since $\#(S \setminus S \cap U) \leq 3d+1-d-2-d-1+1$, $h^1(\mathcal{I}_{S \setminus S \cap U}(d-1)) = 0$ and hence $S \subset U$ (part (b) of Lemma 2). Since $S$ is a finite set, taking a general $U$ containing $W := \langle R \cup L \rangle \supset S$. Since $\langle A \rangle \cap E = \emptyset$, $\dim \langle W = 3$, i.e. $R \cap L = \emptyset$. Since $\mathcal{I}_{L \cup R, W}(2)$ is globally generated and $S$ is a finite set, there is a quadric surface $Q \subset W$ such that $S \cap Q = S \cap (L \cup R)$. Let $Q' \subset \mathbb{P}^n$ be any quadric hypersurface such that $Q' \cap W = Q$. Since $\#(S \setminus S \cap (L \cup R)) \leq 3d+1-d-2-d-1$, we have $h^1(\mathcal{I}_{S \setminus S \cap Q'}(d-2)) = 0$ and hence $S \subset R \cup L$. Thus at least one of the lines $R$ or $L$ contains 2 points of $A$ and hence they contain no point of $E$ by the choice of $B'$, a contradiction.

(b4.2) Assume $n = 3$ and $R \cap L = \emptyset$. We use the quadric $Q$ as in step (b4.1). QED

# 3 Segre varieties

Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $k \ge 1$, $n_i > 0$ for all $i$, be a multiprojective space. Set $N := \prod_{i=1}^{k}(n_i + 1)$. Let $\nu : Y \to \mathbb{P}^N$ be the Segre embedding of $Y$. Set $X := \nu(Y)$. Thus $X$ is a Segre variety. For any $i \in \{1, \dots, k\}$ let $\pi_i : Y \to \mathbb{P}^{n_i}$ denote the projection onto the $i$-th factor of $Y$ and let $\epsilon_i \in \mathbb{N}^k$ be the multiindex $(a_1, \dots, a_k)$ with $a_i = 1$ and $a_h = 0$ for all $h \ne i$.

**Remark 9.** If $k = 1$ Example 2 gives $or_X(q) = 2$ for all $q \in \mathbb{P}^N$.

By Remark 9 it would be sufficient to study the case $k > 1$.

**Remark 10.** Take $q \in \mathbb{P}^N$, which is not concise, i.e. assume the existence of a multiprojective subspace $Y' \subsetneq Y$ such that $q \in \langle \nu(Y') \rangle$ (we allow the case $q \in X$ in which we may take $Y' = \{q\}$). Set $X' := \nu(Y')$. By concision $r_X(q) = r_{X'}(q)$ and $\mathcal{S}(X', q') = \mathcal{S}(X, q)$ ([13, Proposition 3.1.3.1]). Taking $B = Y'$ we get $or_X(q) > r_X(q)$.

The following result is (in a weak form) the opposite of concision for the open rank of symmetric tensors.

**Proposition 2.** *Let $M \subsetneq Y$ be a positive dimensional multiprojective space. Take any $q \in \langle \nu(M) \rangle$. Then $or_X(q) \ge or_{\nu(M)}(q)$.*

*Proof.* Set $a := or_X(q)$. Using induction of the integer $\dim Y - \dim M$ we see that it is sufficient to do the case $\dim M = \dim Y - 1$. Fix a closed subset $B' \subsetneq M$. Write $M = \prod_{i=1}^{m} \mathbb{P}^{m_i}$ with $0 \le m_i \le n_i$ for all $i$ and $\sum_i m_i = \sum_i n_i - 1$. Permuting the factors of $Y$ we may assume $m_1 = n_1 - 1$, thus $M = M_1 \times W$, where $W := \prod_{i=2}^{k} \mathbb{P}^{n_i}$ and $M_1$ is a hyperplane of $\mathbb{P}^{n_1}$. Fix a closed set $B'' \subsetneq Y$. Take any $S \subset Y \backslash B''$ with $\#S \le a$ and $q \in \langle \nu(S) \rangle$. Fix a general $o \in \mathbb{P}^{n-1}$ and let $\ell : \mathbb{P}^{n_1} \backslash \{o\} \to M_1$ denote the linear projection from $o$. The submersion $\ell$ induces a submersion $\mu : Y \setminus \{o\} \times W \to M$. For a general $o$ we have $\{o\} \times W \cap S = \emptyset$. Thus $\mu$ is defined at each point of $S$. Since $\langle \nu(S) \rangle \cap \langle \nu(M) \rangle \subseteq \langle \nu(\mu(S)) \rangle$, we have $q \in \langle \nu(\mu(S)) \rangle$. Since $\#\mu(S) \le a$, to conclude the proof it is sufficient to find $B''$ such that $\mu(S) \cap B' = \emptyset$. Take $B'' := \mu^{-1}(B')$. QED

**Theorem 4.** *We have $or_X(q) = k + 1$ for all $q \in X$.*

*Proof.* By Remark 9 we may assume $k \ge 2$. Fix $q \in X$, say $q = \nu(o)$ with $o = (o_1, \dots, o_k)$. Let $B \subsetneq X$ be a closed subset. Set $B' := \nu^{-1}(B)$. Fix $u = (u_1, \dots, u_k) \in Y \setminus B'$ such that $u_i \ne o_i$ for al $i$. Take $a, b \in \mathbb{P}^1$ such that $a \ne b$. Let $f_i : \mathbb{P}^1 \to \mathbb{P}^{n_i}$ be any degree 1 embedding such that $f_i(a) = u_i$ and $f_i(b) = o_i$. Let $f = (f_1, \dots, f_k) : \mathbb{P}^1 \to Y$ be the embedding such that $\pi_i \circ f = f_i$ for all $i$. Set $D := \nu(f(\mathbb{P}^1))$. Note that $D$ is a degree $k$ rational normal curve in its linear span. Since $f(a) = u$ and $f(b) = o$, $\{u, o\} \subset D$. Since $f(a) \notin B$, $D \cap B$ is a finite set. Fix $S \subset D \setminus D \cap B$ such that $\#S = k+1$. Since $D$ is a degree $k$ rational normal curve of $\langle D \rangle$, $\langle S \rangle = \langle D \rangle$. Thus $or_X(q) \le k+1$.

Assume $or_X(q) \le k$. Set $B' := \cup_{i=1}^{k} \pi_i^{-1}(o_i) \subset Y$ and $B := \nu(B')$. Take $A \subset Y \setminus B'$ such that $\#A \le k$, $q \in \langle \nu(A) \rangle$ and $q \notin \langle \nu(A') \rangle$ for any $A' \subsetneq A$. Write $A = \{a(1), \dots, a(e)\}$ for some $e \le k$ and $a(i) \ne a(j)$ for all $i \ne j$. Let $H_i$, $i = 1, \dots, e$, be a general element of $|\mathcal{O}_Y(\epsilon_i)|$ containing $a(i)$. By the definition of the set $B'$ we have $o_i \notin \pi_i(A)$ for $i = 1, \dots, k$. By the generality of each $H_i$ we have $o \notin H_i$. Thus $o \notin A$ and $q \in \langle \nu(A) \rangle$, $h^1(\mathcal{I}_{A \cup \{o\}}(1, \dots, 1)) > 0$. If $e < k$ take as $H_i$, $e+1, \dots, k$, any element of $|\mathcal{O}_Y(\epsilon_i)|$ not containing $o$. Set $D := H_1 + \cdots + H_k$. Note that $D \cap (\{o\} \cup A) = A$. Since $o \notin A$ and $h^1(\mathcal{I}_o) = 0$, part (b) of Lemma 2 gives a contradiction. QED

**Theorem 5.** *Take $q \in \mathbb{P}^N$ such that $r_X(q) = 2$ and $q$ depends on all $k$ factors of $Y$. Then:*

*(i) $or_X(q) \ge k$;*

*(ii) $or_X(q) = k$ if and only if $q$ is concise, i.e. if $n_i = 1$ for all $i$.*

*Proof.* Fix $A \subset Y$ such that $\nu(A) \in \mathcal{S}(X, q)$. By concision the assumption that $q$ depends on all factors of $X$ is equivalent to $\#\pi_i(A) = 2$ for all $i \in \{1, \dots, k\}$. Since $r_X(q) = 2$ and $q$ depends on all factors, $q$ is concise if and only if $n_i = 1$ for all $i$. We fix 3 distinct points of $\mathbb{P}^1$ and call it 0, 1 and $\infty$. Fix $A = \{a, b\}$ such that $\nu(A) \in \mathcal{S}(X, q)$. Fix a general $u \in Y \setminus B'$. Since $u$ is general $\pi_i(u) \notin \pi_i(A)$ for any $i$.

(a) First assume $n_i = 1$ for all $i$. Fix a closed $B \subsetneq X$ and set $B' := \nu^{-1}(B')$. Let $f_i : \mathbb{P}^1 \to \mathbb{P}^1$ be the only isomorphism such that $f_i(0) = a_i$, $f_i(1) = b_i$ and $f_i(\infty) = u_i$. Thus $f = (f_1, \dots, f_k)$ induces an embedding $f : \mathbb{P}^1 \to Y'$ such that $f(0) = a$, $f(1) = b$ and $f(\infty) = u$. Set $D := f(\mathbb{P}^1)$. Note that $\dim\langle \nu(D) \rangle = k$ and that $\nu(D)$ is a degree $k$ rational normal curve of $\langle \nu(D) \rangle$. Since $u \notin B'$, $D \cap S$ is finite. By Remark 6 there is $S \subset D$ such that $\#S \le k$ and $q \in \langle \nu(S) \rangle$. Thus $or_X(q) \le k$. Assume $or_X(q) \le k-1$ and take $E \subset Y \setminus B'$ such that $\#E \le k-1$ and $q \in \langle \nu(E)$. We assume $B' \supset A$. With this assumption $h^1(\mathcal{I}_{E \cup A}(1, \dots, 1)) > 0$. Since $\#(E \cup A) = k+1$, mimicking the proof of Theorem 10 we get a contradiction.

(b) Now assume $n_i \ge 2$ for some $i$. Let $Y \subsetneq Y$ be the concise Segre of $q$. By concision ([13, Proposition 3.13.1]) every $S \subset Y$ such that $q \in \langle \nu(S) \rangle$ and $S \nsubseteq Y'$ has cardinality $> k$. Taking as closed set $B$ the set $Y'$ we get $or_X(q) > k$. QED

## References

[1] B. Ådlandsvik: *Joins and higher secant varieties*, Math. Scan. 61 (1987), 213–222.

[2] E. Ballico: *Finite subsets of projective spaces with bad postulation in a fixed degree*, Beitrage zur Algebra und Geometrie 54 (2013), no. 1, 81–103.

[3] E. Ballico, A. Bernardi: *Stratification of the fourth secant variety of Veronese variety via the symmetric rank*, Adv. Pure Appl. Math., **4** (2013), 215–250.

[4] E. Ballico, A. Bernardi, M. Christandl, F. Gesmundo: *On the partially symmetric rank of tensor product of W-states and other symmetric tensors*, Rend. Lincei Math. Appl., **30** (2019), 93–124.

[5] E. Ballico, A. de Paris: *Generic power sum decompositions and bounds for the Waring rank*, Discrete Comput. Geom., **57** (2017), n. 4, 896–914.

[6] A. Bernardi, A. Gimigliano, M. Idà: *Computing symmetric rank for symmetric tensors*, J. Symbolic Comput,. **46** (2011), n. 1, 34–53.

[7] A. Białynicki-Birula, A. Schinzel: *Representations of multivariate polynomials by sums of univariate polynomials in linear forms*, Colloq. Math., **112** (2002), 2201–233.

[8] A. Białynicki-Birula, A. Schinzel: *Corrigendum to "Representatons of multivariate polynomials by sums of univariate polynomials in linear forms" (Colloq. Math. 112 (2008), 201–233)*, Colloq. Math.,**125** (2011), n. 1, 139.

[9] J. Harris: Curves in projective space, Séminaire de Mathématiques Supérieures, vol. 85, Presses de l'Université de Montréal, Montreal, Que., 1982, With the collaboration of D. Eisenbud.

[10] A. Hefez, S. L. Kleiman: *Notes on the duality of projective varieties*, Geometry today (Rome, 1984), 143–183, Progr. Math., 60, Birkhäuser Boston, Boston, MA, 1985.

[11] J. Jelisiejew: *An upper bound for the Waring rank of a form*, Arch. Math., **102** (2014), n. 4, 329–336.

[12] S. L. Kleiman: *Tangency and duality*, Proceedings of the 1984 Vancouver conference in algebraic geometry, 163–225, CMS Conf. Proc., 6, Amer. Math. Soc., Providence, RI, 1986.

[13] J. M. Landsberg: Tensors: Geometry and Applications. Graduate Studies in Mathematics, Vol. 128, Amer. Math. Soc. Providence, 2012.

[14] J. M. Landsberg, Z. Teitler: *On the ranks and border ranks of symmetric tensors.* Found. Comput. Math. **10** (2010) n. 3, 339–366.