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Harmonic morphisms of compact homogeneous spaces of positive curvature

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Abstract. In this paper, we show that the projection of every compact Riemannian manifold of positive curvature onto a rank one symmetric space is harmonic. As a corollary, an infinite family of distinct harmonic morphisms with minimal circle fibers from the 7-dimensional homogeneous Aloff-Wallach spaces of positive curvature onto the 6-dimensional flag manifolds is given.

Keywords: Riemannian submersion, homogeneous space, Aloff-Wallach space, positive curvature, harmonic morphism

MSC 2020 classification: primary 58E20, secondary 53C43

Introduction

In 1965, Eells and Sampson [6] initiated a theory of harmonic maps in which variational problems play central roles in geometry; Harmonic map are solutions of important variational problems as critical points of the energy functional $E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 \, v_g$ for smooth maps φ of (M, g) into (N, h). The Euler-Lagrange equations are given by the vanishing of the tension filed $\tau(\varphi)$. On the other hand, Fuglede [7] in 1978 and Ishihara [10] in 1979, introduced independently the alternative notion of harmonic morphism which preserves harmonic functions (see [2]). Harmonic morphisms are one of important examples of harmonic maps.

On the other hand, one of the main issues in differential geometry is to classify all compact homogeneous Riemannian manifolds of positive sectional curvature.

Let us recall several facts on compact homogeneous Riemannian manifolds of positive sectional curvature. The well-known examples are compact simply connected rank one Riemannian symmetric spaces, namely, the n-dimensional sphere S^n, the $2n$-dimensional complex projective space $P^n(\mathbb{C})$, the $4n$-dimensional

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quaternion projective space $P^n(\mathbb{H})$, and the 16-dimensional Cayley projective plane $P^2(\mathfrak{C})$.

In 1961, Berger [3] gave a classification of all normal homogeneous Riemannian manifolds with positive sectional curvature, and gave two odd-dimensional examples of the 7-dimensional space $Sp(2)/SU(2)$ and the 13-dimensional space $SU(5)/(T^1 \cdot Sp(2))$, where T^1 is a one-dimensional torus in $U(4)$ satisfying that $T^1 \cdot Sp(2) \subset T^1 \cdot SU(4) = U(4) \subset SU(5)$.

In 1972, Wallach [16] classified all compact even-dimensional homogeneous Riemannian manifolds with positive sectional curvature, and gave three new examples of homogeneous Riemannian manifolds with positive curvature. These examples are the following: $SU(3)/T$ (dimension 6), $Sp(3)/(Sp(1) \times Sp(1) \times Sp(1))$ (dimension 12), and $F_4/Spin(8)$ (dimension 24), where T is a maximal torus in $SU(3)$, $Spin(8) \subset Spin(9) \subset F_4$ and $F_4/Spin(9) = P^2(\mathfrak{C})$. He also defined the notion of Condition (III) (cf. [16, page 288]) and showed that a homogeneous space G/H admits a G-invariant Riemannian metric with positive sectional curvature if the corresponding pair (G, H) satisfies Condition (III).

In 1975, Aloff and Wallach introduced the new notion of Condition (II) in [1, page 93]. This Condition (II) is weaker than Condition (III), namely, if (G, H) satisfies Condition (III), then it satisfies Condition (II) (see also [4]). They showed (cf. [1]) that the quotient space G/H admits a G-invariant Riemannian metric of positive sectional curvature if the pair (G, H) satisfies the Condition (II). They gave (cf. [1]) an infinite family of compact homogeneous spaces $SU(3)/T_{k,\ell}$ of 7-dimension, where $T_{k,\ell}$ of 7-dimension, where $T_{k,\ell}$ $(k,\, \ell \in \mathbb{Z}, (k, \ell) = 1)$ are one dimensional tori in $SU(3)$, and showed that all pairs $(SU(3), T_{k,\,\ell})$ satisfy Condition (II), and there exist $SU(3)$-invariant Riemannian metrics on $SU(3)/T_{k,\ell}$ with positive sectional curvature. Recently, Wilking, Grove, and Ziller have clarified fine structures and properties of compact Riemannian manifolds with positive sectional curvature (cf. [17], [18], [19], [8], [20]).

Now, it is well-known that the important problems are:

Problems: (a) *Classify all simply connected compact homogeneous Riemannian manifolds $(G/H, g)$ for pairs (G, H) satisfying Condition* (II).

(b) *Classify all simply connected compact homogeneous Riemannian manifolds with positive sectional curvature.*

In this paper, concerning these problems, we will show that:

Theorem 1. (*Theorem 4*) *For any compact simply connected homogeneous Riemannian manifold $(G/H, g)$ such that (G, H) satisfies Condition* (II), *the projection from $(G/H, g)$ onto a Riemannian symmetric space $(G/K, h)$ of rank one: $\pi : (G/H, g) \to (G/K, h)$ is harmonic.* (*See Definition 1 for*

Condition (II).)

Important examples of compact homogeneous Riemannian manifolds of positive sectional curvature are an infinite family of Aloff-Wallach spaces of seven-dimensional homogeneous space of positive sectional curvature. We will show that all the projections from the Aloff-Wallach spaces onto the six-dimensional flag manifold are harmonic morphisms. Namely, we show the following theorem.

Theorem 2. (`Theorem 8`) *Assume that* $(P, g) = (M_{k,\ell}, g_t) = (SU(3)/T_{k,\ell}, g_t)$, $k,\ \ell \in \mathbb{Z}$, $(k,\ \ell) = 1$; $-1 < t < 0$, *or* $0 < t < \frac{1}{3}$, *are infinitely many distinct homogeneous the* 7*-dimensional Aloff-Wallach spaces of positive sectional curvature, and let* (M, h) *be the* 6*-dimensional flag manifold* $(SU(3)/T, h)$. *Then, the Riemannian submersions with circle fibers,* $\pi : (P, g) = (M_{k,\ell}, g_t) \to (M, h) = (SU(3)/T, h)$, *are all harmonic morphisms with minimal fibers.*

Here, the subgroups $T_{k,\ell}$ and T of $SU(3)$ and the homogeneous space $M_{k,\ell}$ are given as follows.

$$
\begin{aligned}
T_{k,\ell} &= \left\{ \begin{pmatrix} e^{2\pi k i\theta} & 0 & 0 \\ 0 & e^{2\pi i\ell\theta} & 0 \\ 0 & 0 & e^{-2\pi i(k+\ell)\theta} \end{pmatrix} \,\middle|\, \theta \in \mathbb{R} \right\} \\
&\subset T = \left\{ \begin{pmatrix} e^{2\pi i\theta_1} & 0 & 0 \\ 0 & e^{2\pi i\theta_2} & 0 \\ 0 & 0 & e^{-2\pi i(\theta_1+\ell_2)} \end{pmatrix} \,\middle|\, \theta_1,\ \theta_2 \in \mathbb{R} \right\} \subset G = SU(3),
\end{aligned}
$$

and $M_{k,\ell} = G/T_{k,\ell} = SU(3)/T_{k,\ell}$.

1 Preliminaries

1.1 Harmonic maps.

Recall the definition of a harmonic map $\varphi : (M, g) \to (N, h)$, of a compact Riemannian manifold (M, g) into another Riemannian manifold (N, h), as an extremal of the *energy functional* defined by

$$E(\varphi) = \int_M e(\varphi)\, v_g,$$

where $e(\varphi) := \frac{1}{2}|d\varphi|^2$ is called the energy density of φ. That is, for any variation $\{\varphi_t\}$ of φ with $\varphi_0 = \varphi$,

$$\frac{d}{dt}\bigg|_{t=0} E(\varphi_t) = -\int_M h(\tau(\varphi), V) v_g = 0, \tag{1.1}$$

where $V \in \Gamma(\varphi^{-1}TN)$ is a variation vector field along φ which is given by $V(x) = \frac{d}{dt}\big|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N$, $(x \in M)$, and the *tension field* is given by $\tau(\varphi) = \sum_{i=1}^m B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN)$, where $\{e_i\}_{i=1}^m$ is a locally defined orthonormal frame field on (M, g), and $B(\varphi)$ is the second fundamental form of φ defined by

$$\begin{aligned} B(\varphi)(X, Y) &= (\widetilde{\nabla} d\varphi)(X, Y) \\ &= (\widetilde{\nabla}_X d\varphi)(Y) \\ &= \overline{\nabla}_X(d\varphi(Y)) - d\varphi(\nabla_X Y), \end{aligned} \tag{1.2}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Here, ∇, and ∇^h, are Levi-Civita connections on TM and TN of (M, g) and (N, h), and $\overline{\nabla}$, and $\widetilde{\nabla}$ are the induced ones on $\varphi^{-1}TN$, and $T^*M \otimes \varphi^{-1}TN$. By (1), φ is *harmonic* if and only if $\tau(\varphi) = 0$.

1.2 Riemannian submersions.

We prepare with several notions on the Riemannian submersions. A C^∞ mapping π of a C^∞ Riemannian manifold (P, g) into another C^∞ Riemannian manifold (M, h) is called a *Riemannia submersion* if (0) π is surjective, (1) the differential $d\pi = \pi_* :\ T_uP \to T_{\pi(u)}M$ $(u \in P)$ of $\pi :\ P \to M$ is surjective for each $u \in P$, and (2) each tangent space T_uP at $u \in P$ has the direct decomposition:

$$T_uP = \mathcal{V}_u \oplus \mathcal{H}_u, \qquad (u \in P),$$

which is orthogonal decomposition with respect to g such that $\mathcal{V} = \mathrm{Ker}(\pi_{*u}) \subset T_uP$ and (3) the restriction of the differential $\pi_* = d\pi_u$ to $\mathcal{H}_u$ is a surjective isometry, $\pi_* :\ (\mathcal{H}_u, g_u) \to (T_{\pi(u)}M, h_{\pi(u)})$ for each $u \in P$ (cf. [2]). A manifold P is the total space of a Riemannian submersion over M with the projection $\pi : P \to M$ onto M, where $p = \dim P = k + m$, $m = \dim M$, and $k = \dim \pi^{-1}(x)$, $(x \in M)$. A Riemannian metric g on P, called *adapted metric* on P which satisfies

$$g = \pi^* h + k \tag{1.3}$$

where k is the Riemannian metric on each fiber $\pi^{-1}(x)$, $(x \in M)$. Then, T_uP has the orthogonal direct decomposition of the tangent space T_uP,

$$T_uP = \mathcal{V}_u \oplus \mathcal{H}_u, \qquad u \in P, \tag{1.4}$$

where the subspace $\mathcal{V}_u = \mathrm{Ker}(\pi_{*u})$ at $u \in P$, the *vertical subspace*, and the subspace $\mathcal{H}_u$ of P_u is called *horizontal subspace* at $u \in P$ which is the orthogonal complement of $\mathcal{V}_u$ in T_uP with respect to g.

In the following, we fix a locally defined orthonormal frame field, called *adapted local orthonormal frame field* to the projection $\pi:\ P \to M$, $\{e_i\}_{i=1}^p$ corresponding to (3) and (4) in such a way that

- $\{e_i\}_{i=1}^m$ is a locally defined orthonormal basis of the horizontal subspace $\mathcal{H}_u$ ($u \in P$), and
- $\{e_i\}_{i=1}^k$ is a locally defined orthonormal basis of the vertical subspace $\mathcal{V}_u$ ($u \in P$).

Corresponding to the decomposition (4), the tangent vectors X_u, and Y_u in T_uP can be defined by

$$X_u = X_u^{\mathrm{V}} + X_u^{\mathrm{H}}, \quad Y_u = Y_u^{\mathrm{V}} + Y_u^{\mathrm{H}}, \tag{1.5}$$

$$X_u^{\mathrm{V}},\ Y_u^{\mathrm{V}} \in \mathcal{V}_u, \quad X_u^{\mathrm{H}},\ Y_u^{\mathrm{H}} \in \mathcal{H}_u \tag{1.6}$$

for $u \in P$.

Then, there exist a unique decomposition such that

$$g(X_u, Y_u) = h(\pi_* X_u, \pi_* Y_u) + k(X_u^{\mathrm{V}}, Y_u^{\mathrm{V}}), \quad X_u,\ Y_u \in T_uP,\ u \in P.$$

1.3 The reduction of the harmonic map equation

Hereafter, we treat with the above problem for the case $\dim(\pi^{-1}(x)) = 1$, $(u \in P,\ \pi(u) = x)$. Let $\{e_1,\ e_2,\ \ldots,\ e_{m+1}\}$ be an adapted local orthonormal frame field with $e_{n+1} = e_m$, vertical. The frame fields $\{e_i:\ i = 1, 2, \ldots, m\}$ are a basic orthonormal frame field on (P, g) corresponding to an orthonormal frame field $\{\epsilon_1,\ \epsilon_2,\ \ldots,\ \epsilon_n\}$ on (M, g). Here, a vector field $Z \in \mathfrak{X}(P)$ is *basic* if Z is horizontal and π-related to a vector field $X \in \mathfrak{X}(M)$, i.e., $d\pi(Z) = X \circ \pi$.

In this section, we prepare the notation to determine the harmonic equation precisely in the case that $p = m+1 = \dim P$, $m = \dim M$, and $k = \dim \pi^{-1}(x) = 1$ ($x \in M$). Since $[V, Z]$ is a vertical field on P if Z is basic and V is vertical (cf. [12, page 461]), for each $i = 1, \ldots, n$, $[e_i, e_{n+1}]$ is horizontal, so we can write as follows.

$$[e_i, e_{m+1}] = \sum_{j=1}^{m} \kappa_i^j\, e_j, \quad i = 1,\ \ldots,\ m, \tag{1.7}$$

where $\kappa_i \in C^\infty(P)$ $(i = 1, \ldots, m)$. To be more precise, for two vector fields $X,\ Y$ on M, let $X^*,\ Y^*$, be the horizontal vector fields on P. Then, $[X^*, Y^*]$ is a vector field on P which is π-related to a vector field $[X, Y]$ on M (for instance, [15, page 143]). Thus, for $i,\ j = 1, \ldots, m$, $[e_i, e_j]$ is π-related to $[\epsilon_i, \epsilon_j]$, and we may write as

$$[e_i, e_j] = \sum_{k=1}^{m+1} D_{ij}^k\, e_k, \tag{1.8}$$

where $D_{ij}^k \in C^\infty(P)$ $(1 \le i,\ j \le m;\ 1 \le k \le m+1)$.

2 Homogeneous Riemannian submersions

In this section, let us recall the work on homogeneous Riemannian manifolds with positive sectional curvature by S. Aloff and N. Wallach [1]. We will consider a compact simply connected homogeneous space $(P, g) = (G/H, g)$ with positive sectional curvature which is the total space of a Riemannian submersion $\pi : (G/H, g) \to (G/K, h)$ over a rank one symmetric space $(M, h) = (G/K, h)$. Recall that

Definition 1. A pair (G, H) of a compact connected Lie group G and a closed subgroup K of G is said to satisfy *Condition* (II) if there is an $\mathrm{Ad}(G)$-invariant inner product $\langle\cdot,\cdot\rangle_0$ on $\mathfrak{g}$, the Lie algebra $\mathfrak{g}$ of G has the orthogonal decomposition with respect to $\langle\cdot,\cdot\rangle_0$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v}, \quad \mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$$

satisfying that (i)

$$[\mathfrak{v}_1, \mathfrak{v}_2] \subset \mathfrak{v}_2, \tag{2.9}$$
$$[\mathfrak{v}_1, \mathfrak{v}_1] \subset \mathfrak{h} \oplus \mathfrak{v}_1, \tag{2.10}$$
$$[\mathfrak{v}_2, \mathfrak{v}_2] \subset \mathfrak{h} \oplus \mathfrak{v}_1 \tag{2.11}$$

and (ii) if $X = X_1 + X_2$, $Y = Y_1 + Y_2 \in \mathfrak{v}$ with $X_i,\ Y_i \in \mathfrak{v}_i$ $(i = 1, 2)$ and if $[X, Y] = 0$ and $\{X,\ Y\}$ is linearly independent, then $[X_1, Y_1] \neq 0$.

We put $\mathfrak{k} := \mathfrak{h} \oplus \mathfrak{v}_1$. Then $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, and $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair of rank one (cf. [4]). Let K be the Lie subgroup of G corresponding to $\mathfrak{k}$. Then, K is a closed subgroup of G (cf. [13, page 58]) and $\pi : (G/H, g) \to (G/K, h)$ is a Riemannian submersion.

We choose a local orthonormal frame field $\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+k}\}$ on $P = G/H$ in such a way that $\{e_1, \ldots, e_m\}$ is a basis of $\mathfrak{v}_2$ and $\{e_{m+1}, \ldots, e_{m+k}\}$ is a basis of $\mathfrak{v}_1$, namely, $\{e_1, \ldots, e_m\}$ are horizontal vectors, and $\{e_{m+1}, \ldots, e_{m+k}\}$ are vertical vectors. Then, we can calculate immediately:

Proposition 1. *(i) For $i = 1, \dots, m$; $a = 1, \dots, k$,*

$$[e_i, e_{m+a}] = \sum_{j=1}^{m} A_{ia}^j \, e_j. \tag{2.12}$$

(ii) For b, $c = 1, \dots, k$,

$$[e_{m+b}, e_{m+c}] \equiv \sum_{a=1}^{k} B_{bc}^a \, e_{m+a} \qquad (mod\ \mathfrak{h}). \tag{2.13}$$

(iii) For i, $j = 1, \dots, m$,

$$[e_i, e_j] \equiv \sum_{d=1}^{k} C_{ij}^d \, e_{m+d} \qquad (mod\ \mathfrak{h}). \tag{2.14}$$

Here, all the A_{ia}^j, B_{bc}^a, C_{ij}^d $(i, j = 1, \dots, m;\ a, b, c, d = 1, \dots, k)$

Furthermore, we can determine the Levi-Civita connection ∇ of $(G/H, g)$ as follows.

Proposition 2. *We denote by $X^*, Y^* \in \mathfrak{X}(P)$, the horizontal lifts of vector fields X, $Y \in \mathfrak{X}(M)$, and V, $W \in \mathfrak{X}(P)$, the vertical vector fields on P. Then, we obtain:*

(i)

$$g(\nabla^g_V W, Z) = -\frac{1}{2} \, Z(g(V, W)), \tag{2.15}$$

for vertical vector fields V, W, $Z \in \mathfrak{X}(P)$.

(ii)

$$g(\nabla^g_V X^*, W) = \frac{1}{2} \, X^*(g(W, V)), \tag{2.16}$$

for vertical vector fields V, W, and a horizontal vector field X^* on P.

(iii)

$$g(\nabla^g{}_{X^*} V, W) = \frac{1}{2} \, X^*(g(V, W)), \tag{2.17}$$

for vertical vector fields V, W and a horizontal vector field X^* on P.

(iv)

$$g(\nabla^g_{X^*}Y^*) = \frac{1}{2}\Big\{ - W(g(X^*, Y^*)) + g(W, [X^*, Y^*]) + g(Y^*, [W, X^*]) + g(X^*, [W, Y^*])\Big\}, \tag{2.18}$$

for a vertical vector field V and horizontal vector fields X^*, Y^* on P.

(v)

$$g(\nabla^g_W X^*, Y^*) = \frac{1}{2}\Big\{ W(g(X^*, Y^*)) + g(Y^*, [W, X^*]) + g(X^*, [Y^*, W]) - g(W, [X^*, Y^*])\Big\}, \tag{2.19}$$

for a vertical vector field W and horizontal vector fields X^*, Y^* on P.

The proof of Proposition 2 can be obtained by a direct computation of the Levi-Civita connection ∇,

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, Y)) - Z(g(X, Y)) \\ g(Z, [X, Y]) + g(Y, [Z, X]) - g(X, [Y, Z]), \quad X,\, Y,\, Z \in \mathfrak{X}(M)., \tag{2.20}$$

So, we omit it. $\boxed{QED}$

With the notations of section 1.2, we compute the tension field $\tau(\pi)$ for the projection $\pi :\, (P, g) \to (M, h)$:

Proposition 3. *Let $\pi :\, (P, g) \to (M, h)$ the Riemannian submersion, where $\{e_i\}_{i=1}^p$, $(p := m + k)$ is a locally defined orthonormal frame field on (P, g) such that $\{e_i\}_{i=1}^m$ $(m = \dim M)$ is a horizontal orthonormal frame field on $)(P, g)$, and $\{e_i'\}_{i=1}^m$ is a local orthonormal frame field on (M, h), $e_i' := \pi_* e_i \in \Gamma(\pi^{-1}TM)$. Then, we have the following:*

(i)

$$\tau(\pi) = -\pi_*\bigg(\sum_{i=m+1}^{m+k} \nabla_{e_i u} e_i\bigg), \tag{2.21}$$

(ii) for all $j = 1, \ldots, , k$,

$$\nabla_{e_{m+j}} e_{m+j} = \sum_{i=1}^{k} B_{ij}^i\, e_{m+i}, \tag{2.22}$$

where $\nabla_{e_{m+j}} e_{m+j}$, $j = 1, \ldots, k$, are all vertical vector fields.

Proof For (i), by the definition of the tension field $\tau(\pi)$ for the projection $\pi : (P, g) \to (M, gh)$, we have

$$\begin{aligned}
\tau(\pi) &= \sum_{i=1}^{p} \left(\widetilde{\nabla}_{e_i} \pi_* e_i - \pi_* \nabla_{e_i} e_i \right) \\
&= \sum_{i=1}^{m} \left(\widetilde{\nabla}_{e_i} \pi_*(e_i) - \pi_* \nabla_{e_i} e_i \right) + \sum_{i=m+1}^{m+k} \left(\widetilde{\nabla}_{e_i} \pi_* e_i - \pi_* \nabla_{e_i} e_i \right) \qquad (2.23) \\
&= -\pi_* \left(\sum_{i=m+1}^{m+k} \nabla_{e_i} e_i \right). \qquad (2.24)
\end{aligned}$$

Because, for the second term of (22) we obtain that

$$\sum_{i=m+1}^{m+k} \left(\widetilde{\nabla}_{e_i} \pi_* e_i - \pi_* \nabla_{e_i} e_i \right) = -\pi_* \left(\sum_{i=m+1}^{m+k} \nabla_{e_i} e_i \right) \qquad (2.25)$$

since we have

$$\widetilde{\nabla}_{e_i} \pi_* e_i = 0 \qquad (2.26)$$

because $\{e_i\}_{i=m+1}^{m+k}$ is a frame of vertical vector fields.

Furthermore, for the first term for (22), we obtain

$$\sum_{i=1}^{m} \left(\widetilde{\nabla}_{e_i} \pi_* e_i - \pi_* \nabla_{e_i} e_i \right) = 0, \qquad (2.27)$$

where we should recall $\pi_* Y \in \Gamma(\pi^{-1}TM)$, and $\widetilde{\nabla}_X \pi_* Y \in \Gamma(\pi^{-1}TM)$ for X, $Y \in \mathfrak{X}(P)$ (cf. [15, pages 126–127, (1.14), (1.14′)]). Then, Equality (26) follows from that

$$\widetilde{\nabla}_{e_i} \pi_* e_i = \nabla^h_{\pi_* e_i} f_i \circ \pi = \nabla^h_{e'_i} e'_i, \qquad (2.28)$$

where $\pi_* e_i = f_i \circ \pi$, and

$$\pi_* \nabla_{e_i} e_i = \nabla^h_{\pi_* e_i} \pi_* e_i = \nabla^h j_{e'_i} e'_i, \qquad (2.29)$$

both of which are obtained from the O'Neill's formula (cf. [12, page 460, Lemma 1], or [15, page 143, Lemma (3.9)]). Thus, we obtain Equation (21), which implies (i).

For (ii), we first show that, for all i, $j = 1, \ldots, k$, at $p \in P$,

$$\begin{aligned}
2g(\nabla_{e_{m+j}} e_{m+j}, e_{m+i}) &= e_{m+j}(g(e_{m+j}, e_{m+i})) + e_{m+j}(g(e_{m+i}, e_{m+j})) \\
&\quad - m_{m+i}(g(m_{m+j}, e_{m+j})) + g(e_{m+i}, [e_{m+j}, e_{m+j}]) \\
&\quad + g(e_{m+j}, [e_{m+i}, e_{m+j}]) - g(e_{m+j}, [e_{m+j}, e_{m+i}]) \\
&= 2g(e_{m+j}, [e_{m+i}, e_{m+j}]). \qquad (2.30)
\end{aligned}$$

Here, notice that, for $\{e_{m+j} : j = 1, \ldots, k\}$, if we write

$$[e_{m+b}, e_{m+c}] = \sum_{a=1}^{k} B^a_{bc}\, e_{m+a} + X_{bc}, \qquad (2.31)$$

where $X_{bc} \in \mathfrak{h}$, we may choose $\{e_{m+a}\}$ $(a, b, c = 1, \ldots, k)$ in such a way that

$$g(e_{m+j\ x}, X_{bc\ x}) = 0 \qquad (\forall\, j = 1, \ldots, k;\ x \in V), \qquad (2.32)$$

in a small neighborhood V of the origin $o = \{H\}$ of G/H.

Notice that the decomposition $\mathfrak{h} \oplus \mathfrak{v}_1 \oplus \mathfrak{v}_2$ is the orthogonal decomposition with respect to $g_o(\cdot, \cdot) = (\cdot, \cdot)_o$. Therefore, we may assume that, for all $j = 1, \ldots, k$,

$$g(e_{m+j\ o}, X_o) = 0 \qquad (\forall\ X \in \mathfrak{h}),$$

since $X_o := \frac{d}{dt}|_{t=0} \exp(tX)$ $(X \in \mathfrak{g})$, and $X_o = 0$ $(X \in \mathfrak{h})$.

On the other hand, we can define a local orthonormal frame field $\{e_{m+j} : j = 1, \ldots, k\}$ in such a that, every vector field e_{m+j} $(j = 1, \ldots, k)$ is parallel on a neighborhood V of the origin $o = \{H\}$ of G/H. Then it follows that

$$g_x(e_{m+j\,x}, X_x) = g_o(e_{m+j\,o}, X_o) = 0 \ \ (\forall\, X \in \mathfrak{h},\ \forall\, x \in V,\ \forall\, j = 1, \ldots, k).$$

Then, we obtain for (30),

$$\begin{aligned}
2g(e_{m+j}, [e_{m+i}, e_{m+j}]) &= 2g(e_{m+j}, \sum_{a=1}^{k} B^a_{ij}\, e_{m+a} + X) \\
&= 2\sum_{a=1}^{k} B^a_{ij}\, \delta_{aj} \\
&= 2\, B^j_{ij}. \qquad (2.33)
\end{aligned}$$

Furthermore, for each $s = 1, \ldots, m$, we have

$$\begin{aligned}
2g(\nabla_{e_{m+j}} e_{m+j}, e_s) &= e_{m+j}(g(e_{m+j}, e_s)) + e_{m+j}(g(e_s, e_{m+j})) - e_s(g(e_{m+j}, e_{m+j})) \\
&\quad + g(e_s, [e_{m+j}, e_{m+j}]) + g(e_{m+j}, [e_s, e_{m+j}]) - g(e_{m+j}, [e_{m+j}, e_s]) \\
&= 2g(e_{m+j}, [e_s, e_{m+j}]). \qquad (2.34)
\end{aligned}$$

Here, let us notice that, for all $s = 1, \ldots, m$; $j = 1, \ldots, k$, $[e_s, e_{m+j}]$ is a horizontal vector field due to (12) in Proposition 1 (i). Therefore, since $\{e_{m+j} : j = 1, \ldots, k\}$ are vertical, we have

$$g(e_{m+j}, [e_s, e_{m+j}]) = 0 \quad (\forall\ s = 1, \ldots, m;\ j = 1, \ldots, k). \tag{2.35}$$

Together with (34) and (35), we obtain, for all $j = 1, \ldots, k$,

$$\nabla_{e_{m+j}} e_{m+j} = \sum_{i=1}^{k} B_{ij}^{i}\, e_{m+i}, \tag{2.36}$$

which are all vertical vector fields.

By (i), we obtain

$$\tau(\pi) = -\sum_{j=1}^{k} d\pi \left(\nabla_{e_{m+j}} e_{m+j}\right) = 0. \tag{2.37}$$

Therefore, we obtain (ii). $\boxed{QED}$

Remark 1. For more related topics in harmonic morphisms, and horizontally weakly harmonic maps, see [2].

By applying Proposition 3, we immediately have the following theorem:

Theorem 3. *Assume that* (G, H) *satisfies Condition* (II), *and* $(M, h) = (G/K, h)$ *is a symmetric space of rank one, where* K *is a closed subgroup of* G *corresponding to* $\mathfrak{k} := \mathfrak{h} \oplus \mathfrak{v}_1$. *Then,*

$$\pi : \, (P, g) = (G/H, g) \to (G/K, h)$$

is harmonic.

Due to Theorem 3, we obtain:

Theorem 4. *For any compact simply connected homogeneous Riemannian manifold* $(G/H), g)$ *with* (G, H) *satisfying Condition* (II), *the projection from* $(G/H, g)$ *onto a Riemannian symmetric space* $(G/K, h)$ *of rank one,* $\pi \colon (G/H, g) \to (G/K, h)$ *is harmonic.*

Remark 2. We have the following two unsolved problems:

(a) Classify all simply connected compact homogeneous Riemannian manifolds $(G/H, g)$ for pairs (G, H) satisfying Condition (II).

(b) Classify all simply connected compact homogeneous Riemannian manifolds with positive sectional curvature.

3 The tension field of the Riemannian submersions

In this section, we need the following theorem which is obtained directly from the several formulas in [2, Chapter 4]. For completeness, we give a proof.

Theorem 5. *Let* $\pi : (P, g) \to (M, h)$ *be a Riemannian submersion over* (M, h) *satisfying* $\dim(\pi^{-1}(x)) = 1$, $(u \in P,\ \pi(u) = x)$. *If the submersion satisfies* (7)*, namely, for every* $i = 1, \dots, n$, $[e_i, e_{n+1}]$ *is horizontal, then* $\pi : (P, g) \to (M, h)$ *is harmonic, i.e., the tension field* $\tau(\pi)$ *of* π *satisfies*

$$\tau(\pi) = -d\pi\left(\nabla_{e_{n+1}} e_{n+1}\right) = 0. \tag{3.38}$$

Proof. We only have to calculate the tension field $\tau(\pi)$ for $\pi : (P, g) \to (M, h)$, a Riemannian submersion over (M, h) satisfying that $\dim(\pi^{-1}(x)) = 1$, $(u \in P, \pi(u) = x)$ to see that

$$\tau(\pi) = -d\pi\left(\nabla_{e_{n+1}} e_{n+1}\right) = 0. \tag{3.39}$$

Indeed, we have

$$\begin{aligned}
\tau(\pi) &= \sum_{i=1}^{m} \left\{\nabla^{\pi}_{e_i} d\pi(e_i) - d\pi(\nabla_{e_i} e_i)\right\} \\
&= \sum_{i=1}^{n} \left\{\nabla^{\pi}_{e_i} d\pi(e_i) - d\pi(\nabla_{e_i} e_i\right\} + \nabla^{\pi}_{e_{n+1}} d\pi(e_{n+1}) - d\pi\big(\nabla_{e_{n+1}} e_{n+1}\big) \\
&= -d\pi\big(\nabla_{n+1} e_{n+1}\big).
\end{aligned} \tag{3.40}$$

Because, for i, $j = 1, \dots, n$, $d\pi(\nabla_{e_i} e_j) = \nabla^h_{\epsilon_i} \epsilon_j$, and $\nabla^{\pi}_{e_i} d\pi(e_i) = \nabla^h_{d\pi(e_i)} f_i \circ \pi = \nabla^h_{\epsilon_i} \epsilon_i$, where $d\pi(e_i) = f_i \circ \pi$. Thus, we have

$$\sum_{i=1}^{n} \left\{\nabla^{\pi}_{e_i} d\pi(e_i) - d\pi(\nabla_{e_i} e_i)\right\} = 0. \tag{3.41}$$

Since $e_{n+1} = e_m$ is vertical, $d\pi(e_{n+1}) = 0$, so that $\nabla^{\pi}_{e_{n+1}} d\pi(e_{n+1}) = 0$.

Furthermore, by definition of the Levi-Civita connection, we have, for $i = 1, \dots, n$,

$$\begin{aligned}
2g(\nabla_{e_{n+1} e_{n+1}}, e_i) &= 2g(e_{n+1}, [e_i, e_{n+1}]) \\
&= 2g\left(e_{n+1}, \sum_{j=1}^{n} \kappa_i^j e_j\right) \\
&= 2\sum_{j=1}^{n} \kappa_i^j\, g(e_{n+1}, e_j) \\
&= 0.
\end{aligned} \tag{3.42}$$

and $2g(\nabla_{e_{n+1}}e_{n+1}, e_{n+1}) = 0$. Therefore, we have

$$\nabla_{e_{n+1}}e_{n+1} = f\, e_{n+1},$$

for some $f \in C^\infty(P)$. Then,

$$d\pi\left(\nabla_{e_{n+1}}e_{n+1}\right) = 0. \tag{3.43}$$

$\boxed{QED}$

4 Harmonic morphisms

Definition 2. (1) A smooth map $\pi : (P, g) \to (M, h)$ is *harmonic* if the tension field vanishes, $\tau(\pi) = 0$, and

(2) $\pi : (P, g) \to (M, h)$ is a *harmonic morphism* (cf. [2, page 106]) if, for every local harmonic function $f : (M, h) \to \mathbb{R}$, the composition $f \circ \pi : (P, g) \to \mathbb{R}$ is also harmonic.

(3) $\pi : (P, g) \to (M, h)$ is *horizontally weakly conformal* (cf. [2, page 46]) if, the differential $\pi_{*\,p} : T_pP \to T_{\pi(p)}M$ is surjective, and

$$(\pi^*h)(X, Y) = \Lambda(p)\, g(X, Y) \qquad (X,\, Y \in \mathcal{H}_p),$$

for some non-zero number $\Lambda(p) \neq 0\, (p \in P)$, or the differential $\pi_{*\,p}$ vanishes. .

Here, let $\mathcal{H}_p$ be the horizontal subspace of T_pP for the Riemannian submersion $\pi : (P, g) \to (M, h)$. Then, we have the following direct decomposition satisfying that

$$T_pP = \mathcal{V}_p \oplus \mathcal{H}_p,$$

where $\mathcal{V}_p = \mathrm{Ker}(\pi_{*\,p})$.

It is well known (cf. [2, page 108]) that

Theorem 6. *(Fuglede, 1978 and Ishihara, 1979) Let $\varphi : (P, g) \to (M, g)$ be a Riemannian submersion. Then, it is harmonic morphism if and only if the following two conditions hold:*

(i) $\varphi : (P, g) \to (M, h)$ is harmonic and

(ii) $\varphi : (P, g) \to (M, h)$ is horizontally weakly conformal.

The following corollary is very useful. We will apply this to prove Theorem 8.

Corollary 1. (cf. [2, page 123]) *A Riemannian submersion $\varphi : (P, g) \to (M, g)$ is a harmonic morphism if and only if φ has minimal fibers.*

5 Positive curvature metrics on Aloff-Wallach spaces

We first prepare the setting of the Aloff-Wallach theorem [1]. Let $G = SU(3)$, and

$$T_{k,\ell} = \left\{ \begin{pmatrix} e^{2\pi k i\theta} & 0 & 0 \\ 0 & e^{2\pi i\ell\theta} & 0 \\ 0 & 0 & e^{-2\pi i(k+\ell)} \end{pmatrix} \mid \theta \in \mathbb{R} \right\}$$
$$\subset T = \left\{ \begin{pmatrix} e^{2\pi i\theta_1} & 0 & 0 \\ 0 & e^{2\pi i\theta_2} & 0 \\ 0 & 0 & e^{-2\pi i(\theta_1+\theta_2)} \end{pmatrix} \mid \theta_1\ \theta_2 \in \mathbb{R} \right\}$$
$$\subset G_1 = \left\{ \begin{pmatrix} x & 0 \\ 0 & \det(x^{-1}) \end{pmatrix} \mid x \in U(2) \right\} \subset G = SU(3),$$

and the Lie algebras of G, $T_{k,\ell}$, T, G_1 by $\mathfrak{g}$, $\mathfrak{t}_{k,\ell}$, $\mathfrak{t}$, $\mathfrak{g}_1$, respectively. Let the $\mathrm{Ad}(G)$-invariant inner product $\langle\, ,\, \rangle_0$ by

$$\langle X, Y\rangle_0 := -\mathrm{Re}(\mathrm{Tr}(XY)), \qquad X, Y \in \mathfrak{g},$$
$$\mathfrak{m} = {\mathfrak{g}_1}^{\perp} := \left\{ \begin{pmatrix} 0 & 0 & z_2 \\ 0 & 0 & z_1 \\ -\overline{z}_2 & -\overline{z}_1 & 0 \end{pmatrix} \mid z_1,\ z_2 \in \mathbb{C} \right\},$$
$$\mathfrak{t}_{k,\ell} := \left\{ \begin{pmatrix} 2\pi i k\theta & 0 & 0 \\ 0 & 2\pi \ell\theta & 0 \\ 0 & 0 & -2\pi i(k+\ell)\theta \end{pmatrix} |\theta \in \mathbb{R} \right\},$$
$$V_1 := {\mathfrak{t}_{k,\ell}}^{\perp} \cap \mathfrak{g}_1, \qquad V_2 := {\mathfrak{g}_1}^{\perp} = \mathfrak{m},$$

and let

$$\mathfrak{g} = \mathfrak{s}u(3) = \mathfrak{t}_{k,\ell} \oplus V_1 \oplus V_2,$$

the orthogonal direct decomposition of $\mathfrak{g}$ with respect to the inner product $\langle\, ,\, \rangle_0$. For $-1 < t < \infty$, let the new inner product $\langle\, ,\, \rangle_t$ by

$$\langle x_1 + x_2, y_1 + y_2\rangle_t := (1+t)\langle x_1, y_1\rangle_0 + \langle x_2, y_2\rangle_0, \tag{5.44}$$

where $x_i,\ y_i \in V_i$ $(i = 1, 2)$, and let g_t, the corresponding G-invariant Riemannian metric on the homogeneous space $G/T_{k,\ell}$ with $(k,\ell) = 1$. The condition that $(k,\ell) = 1$, i.e., that $\{k, \ell\}$ are linearly independent is necessary to get the 1-dimensional tori $T_{k,\ell}$ in the maximal torus T of G. Then,

Theorem 7. (*cf. [1]*) *The homogeneous space* $(G/T_{k,\ell}, g_t)$ *with* $(k,\ell) = 1$ *corresponding to* (6.1) *with* $(-1 < t < 0)$ *or* $(0 < t < \frac{1}{3})$ *has strictly positive sectional curvature.*

We state our main theorem as follows:

Theorem 8. *Let π be the Riemannian submersion of $(SU(3)/T_{k,\ell}, g_t)$ onto $(SU(3)/T, h)$, where $(SU(3)/T, h)$ is a flag manifold with the $SU(3)$-invariant Riemannian metric h corresponding the inner product $\langle\, ,\, \rangle_0$ on $\mathfrak{g}$. Then, it is a harmonic morphism, i.e., for every harmonic function on a neighborhood V in $SU(3)/T$, the composition $f \circ \pi$ is harmonic on a neighborhood $\pi^{-1}(V)$ in $SU(3)/T_{k,\ell}$, and also it has minimal fibers.*

Proof. We take a basis $\{X_0, X_1, X_2\}$ of V_1 and the one $\{X_3, X_4, X_5, X_6\}$ of V_2 as follows:

$$X_0 = \frac{i}{\sqrt{5\Gamma}} \begin{pmatrix} 2\ell + k & 0 & 0 \\ 0 & 2m + \ell & 0 \\ 0 & 0 & 2k + m \end{pmatrix},$$

$$X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$X_5 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad X_6 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}.$$

Here $\Gamma := k^2 + \ell^2 + k\ell$, $m := -k - \ell$, and $\{X_0, X_1, X_2, X_3, X_4, X_5, X_6\}$ is an orthonormal basis of $V_1 \oplus V_2$ with respect to $\langle\, ,\, \rangle_0$, and $\mathfrak{g} = \mathfrak{su}(3) = \mathfrak{t}_{k,\ell} \oplus V_1 \oplus V_2$.

Then, the basis of $V_1 \oplus V_2$

$$\left\{ \frac{1}{\sqrt{1+t}} X_0,\ \frac{1}{\sqrt{1+t}} X_1,\ \frac{1}{\sqrt{1+t}} X_2,\ X_3,\ X_4,\ X_5,\ X_6 \right\} \tag{45}$$

is orthonormal with respect to the inner product $\langle\, ,\, \rangle_t$, $(-1 < t < \infty)$ in (44). We denote $M_{k,\ell} := SU(3)/T_{k,\ell}$ with k and $\ell \in \mathbb{Z}$ with $(k,\ell) = 1$, and the corresponding local unit orthonormal vector fields on $M_{k,\ell} := SU(3)/T_{k,\ell}$ by

$$\{e_0^t,\ e_1^t,\ e_2^t,\ e_3^t,\ e_4^t,\ e_5^t,\ e_6^t\} \tag{46}$$

For the projection $\pi : M_{k,\ell} = SU(3)/T_{k,\ell} \to M = SU(3)/T$, each element e_i^t $(i = 0, 1, \ldots, 6)$ in (45) corresponds by π_* (the differential of π), to each in

$$\left\{ 0,\ \frac{1}{\sqrt{1+t}} e_1',\ \frac{1}{\sqrt{1+t}} e_2',\ e_3',\ e_4',\ e_5',\ e_6' \right\},$$

where $\{e_1', e_2', e_3', e_4', e_5', e_6'\}$ is an orthonormal frame field on $(SU(3)/T, h)$.

By definition of the Levi-Civita connection of a Riemannian metric g_t, for every vector field X on $P = M_{k,\ell}$,

$$\begin{aligned}2g_t(X, \nabla^{g_t}_{e_0^t} e_0^t) &= e_0^t\, g_t(X, e_0^t) + e_0^t\, g_t(X, e_0^t) - X g_t(e_0^t, e_0^t)\\ &\quad + g_t(e_0^t, [X, e_0^t]) + g_t(e_0^t, [X, e_0^t]) - g_t(X, [e_0^t, e_0^t])\\ &= 2\left\{e_0^t\, g_t(X, e_0^t) + g_t(e_0^t, [X, e_0^t])\right\}.\end{aligned}$$

Thus, we have

$$g_t(X, \nabla^{g_t}_{e_0^t} e_0^t) = e_0^t\, g_t(X, e_0^t) + g_t(e_0^t, [X, e_0^t]). \tag{47}$$

On the other hand, we have

$$e_0^t\, g_t(e_i^t, e_0^t) = 0 \qquad (i = 0, 1, \ldots, 6) \tag{48}$$

Indeed,

$$e_0^t g(e_i^t, e_0^t) = 0 \qquad (i = 0, 1, \ldots, 6) \tag{49}$$

and by a straightforward computation, we have the following Lemma:

Lemma 1. *We have*

$$\begin{aligned}&\left[\frac{1}{\sqrt{t+1}}\, X_1,\ \frac{1}{\sqrt{t+1}}\, X_0\right] = -\frac{3(k+\ell)}{(1+t)\sqrt{5\Gamma}}\, X_2,\\ &\left[\frac{1}{\sqrt{t+1}}\, X_2,\ \frac{1}{\sqrt{t+1}}\, X_0\right] = \frac{3(k+\ell)}{(1+t)\sqrt{5\Gamma}}\, X_1,\\ &\left[X_3,\ \frac{1}{\sqrt{1+t}}\, X_0\right] = \frac{-3\ell}{\sqrt{1+t}\,\sqrt{5\Gamma}}\, X_4,\ \left[X_4,\ \frac{1}{\sqrt{1+t}}\, X_0\right] = \frac{3\ell}{\sqrt{1+t}\,\sqrt{5\Gamma}}\, X_3,\\ &\left[X_5,\ \frac{1}{\sqrt{1+t}}\, X_0\right] = \frac{3k}{\sqrt{1+t}\,\sqrt{5\Gamma}}\, X_6,\ \left[X_6,\ \frac{1}{\sqrt{1+t}}\, X_0\right] = \frac{-3k}{\sqrt{1+t}\,\sqrt{5\Gamma}}\, X_5.\end{aligned}$$

By Lemma 1, we have

$$g_t(e_0^t, [X, e_0^t]) = 0 \qquad (\forall\ X = X_i\ (i = 0, 1, \cdots, 6)). \tag{50}$$

By (47), (48), (49), we have

$$g_t(X, \nabla^{g_t}_{e_0^t} e_0^t) = 0 \qquad (\forall X \in \mathfrak{X}(M_{k,\ell})), \tag{51}$$

which implies that

$$\nabla^{g_t}_{e_0^t} e_0^t = 0. \tag{52}$$

Then, we have

$$\tau(\pi) = -d\pi(\nabla^{g_t}_{e_0^t} e_0^t) = 0. \tag{53}$$

Therefore, by (52) and (53), the submersion π is a harmonic map with minimal fibers. Due to Corollary 1, we have Theorem 8. $\boxed{QED}$

Remark 3. (1) In our main theorem, Theorem 8, since $(G/T, h)$ is a flag manifold, so a Kähler manifold, it admits a lot of harmonic function on an open subset V in G/T. For a harmonic function f on an open subset $V \subset G/T$, then $f \circ \pi$ is harmonic on $\pi^{-1}(V)$.

(2) Our fibration $\pi : (SU(3)/T_{k,\,\ell}, g) \to (SU(3)/T, h)$ has close similarities to the Hopf fibration $\pi' : (S^{2n+1}, g_0) \to (\mathbb{C}P^n, h_0)$. Both the total spaces have positive sectional curvature, and both the base spaces are Kähler manifolds.

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