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Positive Solutions of singular semilinear elliptic equation in bounded NTA- Domains

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Abstract. We study the existence, the uniqueness and the sharp estimate of a positive solution of the nonlinear equation

$$\Delta v + \psi(.,v) = 0,$$

in a bounded NTA-domain Ω in $\mathbb{R}^n$ $(n \geq 2)$, when a measurable function $\psi(.,.)$ is continuous and non-increasing with respect to the second variable.

Keywords: Green function, superharmonic function, piecewise Dini-smooth bounded Jordan domain, conformal mapping, NTA-domain, Schauder fixed point theorem.

MSC 2000 classification: primary 30C35, secondary 31A05, 31A35, 35J08, 35J61

Introduction

We work in the Euclidean space $\mathbb{R}^n$ $(n \geq 2)$. We denote by Ω a bounded non-tangentially accessible (NTA for short) domain (see definition in [15]). In this paper we study the existence, the uniqueness and a sharp estimate of a positive continuous solution of the nonlinear elliptic problem

$$\begin{cases} \Delta v(x) + \psi(x, v(x)) = 0, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega, \end{cases} \tag{0.1}$$

where ψ is required to satisfy some appropriate hypothesis related to a functional class. The existence results of problem (0.1) have been extensively studied for the special nonlinearity $\psi(x,t) = p(x)\ q(t)$, for both bounded and unbounded domains in $\mathbb{R}^n$ $(n \geq 2)$ with smooth compact boundary (see for example [4, 8, 9, 10, 12, 13, 14, 18]). In [14] Edelson studied (0.1) in $\mathbb{R}^2$, when $\psi(x,t) = p(x)t^{-\gamma}$, $0 < \gamma < 1$. He proved the existence of an entire positive

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solution with the growth ln$|.|$ near infinity. This result is generalized later by Zeddini in [29], he studied (0.1) in $\mathcal{D} = \{x \in \mathbb{R}^2 \ : \ |x| > 1\}$, and he proved the existence of positive solutions with the same growth as ln$|.|$ near infinity. In [2] Ben Boubaker and Gharbi studied (0.1) on NTA-cones in $\mathbb{R}^n (n \geq 3)$. They study the existence, the uniqueness and the asymptotic behavior of positive solutions. In [10] Crandall, Rabinowitz and Tatar studied (0.1) on a bounded open domain, where they proved the existence of solutions, and continuity properties of the solutions if $\psi(x,t)$ does not depend on x, by using the method of sub- and supersolutions. Lazer and MacKenna [20] also dealt the problem (0.1), when $\psi(x,t) = p(x)t^{-\gamma}$, $\gamma > 0$ on a bounded open domain, with p a continuous function, proving existence and regularity results at the boundary for the solutions. In [18], Lair and Shaker proved the result of [20] in $\mathbb{R}^n$ $(n \geq 3)$. These results were generalized later by Lair and Shaker in [19]. They studied (0.1) on a bounded smooth domain when $\psi(x,t) = p(x)\ q(t)$, q is a positive non-increasing and differentiable function on $]0, +\infty[$ which is integrable near 0. They proved that the problem (0.1) has a unique weak positive solution $v \in H_0^1(\Omega)$, provided that q is a nontrivial, nonnegative $L^2(\Omega)$ function. In [6] Canino, Grandinetti and Sciunzi studied (0.1) on a bounded smooth domain of $\mathbb{R}^n$ $(n \geq 1)$. They considered a jumping problem for singular elliptic equations, by using truncation argument and exploiting minimax methods they proved the existence of solutions to the truncated problem. In [4] Boccardo and Orsina studied (0.1) when $\psi(x,t) = p(x)t^{-\gamma}$, $\gamma > 0$ on a bounded open set of $\mathbb{R}^n$ $(n \geq 2)$, with p is a nonnegative function. They proved existence, regularity and non existence results which depend on the summability of p in some Lebesgue spaces, and on the value of γ. Recently in [8] Canino and Sciunzi prove the uniqueness of the solution for the problem studied by Boccardo and Orsina in [4]. Yet recently in [9] Carmona and Martinez-Aparicio studied (0.1) when $\psi(x,t) = p(x)\ t^{-\gamma(x)}$ on an open bounded set of $\mathbb{R}^n$ $(n \geq 2)$, with $\gamma(x)$ is a positive continuous function and p is a positive function that belongs to a certain Lebesgue space. Inspired by [4], they proved existence results for the problem (0.1). In [28] Wang, Zhao and Zhang studied (0.1), when $\psi(x,t) = \lambda t^{\beta} + p(x)t^{-\gamma}$ in a bounded smooth domain Ω, with $1 < \beta$, $0 < \gamma < 1$ and $p \in \mathcal{C}_0^{\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$. They proved that (0.1) has at least two positive solutions. In [7] Canino, Montoro and Sciunzi studied (0.1) in a bounded smooth domain of $\mathbb{R}^n$ $(n \geq 1)$, when $\psi(x,t) = \frac{1}{t^{\gamma}} + f(x,t)$, $\gamma > 0$ and f is a carathéodory function which is uniformly locally Lipschitz continuous with respect to the second variable. They proved symmetry and monotonicity properties of the solutions under general assumptions on the nonlinearity. In [5] Canino, Esposito and Sciunzi studied (0.1) in a bounded $\mathcal{C}^{2,\alpha}$ domain of $\mathbb{R}^n$, with $0 < \alpha < 1$, $n \geq 1$. They proved a Höpf type boundary lemma via a suitable scaling argument that allows to deal with the lack of regularity of the solutions up to the boundary. Thus, our aim in this paper is to extend the results of [13, 14, 12, 29] and [28] to a more general problem on bounded non-smooth domains in $\mathbb{R}^n$, $n \geq 2$. Moreover we give sharp estimates for the solutions. In order to establish our results we are required to use technical methods, we apply sharp estimates for the Green's function and 3-G inequalities established by Ben Boubaker and Selmi in [3], Hirata in [17], Hansen in [15] and Riahi in [26]. Let z_0 be a fixed point in Ω. By G, we denote the Green function for the Laplacian in Ω and by $g = \min(1, G(., z_0))$.

Definition 1 (**Kato class**)**.** (see [25], p.61)
A Borel measurable function q on Ω belongs to the Kato class $\mathcal{K}(\Omega)$ if

$$\lim_{r \longrightarrow 0} \sup_{x\in\Omega} \int_{\Omega\cap(|x-y|<r)} \frac{g(y)}{g(x)} G(x,y) \, |q(y)| \, dy = 0. \tag{0.2}$$

The following hypothesis on ψ are adopted :

($\mathbf{H_1}$) $\psi : \ \Omega \times (0,\infty) \longrightarrow [0,+\infty)$, is a Borel measurable function which is continuous and nonincreasing with respect to the second variable .

($\mathbf{H_2}$) The function $\psi(.,c)$ belongs to the Kato class $\mathcal{K}(\Omega)$, for every $c > 0$.

($\mathbf{H_3}$) The function $V(\psi(.,c))$ is strictly positive for every $c > 0$.
Where V is the potential kernel associated to Δ ($i.e.$ $V = (-\Delta)^{-1}$).

The following notations will be adopted:

i) Let f and h be two positive functions on Ω. We say that f is comparable to h on Ω and we denote $f \simeq h$, if there exists $C \geq 1$, such that $\forall x \in \Omega$, $\frac{1}{C} \, h(x) \leq f(x) \leq C \, h(x)$.

ii) $B(x,r)$ denotes the open ball of center x and radius r.

iii) $B(\Omega)$ be the set of Borel measurable functions in Ω and $B^+(\Omega)$ be the set of non-negative one.

iv) $C_0(\overline{\Omega})$ will denote the set of continuous functions in $\overline{\Omega}$ vanishing on $\partial\Omega$.

v) We denote by $\delta(x)$ (respectively d), the distance from x to $\partial\Omega$ for all $x \in \Omega$ (respectively the diameter of Ω).

vi) By the symbol C, we denote an absolute positive constant whose value is unimportant and may change from line to line.

vii) For two constants $a, b \in \mathbb{R}$, we denote $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$.

We define the potential kernel V on $B^+(\Omega)$ by $V\Psi(x) = \int_\Omega G(x,y)\Psi(y)\,dy$. We note that, for any $\Psi \in B^+(\Omega)$ such that $\Psi \in L^1_{loc}(\Omega)$ and $V\Psi \in L^1_{loc}(\Omega)$, we have in the distributional sense

$$\Delta(V\Psi) = -\Psi \qquad \text{in } \Omega. \tag{0.3}$$

We point out that for any $\Psi \in B^+(\Omega)$ such that $V\Psi \not\equiv \infty$, we have $V\Psi \in L^1_{loc}(\Omega)$ (see [11], p51). Let us recall that V satisfies the complete maximum principle, i.e for each $\Psi \in B^+(\Omega)$ and a nonnegative superharmonic function v on Ω such that $V\Psi \leq v$ in $\{\Psi > 0\}$ we have $V\Psi \leq v$ in Ω, (cf. [24], Theorem 3.6, p 175). Our main result is the following :

Theorem 1. *Assume* $(\mathbf{H_1})$,$(\mathbf{H_2})$ *and* $(\mathbf{H_3})$ *are satisfied. Then the problem (0.1) has a unique positive solution* v *continuous on* $\overline{\Omega}$*, satisfying for all* $x \in \Omega$,

$$Cg(x) \leq v(x) \leq \min\left(\nu, \int_\Omega G(x,y)\ \psi(y, Cg(y))dy\right),$$

where $\nu = \inf_{\sigma>0} \left(\sigma + \|V\psi(.,\sigma)\|_\infty\right)$.

This paper consists of 4 sections devoted to the following topics. In section 2 we recall and establish some results which will be the basic tools to prove Theorem 1 in section 3. In section 4, by using some results that we established recently in [1], we give an interesting example. (see Example 1). More precisely, we study the problem (0.1), in a bounded simply connected piecewise Dini-smooth Jordan domain in $\mathbb{R}^2$. By using conformal mapping and taking inspiration from [[22], Proposition 4, p.403], we establish sharp estimates for the solution of problem (0.1). Note that a bounded simply connected piecewise Dini-smooth Jordan domain in $\mathbb{R}^2$ is an NTA- domain.

1 Preliminary results

In this section we need to recall and prove some results which are the basic tools to prove Theorem 1. The following set is introduced by K. Hirata in [17]. For each pair of points $x, y \in \overline{\Omega}$, let

$$\mathcal{B}(x,y) = \left\{ b \in \Omega\ :\ \frac{1}{C}\Big(|x-b| \wedge |b-y|\Big) \leq |x-y| \leq 2C\ \delta(b) \right\}, \tag{1.1}$$

where C is a constant strictly grater than 1 depending only on Ω.

Lemma 1. (see [17]) *For each $x, y \in \Omega$ and $b \in \mathcal{B}(x, y)$,*

$$G(x,y) \simeq \begin{cases} \frac{g(x)g(y)}{g(b)^2}\left(1+\ln^+ \frac{(\delta(x)\wedge\delta(y))}{|x-y|}\right) & \text{if } n=2, \\ \\ \dfrac{g(x)g(y)}{g(b)^2}\,|x-y|^{2-n} & \text{if } n\geq 3, \end{cases} \tag{1.2}$$

where $\ln^+ t = (0 \vee \ln t)$ and the constant of comparison depends only on n and Ω.

By using (1.1) and (1.2), we get :

Lemma 2. *There exists a constant $C > 0$ depending only on Ω such that for all $x, y \in \Omega$*

$$G(x,y) \geq C g(x) g(y).$$

Proposition 1. *Let $r > 0$, then there exists a constant C depending only on r and Ω such that for all $x, y \in \Omega$, satisfying $|x-y| \geq r$,*

$$G(x,y) \leq C\ g(x)\ g(y) \tag{1.3}$$

The following 3G-Theorem is established for NTA-domains by W. Hansen in [15] $\mathbb{R}^n, n \geq 3$ and for Jordan domains by L. Riahi in $\mathbb{R}^2$ [26]. It is also true for NTA-domains in $\mathbb{R}^2$ by similar arguments as in [15].

Theorem 2 (**3G-Theorem**). *There exists a constant $C > 0$ depending only on n and Ω such that for all $x, y, z \in \Omega$,*

$$\frac{G(x,z)G(z,y)}{G(x,y)} \leq C\left(\frac{g(z)}{g(x)}G(x,z) + \frac{g(z)}{g(y)}G(z,y)\right). \tag{1.4}$$

Proposition 2. (see [25]) *If $q \in \mathcal{K}(\Omega)$, then*

$$\|q\| = \sup_{x\in\Omega}\int_\Omega \frac{g(y)\ G(x,y)}{g(x)}\ |q(y)|\ dy < +\infty.$$

Corollary 1. (see [25]) *Let $q \in \mathcal{K}(\Omega)$, h a positive superharmonic function h on Ω and C as in Theorem 2, we have*

$$\sup_{x\in\Omega}\int_\Omega \frac{h(y)}{h(x)}G(x,y)\ \ |q(y)|\ dy \leq 2C\|q\|. \tag{1.5}$$

In particular

$$\sup_{x\in\Omega}\int_\Omega G(x,y)\,|q(y)|\ dy \leq 2C\|q\|. \tag{1.6}$$

From (1.6), we deduce

Corollary 2. *Let q be a function in $\mathcal{K}(\Omega)$, then*

$$\int_{\Omega} g(y)\ |q(y)|\ dy < +\infty. \tag{1.7}$$

In particular $q \in L^1_{loc}(\Omega)$

Proposition 3. *Let $q \in \mathcal{K}(\Omega)$ and h be a positive superharmonic function on Ω, then for each $x_0 \in \overline{\Omega}$,*

$$\lim_{r \longrightarrow 0} \sup_{x \in \Omega} \frac{1}{h(x)} \int_{B(x_0,r)\cap\Omega} h(y)\ G(x,y)\ |q(y)|\ dy = 0. \tag{1.8}$$

Proof. Let h be a positive superharmonic function in Ω. Then by ([18],Theorem 2.1, p.164), there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of positive measurable functions in Ω such that

$$h(y) = \sup_{n\in\mathbb{N}} \int_{\Omega} G(y,z)\ f_n(z)\ dz.$$

Hence we need to verify (1.8) only for $h(y) = G(y,z)$, uniformly for $z \in \Omega$. Let $r > 0$. By using Theorem 2, we get

$$\frac{1}{G(x,z)} \int_{B(x_0,r)\cap\Omega} G(x,y)\ G(y,z)\ |q(y)|\ dy \leq 2C \sup_{z\in\Omega} \int_{B(x_0,r)\cap\Omega} \frac{g(y)\ G(z,y)}{g(z)}\ |q(y)|\ dy.$$

Let $\eta > 0$, then

$$\int_{B(x_0,r)\cap\Omega} \frac{g(y)\ G(z,y)}{g(z)} |q(y)|\ dy \leq \int_{\Omega\cap B(z,\eta)} \frac{g(y)\ G(z,y)}{g(z)}\ |q(y)|\ dy + \int_{B(x_0,r)\cap\Omega\setminus B(z,\eta)} \frac{g(y)\ G(z,y)}{g(z)}\ |q(y)|\ dy.$$

Since q is in $\mathcal{K}(\Omega)$, it follows from (0.2), that for all $\varepsilon > 0$, we can find some $\eta_0 > 0$, such that

$$\sup_{z\in\Omega} \int_{\Omega\cap B(z,\eta_0)} \frac{g(y)\ G(z,y)}{g(z)}\ |q(y)|\ dy < \varepsilon.$$

On the other hand it follows from Proposition 1, that for all $y \in \big(B(x_0,r)\cap\Omega\big) \setminus B(z,\eta_0)$

$$\frac{g(y)\ G(z,y)}{g(z)} \leq C\ g^2(y) \leq C\ g(y).$$

Thus the result follows by using (1.7). $\boxed{QED}$

2 Proof of Theorem 1

For $\sigma \geq 0$, we denote by P_σ the following nonlinear boundary value problem

$$P_\sigma = \begin{cases} \Delta u(x) + \psi(x, u(x)) = 0 \quad x \in \Omega, \\ u(x) = \sigma \quad x \in \partial\Omega, \\ u \in C(\overline{\Omega}). \end{cases}$$

Proposition 4. *Assume that (H_1) is satisfied. Then, for each $\sigma \geq 0$, the problem P_σ has at most one positive solution.*

Proof. (see [13])Assume that there exist two positive solutions u, v of (P_σ) with $u \neq v$. Suppose that there exists $x_0 \in \Omega$ such that $v(x_0) > u(x_0)$.
Put $w = v - u \in C_0(\overline{\Omega})$. Then we have

$$\begin{cases} \Delta w + \psi(., v) - \psi(., u) = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $U = \{x \in \Omega,\ w(x) > 0\}$. Then U is an open nonempty set. Since the function ψ satisfies (H_1), we deduce that

$$\begin{cases} \Delta w \geq 0 & \text{in } U, \\ w = 0 & \text{on } \partial U. \end{cases}$$

Hence by the maximum principle (see [12], pages 465-466), we get $w \leq 0$ in U. Which is in contradiction with the definition of U $\boxed{QED}$

Theorem 3. *Let $\sigma > 0$. Then the problem P_σ has a unique positive solution u_σ.*

Proof. Let $\sigma > 0$ and $C_\sigma = \{v \in C(\overline{\Omega}) \ : \ v \geq \sigma\}$. We define the operator T on C_σ by

$$Tv(x) = \sigma + \int_\Omega G(x,y)\psi(y, v(y))dy \quad , \ x \in \Omega.$$

We propose to prove the equicontinuity of $T(C_\sigma)$ in $\overline{\Omega}$. Let $x_0 \in \overline{\Omega}$ and $\eta > 0$. Let $x, x' \in B(x_0, \frac{\eta}{2}) \cap \Omega$. Let $v \in C_\sigma$ then

$$\left|Tv(x) - Tv(x')\right| = \int_\Omega \left|G(x,y) - G(x',y)\right| \psi(y, v(y))\, dy.$$

Since for all $v \in C_\sigma$, $\psi(.,v) \le \psi(.,\sigma)$ then

$$\begin{aligned}\left|Tv(x) - Tv(x')\right| &\le \int_\Omega \left|G(x,y) - G(x',y)\right| \ \psi(y,\sigma)\ dy \\ &\le 2 \sup_{\zeta\in\Omega} \int_{\Omega\cap B(x_0,\eta)} G(\zeta,y)\ \psi(y,\sigma)\ dy \\ &+ \int_{\Omega\setminus B(x_0,\eta)} \left|G(x,y) - G(x',y)\right| \ \psi(y,\sigma)\ dy.\end{aligned}$$

By (1.8), the first quantity of the right hand side is bounded by ε whenever η is sufficiently small. For η sufficiently small, $G(.,y)$ is continuous on $B(x_0,\frac{\eta}{2})\cap\overline{\Omega}$, whenever $y \in \Omega\setminus B(x_0,\eta)$. Moreover, by (1.3), there exists $C > 0$ depending only on Ω such that

$$G(x,y) \le C\ g(y), \quad \forall (x,y) \in \Big(B(x_0,\frac{\eta}{2})\cap\Omega\Big) \times \Big(\Omega\setminus B(x_0,\eta)\Big).$$

Since $\psi(.,\sigma) \in \mathcal{K}(\Omega)$, it follows from (1.7) and Lebesgue's theorem that

$$\int_{\Omega\setminus B(x_0,\eta)} \left|G(x,y) - G(x',y)\right| \ \psi(y,\sigma)\ dy \underset{|x-x'|\to 0}{\longrightarrow} 0.$$

Hence Tv is continuous in $\overline{\Omega}$ uniformly on C_σ.
Now, we will show that $\lim\limits_{x\longrightarrow\partial\Omega} V\psi(.,\sigma)(x) = 0$. Let $x_0 \in \partial\Omega$, $\eta \in]0,1[$ and $x \in B(x_0,\frac{\eta}{2})\cap\Omega$. Then

$$\begin{aligned}|V\psi(.,\sigma)(x)| &\le \int_\Omega G(x,y)\ |\psi(y,\sigma)|\ dy \\ &\le \sup_{x\in\Omega}\int_{B(x_0,\eta)\cap\Omega} G(x,y)\ |\psi(y,\sigma)|\ dy + \int_{\Omega\setminus B(x_0,\eta)} G(x,y)\ |\psi(y,\sigma)|\ dy.\end{aligned}$$

By Proposition 1, we get

$$\int_{\Omega\setminus B(x_0,\eta)} G(x,y)\ |\psi(y,\sigma)|\ dy \le C\ g(x)\ \int_\Omega g(y)\ |\psi(y,\sigma)|\ dy.$$

Thus, we obtain from (1.8) and (1.7), that $\lim\limits_{x\longrightarrow\partial\Omega} V\psi(.,\sigma)(x) = 0$. In the sequel, we deduce that $V\psi(.,\sigma) \in C_0(\overline{\Omega})$. On the other hand, for all $v \in C_\sigma$ we have $\|Tv\|_\infty \le \sigma + \|V\psi(.,\sigma)\|_\infty$. Thus, the family $\{Tv(x), v \in C_\sigma\}$ is uniformly bounded on $\overline{\Omega}$. It follows, by Ascoli's theorem, that $T(C_\sigma)$ is relatively compact in $C(\overline{\Omega})$.
Next, we propose to prove the continuity of T on C_σ.

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in C_σ which converges uniformly to a function u in C_σ. Since ψ is continuous with respect to the second variable and

$$|\psi(y,u_n(y)) - \psi(y,u(y))| \leq 2\,\psi(y,\sigma).$$

It follows, by the dominated convergence theorem, that

$$\forall x \in \overline{\Omega}, \quad Tu_n(x) - Tu(x) \underset{n\to\infty}{\longrightarrow} 0.$$

Since $T(C_\sigma)$ is relatively compact in $C(\overline{\Omega})$, then Tu_n converges uniformly to Tu.
Finally, we deduce, by the Schauder's fixed point theorem, that there exists $v_\sigma \in C_\sigma$ such that

$$v_\sigma = \sigma + \int_\Omega G(x,y)\psi(y,v_\sigma(y))\,dy. \tag{2.1}$$

Hence v_σ is a positive solution of P_σ. The uniqueness follows from Proposition 4. $\boxed{QED}$

Proposition 5. *Let $0 < \mu \leq \sigma$. Then we have $0 \leq v_\sigma - v_\mu \leq \sigma - \mu$.*

Proof. Let f be the function defined on Ω by

$$f(x) = \begin{cases} \dfrac{\psi(x,v_\mu(x)) - \psi(x,v_\sigma(x))}{v_\sigma(x) - v_\mu(x)} & \text{if } v_\mu(x) \neq v_\sigma(x), \\ 0 & \text{if } v_\mu(x) = v_\sigma(x). \end{cases}$$

Let us put $g = v_\sigma - v_\mu$, and denote by $g^+ = \max(g,0)$ and $g^- = \max(-g,0)$.
It is easily to see that $g \in B(\Omega)$, $f \in B^+(\Omega)$ and $g + V(fg) = \sigma - \mu$.
Since $V\psi(.,\sigma)$ and $V\psi(.,\mu) \in C_0(\overline{\Omega})$, we deduce by (H_2) that $V(f\,|g|) \leq \infty$.
Which implies that

$$g^+ + V(fg^+) = (\sigma - \mu) + g^- + V(fg^-).$$

In consequence, we get

$$V(fg^+) \leq (\sigma - \mu) + V(fg^-) \quad \text{on the set} \quad \left\{g^+ > 0\right\}.$$

Since $(\sigma - \mu) + V(fg^-)$ is a nonnegative superharmonic function on Ω, then the complete maximum principle implies that

$$V(fg^+) \leq (\sigma - \mu) + V(fg^-) \quad \text{on } \Omega.$$

And so,

$$V(fg) \leq (\sigma - \mu) = g + V(fg) \quad \text{on } \Omega.$$

Thus, we deduce that

$$0 \leq g \leq \sigma - \mu.$$

$\boxed{QED}$

Proof. **(of Theorem 1)**Let $(\sigma_n)_{n\geq 0}$ be a positive sequence of real numbers that decreases to zero. Let us put for all $n \in \mathbb{N}$, $\theta_n = \sigma_n + \|V(\psi(., \sigma_n))\|_\infty$ and and let v_n be the unique positive solution of the problem P_{σ_n} given in Theorem 3. It follows by Proposition 5, that the sequence$(v_n - \sigma_n)_{n\geq 0}$ increases to a function v. By (H_3), we deduce that, for each $x \in \Omega$,

$$v(x) \geq v_n(x) - \sigma_n \geq \int_\Omega G(x,y)\psi(y, \theta_n)\, dy > 0.$$

So, using the monotone convergence theorem, we obtain

$$v(x) = \int_\Omega G(x,y)\ \psi(y, v(y))\ dy, \quad \forall x \in \Omega. \tag{2.2}$$

Now, we shall prove that v is continuous on Ω. By using (2.2), we see that v is a lower semicontinuous function on Ω. On the other hand, $(v_n)_{n\geq 0}$ is a decreasing sequence of positive continuous functions on Ω, then $v = \inf_{n\geq 0} v_n$ is an upper semicontinuous function on Ω. Which implies that v is continuous on Ω. Hence, it follows from (2.2) that $V\psi(., v) \in L^1_{loc}(\Omega)$. Since v is continuous and positive on Ω, we deduce, by hypothesis (H_2) and corollary 2, that the function $y \longmapsto \psi(y, v(y))$ is in $L^1_{loc}(\Omega)$. Applying Δ on both sides of equality (2.2) and using (0.3), we conclude that v satisfies the equation

$$\Delta v + \psi(., v) = 0 \quad \text{in } \Omega.$$

Finally, since for each $x \in \Omega$ and $n \in \mathbb{N}$, $0 < v(x) \leq v_n(x)$, then $\lim\limits_{x \longrightarrow \partial\Omega} v(x) = 0$. Thus, $v \in C(\overline{\Omega})$ and v is a positive solution of the problem (0.1). The uniqueness follows from Proposition 4.

Now we prove the sharp estimates for the solution.

By (H_1) and (H_3), we see that $\nu = \inf\limits_{\sigma>0} \left(\sigma + \|V\psi(., \sigma)\|_\infty\right) > 0$. Let $\sigma > 0$, from Proposition 5, (H_1) and (2.1), we get

$$v(x) \leq v_\sigma(x) \leq \sigma + V\psi(., \sigma)(x), \quad \text{for all } x \in \Omega.$$

This implies that for all $x \in \Omega$

$$v(x) \leq \nu.$$

By (H_1), we deduce that for all $x \in \Omega$

$$\int_\Omega G(x,y)\psi(x,\nu)dy \leq v(x) \leq \nu.$$

Then it follows from Lemma 2, that for all $x \in \Omega$

$$Cg(x)\int_\Omega g(y)\psi(y,\nu)\ dy \leq v(x) \leq \nu. \tag{2.3}$$

In particular if $x = z_0$, $g(x) = \min(1, +\infty) = 1$, which give

$$\int_\Omega g(y)\psi(y,\nu)\ dy \leq \frac{\nu}{C} < +\infty. \tag{2.4}$$

Hence, we deduce from (2.3) and (2.4) that for all $x \in \Omega$

$$Cg(x) \leq v(x).$$

Since ψ is non-increasing with respect to second variable, we get for all $x \in \Omega$

$$Cg(x) \leq v(x) \leq \min\left(\nu, \int_\Omega G(x,y)\ \psi(y, Cg(y))dy\right).$$

$\boxed{QED}$

3 Study of a particular case

The aim of this section is to establish the following example

Example 1. Let Ω be a bounded simply connected piecewise Dini-smooth Jordan domain in $\mathbb{R}^2$ (see definition in [3] or [23],) having n Dini -smooth corners at $a_1, \ldots, a_n$ of opening angles respectively $\frac{\pi}{\alpha_1}, \ldots, \frac{\pi}{\alpha_n}$, $\alpha_i \in]\frac{1}{2}, +\infty[\setminus\{1\}$.
Let λ, γ a strictly positive constant such that $\tau = \gamma + \lambda < 2$, and for all $y \in \Omega$, $\delta_\lambda(y) = \dfrac{1}{\prod_{i=1}^{n} |y - a_i|^{(\lambda-2)(\alpha_i - 1)} (\delta(y))^\lambda}$, then the problem

$$\begin{cases} \Delta v(x) + \ (v(x))^{-\gamma}\,\delta_\lambda(x) = 0, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega, \end{cases}$$

has a unique positive solution v in $C(\overline{\Omega})$, satisfying for all $x \in \Omega$,

$$C\prod_{i=1}^{n}|x-a_i|^{\alpha_i-1}\,\delta(x)\le v(x)\le$$
$$C\begin{cases}\left(\displaystyle\prod_{i=1}^{n}|x-a_i|^{\alpha_i-1}\,\delta(x)\right)^{2-\tau} & ,\ \text{for } 1<\tau<2\\ \displaystyle\prod_{i=1}^{n}|x-a_i|^{\alpha_i-1}\,\delta(x)\ \ln\left(\frac{e}{\displaystyle\prod_{i=1}^{n}|x-a_i|^{\alpha_i-1}\,\delta(x)}\right) & ,\ \text{for } \tau=1\\ \displaystyle\prod_{i=1}^{n}|x-a_i|^{\alpha_i-1}\,\delta(x) & ,\ \text{for } 0<\tau<1.\end{cases}\tag{3.1}$$

In the beginning let us recall some results that will be necessary to prove inequality (3.1).

Theorem 4. (see [3]) *Let Ω be a bounded simply connected piecewise Dini smooth Jordan domain in $\mathbb{R}^2$ having n Dini-smooth corners at $a_1, a_2, \dots, a_n$ of opening angle respectively $\frac{\pi}{\alpha_1}, \dots, \frac{\pi}{\alpha_n}$, $\alpha_i \in]\frac{1}{2}, +\infty[\setminus 1$, ϕ a conformal mapping from Ω onto the unit disk $D = D(0,1)$. Then we have the following results :*

$$G(x,y)\simeq\ln\left(1+\prod_{k=1}^{n}\left(\frac{(|x-a_k|\wedge|y-a_k|)}{(|x-a_k|\vee|y-a_k|)}\right)^{\alpha_k-1}\frac{\delta(x)\delta(y)}{|x-y|^2}\right),\quad\forall x,y\in\Omega\,,\tag{3.2}$$

$$\left|\phi'(x)\right|\simeq\prod_{i=1}^{n}|x-a_i|^{\alpha_i-1},\quad\forall x\in\Omega\,,\tag{3.3}$$

$$\delta(\phi(x))\simeq\prod_{i=1}^{n}|x-a_i|^{\alpha_i-1}\ \delta(x),\quad\forall x\in\Omega\,,\tag{3.4}$$

where $\delta(\phi(x))$ is the distance of $\phi(x)$ to $\partial D(0,1)$.

Remark 1. Let ϕ be a conformal mapping from Ω onto D, then by using relation (3.4) in Theorem 4, we can see that

$$g(x)\simeq\delta(\phi(x))\qquad\text{on }\Omega\,.\tag{3.5}$$

Proposition 6. (see [1]) *Let $0<\tau<2$. Then the function defined on Ω by $\delta_\tau(y)=\dfrac{1}{\displaystyle\prod_{i=1}^{n}|y-a_i|^{(\tau-2)(\alpha_i-1)}\,(\delta(y))^{\tau}}$ belong to the Kato class $\mathcal{K}(\Omega)$.*

Proof. **of inequality (3.1)** From Theorem 1, (3.4) and (3.5), we deduce that for all $x \in \Omega$,

$$C \prod_{i=1}^{n} |x - a_i|^{\alpha_i - 1}\, \delta(x) \leq v(x) \leq C\ V\ \delta_\tau(x).$$

On the other hand,

$$\begin{aligned} V\ \delta_\tau(x) &= \int_\Omega G(x,y)\ \delta_\tau(y)\ dy \\ &= \int_\Omega G(x,y) \frac{1}{\left(\displaystyle\prod_{i=1}^{n} |y - a_i|^{\alpha_i - 1}\, \delta(y)\right)^\tau} \prod_{i=1}^{n} |y - a_i|^{2(\alpha_i - 1)}\ dy. \end{aligned}$$

Let ϕ be the conformal mapping from Ω to D defined in Theorem 4. By using the variable change $z = \phi(y)$,(3.3) and (3.4), we obtain

$$V\ \delta_\tau(y) \simeq \int_D \ln(1 + \frac{\delta_D(x^*)\delta_D(z)}{|x^* - z|^2})\ \frac{1}{(\delta_D(z))^\tau}\ dz,$$

where $x^* = \phi(x)$ and $\delta_D(z) =$ is the distance from z to ∂D .
In the following, let $R = |x^*|$. We have to discuss two cases.
Case 1 : If $R \leq \frac{1}{4}$. For $x \in D$, we have

$$\begin{aligned} V\delta_\tau(x) \leq C \Big(& \underbrace{\int_{(|z|\leq\frac{1}{2})} \ln(1 + \frac{\delta_D(x^*)\delta_D(z)}{|x^* - z|^2})\ \frac{1}{(\delta_D(z))^\tau}\ dz}_{I_1} + \\ & + \underbrace{\int_{(\frac{1}{2}\leq|z|\leq 1)} \ln(1 + \frac{\delta_D(x^*)\delta_D(z)}{|x^* - z|^2})\ \frac{1}{(\delta_D(z))^\tau}\ dz}_{I_2} \Big). \end{aligned}$$

Since, for $z \in (|z| \leq \frac{1}{2})$ we have $\delta_D(z) \geq \frac{1}{2}$ then,

$$\begin{aligned} I_1 &\leq C \int_0^{\frac{1}{2}} \ln(\frac{1}{(R \vee r)})\ r\ dr \\ &\leq C\ (\ln(\frac{1}{R}) + \int_R^{\frac{1}{2}} r\ \ln(\frac{1}{r})\ dr). \end{aligned}$$

On the other hand, $\ln(1+\dfrac{\delta_D(x^*)\delta_D(z)}{|x^* - z|^2}) \leq C\ \dfrac{\delta_D(z)}{|x^* - z|^2}$ and for $z \in (\frac{1}{2} \leq |z| \leq 1)$, we have $|x^* - z| \geq |z| - |x^*| \geq \frac{1}{4}$. It follows that

$$I_2 \leq C \int_D \frac{1}{(\delta_D(z))^{\tau - 1}}\ dz.$$

Now, using ([20], Lemma, page 726), we have $\int_D \frac{1}{(\delta_D(z))^{\tau-1}}\, dz < \infty$ for $\tau < 2$. Hence,

$$V\ \delta_\tau(x) \leq C\ \left(\ln(\frac{1}{R}) + \int_R^{\frac{1}{2}} \ln(\frac{1}{r})\ r\ dr + 1 \right) \leq\ C\ .$$

Case 2 : If $R \geq \frac{1}{4}$. For $x \in D$, we have

$$V\ \delta_\tau(x) \leq C\ \Big(\underbrace{\int_{(|z|\leq\frac{1}{8})} \ln(1 + \frac{\delta_D(x^*)\delta_D(z)}{|x^* - z|^2})\ \frac{1}{(\delta_D(z))^\tau}\, dz}_{J_1} + \underbrace{\int_{(\frac{1}{8}\leq|z|\leq 1)} \ln(1 + \frac{\delta_D(x^*)\delta_D(z)}{|x^* - z|^2})\ \frac{1}{(\delta_D(z))^\tau}\, dz}_{J_2} \Big).$$

For $z \in (|z| \leq \frac{1}{8})$, we have $|x^* - z| \geq |x^*| - |z| \geq \frac{1}{8}$ and $\frac{7}{8} \leq \delta_D(z) \leq 1$. It follows that

$$J_1 \leq\ C\ \delta_D(x^*) \int_0^{\frac{1}{8}} r\ dr \leq C\ \delta_D(x^*).$$

So using ([21], Proposition 2.10, page 290), we have

i) If $1 < \tau < 2$, then

$$V\ \delta_\tau(x) \leq C\ (\delta_D(x^*) + (\delta_D(x^*))^{2-\tau}) \leq C\ (\delta_D(x^*))^{2-\tau}\ .$$

ii) If $\tau = 1$, then

$$V\ \delta_\tau(x) \leq C\ \left(\delta_D(x^*) + \delta_D(x^*)\ \ln\left(\frac{e}{\delta_D(x^*)}\right)\right) \leq C\ \delta_D(x^*)\ \ln\left(\frac{e}{\delta_D(x^*)}\right)\ .$$

iii) If $0 < \tau < 1$, then

$$V\ \delta_\tau(x) \leq C\ (\delta_D(x^*) + \delta_D(x^*)) \leq C\ \delta_D(x^*)\ .$$

Thus, the result holds, from relation (3.4).

$\boxed{QED}$

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