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Some results on almost Kenmotsu manifolds

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Abstract. First we consider almost Kenmotsu manifolds which satisfy Codazzi condition for h and φh, and we prove that in such cases the tensor h vanishes. Next, we prove that an almost Kenmotsu manifold having constant ξ-sectional curvature K which is locally symmetric is a Kenmotsu manifold of constant curvature $K = -1$. We also prove that, for a $(\kappa, \mu)'$-almost Kenmotsu manifold of $dim > 3$ with $h' \neq 0$, every conformal vector field is Killing. Finally, we prove that if M is a $(\kappa, \mu)'$-almost Kenmotsu manifold with $h' \neq 0$ and $\kappa \neq -2$, then the vector field V which leaves the curvature tensor invariant is Killing.

Keywords: Almost Kenmotsu manifold, Locally symmetric spaces, Infinitesimal contact transformation, Conformal vector field.

MSC 2000 classification: primary 53C15, 53C25, secondary 53D10.

1 Introduction

Geometry of almost Kenmotsu manifolds was first introduced by Janssens and Vanhecke in [4], and became an interesting area of research in the field of differential geometry. They provide a special class of almost contact metric manifolds. An almost Kenmotsu manifold M is an almost contact metric manifold of dimension $2n+1$ with structure (φ, ξ, η, g) such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, where Φ is the fundamental 2-form associated to the structure. The warped products of an almost Kählerian manifold and a real line give examples of almost Kenmotsu manifolds. Further if the structure is normal, M is called a Kenmotsu manifold [5]: they set up one of the three classes of almost contact

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metric manifolds, whose automorphism group attains the maximum dimension [11].

The present paper is organized as follows. In section 2, we give the basics of almost Kenmotsu manifolds. In section 3, we study almost Kenmotsu manifolds which satisfy Codazzi condition for h and φh, and we prove that in both cases the tensor h vanishes. Section 4 deals with almost Kenmotsu manifolds which are locally symmetric, and we prove that a locally symmetric almost Kenmotsu manifold M of constant ξ-sectional curvature K is a Kenmotsu manifold of constant curvature $K = -1$. Geometric vector fields such as infinitesimal contact transformation and conformal vector fields are considered in Section 5. We prove that every conformal vector field on a $(\kappa, \mu)'$-almost Kenmotsu manifold of dimension > 3 with $h' \neq 0$ is necessarily Killing. Also, it has been shown that if M is a $(\kappa, \mu)'$-almost Kenmotsu manifold with $h' \neq 0$ and $\kappa \neq -2$, then any vector field V which leaves the curvature tensor invariant is Killing.

2 Preliminaries

Let M be a smooth manifold of dimension $2n+1$. We say M has an *almost contact structure* if there is a tensor field φ of type $(1,1)$, a vector field ξ (called the *Reeb vector field* or *characteristic vector field*), and a 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{2.1}$$

Further (2.1) imply that $\varphi\xi = 0$, $\eta \circ \varphi = 0$, and the rank of φ is $2n$. For more details, we refer to [1].

If M is endowed with a (φ, ξ, η)-structure and a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all $X, Y \in TM$, then $(M, \varphi, \xi, \eta, g)$ is called an *almost contact metric manifold.* The relation (2.2) implies that $\eta(X) = g(X, \xi)$ and is equivalent to

$$g(\varphi X, Y) = -g(X, \varphi Y). \tag{2.3}$$

The curvature operator R is given by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, where ∇ is the Levi-Civita connection of g.

The *fundamental* 2*-form* of an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is defined by

$$\Phi(X, Y) = g(X, \varphi Y),$$

which satisfies $\eta \wedge \Phi^n \neq 0$. An almost contact metric manifold with (φ, ξ, η, g) structure such that the 1-form η is closed and $d\Phi = 2\eta \wedge \Phi$ is said to be an *almost Kenmotsu manifold* [4]. A normal almost Kenmotsu manifold is called a Kenmotsu manifold.

Next, we consider two tensor fields $h := (1/2)\pounds_\xi\varphi$ and $h' := h\circ\varphi$, and both are known to be symmetric and satisfy

$$h\xi = h'\xi = 0, \quad trh = trh' = 0, \quad h^2 = h'^2, \quad h\varphi + \varphi h = 0, \tag{2.4}$$

where tr denotes the trace. Further, one has the following formulas:

$$\nabla\xi = -\varphi^2 - \varphi h, \tag{2.5}$$

$$\nabla_\xi h = -\varphi - 2h - \varphi h^2 - \varphi R(\cdot,\xi)\xi, \tag{2.6}$$

$$R(\cdot,\xi)\xi - \varphi R(\varphi\cdot,\xi)\xi = 2(\varphi^2 - h^2), \tag{2.7}$$

$$R(X,Y)\xi = \eta(X)(Y - \varphi hY) - \eta(Y)(X - \varphi hX) + (\nabla_Y\varphi h)X - (\nabla_X\varphi h)Y. \tag{2.8}$$

An almost Kenmotsu manifold such that $h = 0$ is locally isometric to a warped product $I \times_{f^2} N$, where I is an interval tangent to ξ, N is an almost Kähler manifold, which is an integral submanifold of the horizontal distribution, and $f^2 = ce^{2t}$, $c > 1$ [2]. If furthermore M is normal, i.e. M is a Kenmotsu manifold, then N is Kähler.

The $(\kappa,\mu)'$-nullity distribution was introduced by Dileo and Pastore in [3] which is defined by

$$\begin{aligned} N_p(\kappa,\mu)' = \{Z \in T_pM : R(X,Y)Z =& \kappa[g(Y,Z)X - g(X,Z)Y] \\ &+ \mu[g(Y,Z)h'X - g(X,Z)h'Y]\}, \end{aligned}$$

for any $p \in M$, where $h' = h\circ\varphi$ and $\kappa,\mu \in \mathbb{R}$. An almost Kenmotsu manifold such that $\xi \in N(\kappa,\mu)'$, that is,

$$R(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)h'X - \eta(X)h'Y\}, \tag{2.9}$$

is called $(\kappa,\mu)'$-almost Kenmotsu manifold. Also we have $h'^2 = (\kappa+1)\varphi^2$.

If $h' \neq 0$ in a $(\kappa,\mu)'$-almost Kenmotsu manifold, then $\kappa < -1, \mu = -2$ and the manifold is locally isometric to the warped product

$$\mathbb{H}^{n+1}(\kappa - 2\lambda) \times_f \mathbb{R}^n,$$

where $\lambda = \sqrt{-(\kappa+1)}, \mathbb{H}^{n+1}(\kappa-2\lambda)$ is the hyperbolic space of constant curvature $\kappa - 2\lambda < -1$, $f = ce^{(1-\lambda)t}, c > 0$. In particular the base of the warped product is tangent to the distribution spanned by ξ and eigenvectors of h' with eigenvalue λ, the fibers are tangent to the eigendistribution of h' with eigenvalue λ. When $\lambda = 1$, or equivalently $\kappa = -2$, M is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ [3].

3 The Codazzi condition for h and φh

Let A be a tensor of type $(1,1)$ which is self-adjoint. Then A is said to be Codazzi if

$$(\nabla_X A)Y = (\nabla_Y A)X$$

for all $X, Y \in TM$. For contact metric manifolds M, Sharma [7] proved that, if h (or φh) is Codazzi on M, then M is K-contact (that is, $h = 0$). Now, we give the Kenmotsu version of this result for almost Kenmotsu manifolds.

Theorem 1. *Let M be an almost Kenmotsu manifold.*

(i) If h is a Codazzi tensor, then $h = 0$.

(ii) If φh is a Codazzi tensor, then $h = 0$.

Proof. Suppose that h is Codazzi, that is,

$$(\nabla_X h)Y = (\nabla_Y h)X, \quad X, Y \in TM.$$

For $Y = \xi$, using (2.5) we find

$$(\nabla_\xi h)X = -hX - \varphi h^2 X.$$

In view of (2.6), the above equation turns into

$$\varphi R(X,\xi)\xi = -\varphi X - hX. \tag{3.1}$$

Operating φ on both sides, it follows that

$$R(X,\xi)\xi = \varphi^2 X + \varphi h X. \tag{3.2}$$

Use of (3.1) and (3.2) in (2.7) shows that $h^2 = 0$, and thus, since h is symmetric, then $h = 0$, which proves (i). To prove (ii), define a tensor $\mathcal{A}$ of type $(1,1)$ by

$$\mathcal{A} = \nabla\xi.$$

Then $\mathcal{A} : X \to \nabla_X \xi$, and we have

$$R(X,Y)\xi = (\nabla_X \mathcal{A})Y - (\nabla_Y \mathcal{A})X,$$

which together with (2.5) gives

$$R(X,Y)\xi = -(\nabla_X \varphi^2)Y - (\nabla_X \varphi h)Y + (\nabla_Y \varphi^2)X + (\nabla_Y \varphi h)X.$$

Thus, it shows that φh is Codazzi, that is,

$$(\nabla_X \varphi h)Y = (\nabla_Y \varphi h)X$$

if and only if

$$R(X,Y)\xi = (\nabla_Y \varphi^2)X - (\nabla_X \varphi^2)Y. \tag{3.3}$$

Taking $Y = \xi$ in (3.3) and since $\nabla_\xi \varphi = 0$, we get

$$R(X,\xi)\xi = \varphi^2(\nabla_X \xi) = \varphi^2 X + \varphi h X. \tag{3.4}$$

From which we have

$$\varphi R(\varphi X,\xi)\xi = -\varphi^2 X + \varphi h X. \tag{3.5}$$

Making use of (3.4) and (3.5) in (2.7), it yields $h^2 = 0$, and so $h = 0$. QED

Remark 1. In an almost Kenmotsu manifold M, if h (or φh) is Codazzi on M, then M is locally isometric to a warped product $I \times_{f^2} N$, where I is an interval tangent to ξ, N is an almost Kähler manifold, which is an integral submanifold of the horizontal distribution, and $f^2 = ce^{2t}$, $c > 1$ (see Section 2).

4 Locally symmetric almost Kenmotsu manifolds

Proposition 1. *For a locally symmetric almost Kenmotsu manifold having non-vanishing ξ-sectional curvature, $Q\xi$ is collinear with ξ.*

Proof. Note that $\nabla R = 0$ implies $\nabla Q = 0$, and so

$$R(X,Y)QZ = QR(X,Y)Z.$$

Hence we have

$$g(R(X,Y)Z, QW) = -g(R(X,Y)QW, Z) = -g(R(X,Y)W, QZ).$$

Then we get

$$\begin{aligned} g(R(Z,QW)X,Y) &= g(R(X,Y)Z,QW) = -g(R(X,Y)W,QZ) \\ &= -g(R(W,QZ)X,Y). \end{aligned}$$

Setting $X = Z = \xi$ in above,

$$g(R(\xi,QW)\xi,Y) + g(R(W,Q\xi)\xi,Y) = 0.$$

For $W = \xi$ and $Y = Q\xi$, above equation becomes

$$g(R(\xi,Q\xi)\xi,Q\xi) = 0.$$

If $Q\xi$ is not collinear with ξ at some point, then the last equation shows that $K(\xi,Q\xi) = 0$, which contradicts our hypothesis. Thus, $Q\xi = f\xi$, for some scalar function f on M. QED

We recall the following lemmas which were obtained by Dileo and Pastore in [2].

Lemma 1. *[2, Proposition 6] Let M be be a locally symmetric almost Kenmotsu manifold. Then, $\nabla_\xi h = 0$.*

Lemma 2. *[2, Theorem 3] Let M be a locally symmetric almost Kenmotsu manifold. Then, the following conditions are equivalent:*

(i) M is a Kenmotsu manifold;

(ii) $h = 0$.

Moreover, if any of the above conditions holds, M has constant sectional curvature $K = -1$.

Theorem 4 in [2] implies that an almost Kenmotsu manifold of constant curvature K is a Kenmotsu manifold and $K = -1$. Here we prove:

Theorem 2. *A locally symmetric almost Kenmotsu manifold M of constant ξ-sectional curvature K is a Kenmotsu manifold of constant curvature $K = -1$.*

Proof. As the contact manifold M is locally symmetric, it follows from Lemma 1 and the equation (2.6) that

$$R(X,\xi)\xi = \varphi^2 X + 2\varphi h X - h^2 X, \tag{4.1}$$

for all $X \in TM$. Since M has constant ξ-sectional curvature K, for any X orthogonal to ξ, we have

$$R(X,\xi,\xi,X) = Kg(X,X). \tag{4.2}$$

Taking inner product with $X \perp \xi$ in (4.1) and using (4.2) we get

$$Kg(X,X) = g(\varphi^2 X, X) + 2g(\varphi h X, X) - g(h^2 X, X).$$

Replacing X by $X + Y$, where X and Y are orthogonal to ξ, we obtain

$$Kg(X,Y) = g(\varphi^2 X, Y) + 2g(\varphi h X, Y) - g(h^2 X, Y). \tag{4.3}$$

Now if $X, Y \in TM$, then replacing X, Y by $\varphi X, \varphi Y$ in (4.3), we find

$$h^2 = -(K+1)(I - \eta \otimes \xi) - 2\varphi h,$$

and so (4.1) becomes

$$R(\cdot,\xi)\xi = K(I - \eta \otimes \xi) + 4\varphi h. \tag{4.4}$$

Let X be a unit vector field orthogonal to ξ. Then using (4.4) in (2.7), we get

$$h^2X = -(K+1)X.$$

Now, if X is a unit eigenvector of h corresponding to the eigenvalue λ, then we get $\lambda^2X = -(K+1)X$, and so

$$\lambda^2 = -(K+1). \tag{4.5}$$

Since $\lambda^2 \geq 0$ it follows that $K \leq -1$. Computing $R(X,\xi)\xi$ from (4.1), we get

$$KX + 4\lambda\varphi X = -X + 2\lambda\varphi X - \lambda^2X,$$

and thus it follows that

$$KX + 2\lambda\varphi X = -(1+\lambda^2)X. \tag{4.6}$$

Now taking inner product with φX in (4.6) gives $\lambda = 0$, and so from (4.5) we obtain $K = -1$. Being $h = 0$, from Lemma 2 the result follows. $\boxed{QED}$

Remark 2. It has been kindly brought to my attention by the referee that the result also follows from Proposition 8 of [2].

5 Geometric vector fields on $(\kappa,\mu)'$-almost Kenmotsu manifold

Let (M,η) be a contact manifold. If there exists a certain $\sigma \in C^\infty(M)$ such that

$$\pounds_V\eta = \sigma\eta, \tag{5.1}$$

then we say that the vector field V is an *infinitesimal contact transformation* (see Tanno [10]). If $\sigma = 0$, then V is called *strictly infinitesimal contact transformation.* Note that the 1-form η of an almost Kenmotsu manifold is not a contact form. Here, in this section an analogous notion, with the same terminology, is considered for almost contact manifolds (M,φ,ξ,η). Now, we prove the following:

Proposition 2. *Any infinitesimal contact transformation on $(\kappa,\mu)'$-almost Kenmotsu manifold leaving the Ricci tensor invariant is strictly infinitesimal contact transformation.*

Proof. Let V be an infinitesimal contact transformation which leaves the Ricci tensor invariant, that is, $(\pounds_V S)(X,Y) = 0$ for all $X,Y \in TM$, which implies

$$\pounds_V S(X,\xi) = S(\pounds_V X,\xi) + S(X,\pounds_V\xi). \tag{5.2}$$

Note that from (2.9) one would easily obtain

$$S(X,\xi) = 2n\kappa\eta(X), \tag{5.3}$$

because $tr(h') = 0$. Using (5.3) in (5.2) gives $2n\kappa(\pounds_V\eta)(X) = S(X, \pounds_V\xi)$, and so from (5.1) we obtain $2n\kappa\sigma\eta(X) = S(X, \pounds_V\xi)$. For $X = \xi$, it follows from (5.3) that

$$\sigma = \eta(\pounds_V\xi). \tag{5.4}$$

Note that (5.1) gives

$$\sigma = (\pounds_V\eta)(\xi) = -\eta(\pounds_V\xi). \tag{5.5}$$

Hence (5.4) and (5.5) shows that $\sigma = 0$. This completes the proof. $\boxed{QED}$

Let (M, g) be a Riemannian manifold. If there exists a certain $\rho \in C^\infty(M)$, called the potential function, such that

$$\pounds_V g = 2\rho g \tag{5.6}$$

then we say that the vector field V is a *conformal vector field.* V is homothetic when ρ is constant, whereas Killing when $\rho = 0$. The conformal vector fields on contact metric manifolds are already considered and characterized on it. In particular, Okumura [6] proved that every non-Killing conformal vector field on a Sasakian manifold of dimension > 3 is special concircular. In [9] Sharma and Vrancken proved that a (κ, μ)-contact manifold admitting a non-Killing conformal vector field is either Sasakian or has $\kappa = -n-1, \mu = 1$ in dimension $>$ 3; and Sasakian or flat in dimension 3 which generalize the result of Sharma and Blair [8]. Here we investigate conformal vector fields on $(\kappa, \mu)'$-almost Kenmotsu manifolds.

First we need the following lemmas.

Lemma 3. *If M is a $(\kappa, \mu)'$-almost Kenmotsu manifold endowed with a conformal vector field V, then*

$$\sum_{i=1}^{2n+1} g((\pounds_V h')E_i, E_i) = 0 \text{ and } \sum_{i=1}^{2n+1} g((\pounds_V h')h'E_i, E_i) = 0, \tag{5.7}$$

where $\{E_i\}$ is a local orthonormal basis on M.

Proof. Let $\{E_i\}$ be a φ-basis, that is, $\{E_i = e_i, E_{n+i} = \varphi e_i, E_{2n+1} = \xi\}_{i=1}^n$, such that $he_i = \lambda e_i (\Rightarrow h\varphi e_i = -\lambda\varphi e_i)$. Now $trh' = 0$ implies

$$\sum_{i=1}^{2n+1} g(h'E_i, E_i) = 0.$$

Taking its Lie derivative along V gives

$$\sum_{i=1}^{2n+1}[g((\pounds_V h')E_i,E_i)+2g(\pounds_V E_i,h'E_i)]=0. \tag{5.8}$$

But we have

$$\begin{aligned}\sum_{i=1}^{2n+1} g(\pounds_V E_i,h'E_i)&=\sum_{i=1}^{n}[g(\pounds_V e_i,h'e_i)+g(\pounds_V\varphi e_i,h'\varphi e_i)]\\ &=\lambda\sum_{i=1}^{n}[g(\pounds_V e_i,e_i)-g(\pounds_V\varphi e_i,\varphi e_i)]=0,\end{aligned}$$

and so (5.8) leads to the first equation in (5.7). To prove the second, consider

$$\sum_{i=1}^{2n+1} g((\pounds_V h')E_i,h'E_i)=\sum_{i=1}^{2n+1}[g(\pounds_V h'E_i,h'E_i)-g(\pounds_V E_i,h'^2E_i)]=0, \tag{5.9}$$

where we used (5.6) and $h'^2=(k+1)\varphi^2$. Since V is conformal and h' is self-adjoint, we may note that $g((\pounds_V h')X,Y)=g((\pounds_V h')Y,X)$. Therefore

$$g((\pounds_V h')h'E_i,E_i)=g((\pounds_V h')E_i,h'E_i),$$

and so from (5.9) the second equation of (5.7) follows. $\boxed{QED}$

Lemma 4. *If M is a $(\kappa,\mu)'$-almost Kenmotsu manifold with $h'\neq 0$ endowed with a conformal vector field V, then the Laplacian of the potential function ρ satisfies*

$$\Delta\rho=4n\kappa\rho+(2n-1)(\nabla\nabla\rho)(\xi,\xi). \tag{5.10}$$

Proof. For a conformal vector field V, that is, which satisfies (5.6) we have (see Yano [14])

$$(\pounds_V S)(X,Y)=-(2n-1)(\nabla\nabla\rho)(X,Y)+(\Delta\rho)g(X,Y), \tag{5.11}$$

which gives

$$\Delta\rho=(\pounds_V S)(\xi,\xi)+(2n-1)(\nabla\nabla\rho)(\xi,\xi), \tag{5.12}$$

where $\Delta=-divD$. Note that from (2.9) we can easily obtain $S(X,\xi)=2n\kappa\eta(X)$. Making use of this in (5.12), we get

$$\Delta\rho=-2S(\pounds_V\xi,\xi)+(2n-1)(\nabla\nabla\rho)(\xi,\xi). \tag{5.13}$$

Since (5.3) holds for any $(\kappa,\mu)'$-almost Kenmotsu manifold with $h'\neq 0$, we have

$$S(\pounds_V\xi,\xi)=2n\kappa\eta(\pounds_V\xi). \tag{5.14}$$

Now taking the Lie derivative of $g(\xi,\xi)=1$ along V and using equation (5.6) we get

$$\eta(\pounds_V\xi)=-\rho. \tag{5.15}$$

Use of (5.14) and (5.15) in (5.13) gives (5.10), completing the proof. $\boxed{QED}$

The following lemma is proved in [3] by Dileo and Pastore.

Lemma 5. *[3, Proposition 4.1] Let M be an almost Kenmotsu manifold such that $h'\neq 0$ with $\xi\in N(\kappa,\mu)'$. Then $\kappa<-1$, $\mu=-2$ and spect$(h')=\{0,\lambda,-\lambda\}$ with 0 as simple eigenvalue and $\lambda=\sqrt{-1-k}$.*

It is the time to prove one of the main result of the paper.

Theorem 3. *Let M is a $(\kappa,\mu)'$-almost Kenmotsu manifold of dim >3 with $h'\neq 0$. Then every conformal vector field V on M is Killing.*

Proof. For a conformal vector field V we have the following integrability condition (see Yano [14])

$$\begin{aligned}(\pounds_V R)(X,Y,Z)=&-(\nabla\nabla\rho)(Y,Z)X+(\nabla\nabla\rho)(X,Z)Y\\&-g(Y,Z)\nabla_X D\rho+g(X,Z)\nabla_Y D\rho,\end{aligned} \tag{5.16}$$

where $D\rho$ denotes the gradient of ρ. Now taking the Lie derivative of (2.9) along the conformal vector field V, using equations (5.16), (5.6) and Lemma 5 we obtain

$$\begin{aligned}R(X,Y)\pounds_V\xi=&g(\nabla_\xi D\rho,Y)X-g(\nabla_\xi D\rho,X)Y+\eta(Y)\nabla_X D\rho-\eta(X)\nabla_Y D\rho\\&+\kappa\{2\rho g(\xi,Y)X+g(\pounds_V\xi,Y)X-2\rho g(\xi,X)Y-g(\pounds_V\xi,X)Y\}\\&-2\{2\rho g(\xi,Y)h'X+g(\pounds_V\xi,Y)h'X+g(\xi,Y)(\pounds_V h')X\\&-2\rho g(\xi,X)h'Y-g(\pounds_V\xi,X)h'Y-g(\xi,X)(\pounds_V h')Y\}.\end{aligned}$$

Putting $X=\xi$ in above equation, and using (2.9) we find

$$\begin{aligned}\nabla_Y D\rho=&g(\nabla_\xi D\rho,Y)\xi-g(\nabla_\xi D\rho,\xi)Y+\eta(Y)\nabla_\xi D\rho\\&+2g(h'\pounds_V\xi,Y)\xi+2\kappa\rho\eta(Y)\xi-2\kappa\rho Y\\&-2\eta(Y)(\pounds_V h')\xi+4\rho h'Y+2(\pounds_V h')Y.\end{aligned}$$

Substituting $h'E_i$ for Y in above equation, taking inner product with E_i, summing over i, using $h'^2=(\kappa+1)\varphi^2$ and (5.7) we obtain

$$\sum_{i=1}^{2n+1} g(\nabla_{h'E_i}D\rho,E_i)=-8n\rho(\kappa+1). \tag{5.17}$$

For $(\kappa,\mu)'$-almost Kenmotsu manifold with $h'\neq 0$, we know the following formula for Ricci tensor for $n>1$ [13]

$$S(X,Y)=-2ng(X,Y)+2n(\kappa+1)\eta(X)\eta(Y)-2ng(h'X,Y). \tag{5.18}$$

Taking Lie derivative of (5.18) along V, using (5.11) and (5.6) one would get

$$\begin{aligned}(2n-1)g(\nabla_X D\rho,Y)=&(4n\rho(\kappa+1)+(2n-1)(\nabla\nabla\rho)(\xi,\xi))g(X,Y)\\ &-2n(\kappa+1)\{\eta(Y)(\pounds_V\eta)(X)+\eta(X)(\pounds_V\eta)(Y)\}\\ &+2n\{2\rho g(h'X,Y)+g((\pounds_V h')X,Y)\}.\end{aligned}$$

Substituting $h'E_i$ for X and E_i for Y in above, summing over i, and making use of (5.17) and (5.7) we have

$$8n\rho(\kappa+1)(n-1)=0.$$

Since $n>1$ and $\kappa<-1$, we have $\rho=0$, and so V is Killing. This completes the proof. $\boxed{QED}$

Now we prove:

Theorem 4. *Let M be a $(\kappa,\mu)'$-almost Kenmotsu manifold with $h'\neq 0$ and $\kappa\neq -2$. If V is a vector field which leaves the curvature tensor invariant, then V is Killing.*

Proof. Observe that, the operators h and h' admit the same eigenvalues. Indeed, if X is an eigenvector of h with eigenvalue λ, and thus $h\varphi X=-\lambda\varphi X$, then $X+\varphi X$ is an eigenvector of h' with eigenvalue $-\lambda$, while $X-\varphi X$ is eigenvector with eigenvalue λ.

Let X be a unit vector field such that $X\perp\xi$ and $hX=\lambda X$. Then as $\kappa\neq -2$ from Lemma 5, we see λ is different from $+1$ and -1.

Now the condition $\pounds_V R=0$ implies

$$(\pounds_V g)(R(X,Y)Z,W)+(\pounds_V g)(R(X,Y)W,Z)=0.$$

Let G be a $(1,1)$-tensor field defined by $g(GX,Y)=(\pounds_V g)(X,Y)$. Then

$$g(R(X,Y)Z,GW)+g(R(X,Y)W,GZ)=0. \tag{5.19}$$

Taking $Y=Z=W=\xi$ in above it follows that

$$g(R(X,\xi)\xi,G\xi)=0. \tag{5.20}$$

By [3, Lemma 4.1], one has $\nabla_\xi h'=0$. Since $h=\varphi\circ h'$ and $\nabla_\xi\varphi=0$, we have $\nabla_\xi h=0$. Note that (2.6) takes the form

$$R(\cdot,\xi)\xi=\varphi^2+2\varphi h-h^2, \tag{5.21}$$

because $\nabla_\xi h = 0$. Using (5.21) in (5.20), one can easily obtain

$$(1+\lambda^2)g(X, G\xi) - 2\lambda g(\varphi X, G\xi) = 0,$$
$$(1+\lambda^2)g(\varphi X, G\xi) - 2\lambda g(X, G\xi) = 0,$$

from which we obtain

$$\{(1+\lambda^2)^2 - 4\lambda^2\}g(X, G\xi) = 0,$$
$$\{(1+\lambda^2)^2 - 4\lambda^2\}g(\varphi X, G\xi) = 0.$$

Since λ is different from $+1$ and -1, we get $g(X, G\xi) = 0$ for any $X \perp \xi$. Hence $G\xi = g(G\xi, \xi)\xi$. Now putting $Y = Z = \xi$ in (5.19) gives

$$g(R(X,\xi)\xi, GW) - g(G\xi, \xi)g(R(X,\xi)\xi, W) = 0, \tag{5.22}$$

and using (5.21) in above taking a unit vector field X such that $X \perp \xi$ and $hX = \lambda X$, we have

$$\{(1+\lambda^2)^2 - 4\lambda^2\}(g(GX, W) - g(G\xi, \xi)g(X, W)) = 0,$$
$$\{(1+\lambda^2)^2 - 4\lambda^2\}(g(G\varphi X, W) - g(G\xi, \xi)g(\varphi X, W) = 0.$$

So that we have $GX = g(G\xi, \xi)X$ for any $X \perp \xi$, and hence

$$G = g(G\xi, \xi)I,$$

that is,

$$\pounds_V g = 2\rho g$$

for some function ρ. Thus V is a conformal vector field and by Theorem 3, V is Killing for $n > 1$. Now suppose that $n = 1$. Then (5.11) can be written as

$$(\pounds_V S)(X, Y) = -(\nabla_X d\rho)Y + (\Delta\rho)g(X, Y).$$

Since $\pounds_V R = 0$ implies $\pounds_V S = 0$,

$$(\nabla_X d\rho)Y = (\Delta\rho)g(X, Y),$$

which is equivalent to $g(Y, \nabla_X D\rho) = (\Delta\rho)g(X, Y)$. Contracting this with respect to X and Y and the fact that $\Delta = -divD$, we have $\Delta\rho = 0$. Thus $\nabla d\rho = 0$, and hence

$$\nabla(d\rho \otimes d\rho) = 0.$$

Since $d\rho \otimes d\rho$ is a $(0, 2)$-tensor, it follows from Theorem 4.2 of [12] that $d\rho \otimes d\rho = cg$, for some constant c. Thus

$$(Y\rho)D\rho = cY, \tag{5.23}$$

which for $Y = D\rho$ gives $\|D\rho\|^2 D\rho = cD\rho$. Now (5.23) can be written as

$$g(Y, D\rho)g(D\rho, X) = cg(Y, X)$$

Contracting this equation over X and Y yields

$$\|D\rho\|^2 = 3c.$$

Consequently, we obtain ρ is constant and so V is homothetic. Since ρ is constant (5.10) gives $4\kappa\rho = 0$. As $h' \neq 0$, it follows that $\rho = 0$ and hence V is Killing. This finishes the proof. $\boxed{QED}$

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