Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
*Note Mat.* **40** (2020) no. 1, 45–55. doi:10.1285/i15900932v40n1p45

# On generalization of two theorems pertaining to integrability of cosine and sine trigonometric series

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Received: 16.4.2019; accepted: 8.2.2020.

**Abstract.** In this paper we have generalized two classes of sequences introduced previously by others. In addition, we have employed these two classes to study the integrability of cosine and sine trigonometric series.

**Keywords:** Quasi-convexity, Integrability, Fourier series, Sidon's class.

**MSC 2000 classification:** Primary 42A16, Secondary 42A32.

## 1 Introduction

The problem when the sum-functions of a cosine series

$$\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k \cos kx \tag{1.1}$$

and sine series

$$\sum_{k=1}^{\infty} a_k \sin kx, \tag{1.2}$$

represent integrable functions or are Fourier series, has a long history and still receives considerable attention.

Many researchers have studied integrability of both cosine and sine series, while only few of them have studied the integrability of sine series. All conditions for their integrability are established in terms of their coefficients and throughout this paper we assume that $a_k \to 0$ when $k \to \infty$. As a starting point, let $a_k$ be monotone decreasing coefficients, then condition $\sum_{k=1}^{\infty} \frac{a_k}{k} < \infty$ is a necessary and sufficient condition for integrability of sine series. For cosine series this

[i]This work is partially supported by...

condition is only sufficient but it is not a necessary one. Further, for the integrability problem of cosine series is used the convexity property of its coefficients $a_k$ (see [17]). If the coefficients $a_k$ are quasi-convex i.e. $\sum_{k=1}^{\infty} k|\Delta^2 a_k| < \infty$, where $\Delta^2 a_k := \Delta a_k - \Delta a_{k+1}$ and $\Delta a_k := a_k - a_{k+1}$, then cosine series is an integrable series (see [5]). A cosine series with quasi-convex coefficients is integrable if and only if its coefficients satisfy condition $\sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty$. This fact has been showed in [13]. The integrability conditions of cosine and sine series has been studied, later on, in more general cases, when coefficients belong to the class $\mathbf{S}$ of Sidon [9], which indeed has been reintroduced by Telyakovskii in [12] in a more appropriate way, and that is why it is usually called the Sidon–Telyakovskii class: A sequence $(a_k)$ belongs to the class $\mathbf{S}$ if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $A_k$ such that $\sum_{k=1}^{\infty} A_k < \infty$ and $|\Delta a_k| \leq A_k$ for all $k$. These integrability conditions for cosine and sine series has been studied almost thirty years later by Tomovski [14], except others, who generalized the Sidon–Telyakovskii class $\mathbf{S}$ in the following manner: A sequence $(a_k)$ belongs to the class $\mathbf{S}_r$, $r = 0, 1, 2, \dots$, if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $A_k$ such that $\sum_{k=1}^{\infty} k^r A_k < \infty$ and $|\Delta a_k| \leq A_k$ for all $k$. In that paper the author have extended of corresponding Sidon's and Telyakovskii's results. The interested reader, some other results on the topic under consideration, can be found in [16]. Two years later, Leindler [7] further generalized the class $\mathbf{S}_r$ and showed that his result and Telyakovskii's result imply Tomovski's result proved in [14]. We have to note here that some other authors tried to generalize further Sidon–Telyakovskii class $\mathbf{S}$, however the new defined classes were indeed equivalent classes with the class $\mathbf{S}$.

One should note that many questions regarding to Fourier series are studied by several authors (see as examples of recent studies of H. Bor [1]-[3], references therein, and a joint paper of the present author [4]).

For our further investigations on integrability of cosine and sine series we are going to recall only two of above mentioned results. One of them has been proved by R. Kano in [6] and the other one has been proved by S. A. Telyakovskii [10].

Namely,

**Theorem 1** ([6])**.** *If $(a_k)_{n=1}^{\infty}$ is a null sequence such that*

$$\sum_{k=1}^{\infty} k^2 \left|\Delta^2 \left(\frac{a_k}{k}\right)\right| < \infty, \tag{1.3}$$

*then the series (1.1) and (1.2) are Fourier series, or equivalently, they represent integrable functions.*

**Theorem 2** ([10])**.** *Let $b_k := \frac{a_k}{k}$; $k \in \{1, 2, \dots\}$, and $B_k := \max_{j \geq k} |\Delta b_j|$.*

*If*

$$\sum_{k=1}^{\infty} kB_k < \infty, \tag{1.4}$$

*then the series (1.2) converges for all $x$, its sum-function is itegrable one, and the inequality*

$$\int_0^{\pi} \left| \sum_{k=1}^{\infty} a_k \sin kx \right| dx \leq CkB_k,$$

*holds true, where $C$ is a positive constant.*

This paper has two objectives. The first one is to prove a more general theorem then Theorem 1, replacing condition (1.3) with condition

$$\sum_{k=1}^{\infty} k^{m+1} \left| \Delta^2 \left( \frac{a_k}{k^m} \right) \right| < \infty, \tag{1.5}$$

where $m \in \{1, 2, \dots\}$.

The second objective of is the following: Let $b_{k,m} := \frac{a_k}{k^m}$; $k \in \{1, 2, \dots\}$, $m \in \{1, 3, 5, \dots\}$, and $B_{k,m} := \max_{j \geq k} |\Delta b_{j,m}|$. We aim to prove Theorem 2, replacing condition (1.4) with condition

$$\sum_{k=1}^{\infty} k^m B_{k,m} < \infty.$$

We note that for $m = 1$ the conditions (1.3) and (1.5) are the same. Also we have to note that for $m = 1$ all conditions propose here are the same with those in Theorem 2.

To accomplish these objectives we need some auxiliary statements given in next section.

## 2 Helpful Lemmas

**Lemma 1** ([14])**.** *Let $m$ be a non-negative integer. Then for all $0 < |x| \leq \pi$ and all $n \geq 1$ the estimate $|\widetilde{D}_n^{(m)}(x)| \leq \frac{4n^m \pi}{|x|}$ holds true, where $\widetilde{D}_n^{(m)}(x)$ denotes $m$-th derivative of the conjugate Dirichlet kernel*

$$\widetilde{D}_k(x) = \sum_{j=1}^{k} \sin jx = \frac{\cos \frac{x}{2} - \cos \left(k + \frac{1}{2}\right) x}{2 \sin \frac{x}{2}}.$$

**Lemma 2** ([8]). *Let $m$ be a non-negative integer. Then for all $0<\varepsilon\le x\le\pi$ and all $n\ge 1$ the estimate $|D_n^{(m)}(x)|\le\frac{Cn^m}{x}$ holds true, where $C$ denotes a positive constant and $D_n^{(m)}(x)$ denotes $m$-th derivative of the Dirichlet kernel*

$$D_k(x)=\frac{1}{2}+\sum_{j=1}^{k}\cos jx=\frac{\sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}.$$

**Lemma 3** ([11]). *Let the real numbers $\alpha_i$, $i=1,2,\dots,k$, satisfy conditions $|\alpha_i|\le 1$. Then the following estimations hold true*

$$\int_0^{\pi}\left|\sum_{i=0}^{k}\alpha_i\frac{\sin\left(i+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\right|dx\le C(k+1),$$

*where $C$ is a positive constant.*

**Lemma 4** ([15]). *For all $m\in\{1,2,\dots\}$,*

$$\int_0^{\pi}\left|\widetilde{F}_k^{(m)}(x)\right|dx=\mathcal{O}(k^m),$$

*holds true, where*

$$\widetilde{F}_k(x)=\frac{1}{k+1}\sum_{j=1}^{k}\widetilde{D}_j(x).$$

## 3 Main Results

At first, we prove the following generalization of Theorem 1.

**Theorem 3.** *If $(a_k)_{n=1}^{\infty}$ is a null sequence such that*

$$\sum_{k=1}^{\infty}k^{m+1}\left|\Delta^2\left(\frac{a_k}{k^m}\right)\right|<\infty, \tag{3.1}$$

*then the series (1.1) and (1.2) are Fourier series, or equivalently, they represent integrable functions.*

*Proof.* Let us start first with sine series. Namely, let

$$S_n^s(x)=\sum_{k=1}^{n}a_k\sin kx$$

the partial sums of sine series and $b_{k,m}:=\frac{a_k}{k^m}$, where $n,m\in\{1,2,\dots\}$.

After some transformations we have found that

$$S_n^s(x) = \begin{cases} -\sum_{k=1}^{n} b_{k,m}(\cos kx)^{(m)}, & \text{if } m = 4p-3; \\ -\sum_{k=1}^{n} b_{k,m}(\sin kx)^{(m)}, & \text{if } m = 4p-2; \\ +\sum_{k=1}^{n} b_{k,m}(\cos kx)^{(m)}, & \text{if } m = 4p-1; \\ +\sum_{k=1}^{n} b_{k,m}(\sin kx)^{(m)}, & \text{if } m = 4p, \end{cases} \tag{3.2}$$

where $p \in \mathbb{N}$.

Applying Abel's transformation in (3.2), we get

$$S_n^s(x) = \begin{cases} -\sum_{k=1}^{n-1} \Delta b_{k,m} D_k^{(m)}(x) - b_{n,m} D_n^{(m)}(x), & \text{if } m = 4p-3; \\ -\sum_{k=1}^{n-1} \Delta b_{k,m} \widetilde{D}_k^{(m)}(x) - b_{n,m} \widetilde{D}_n^{(m)}(x), & \text{if } m = 4p-2; \\ +\sum_{k=1}^{n-1} \Delta b_{k,m} D_k^{(m)}(x) + b_{n,m} D_n^{(m)}(x), & \text{if } m = 4p-1; \\ +\sum_{k=1}^{n-1} \Delta b_{k,m} \widetilde{D}_k^{(m)}(x) + b_{n,m} \widetilde{D}_n^{(m)}(x), & \text{if } m = 4p, \end{cases} \tag{3.3}$$

for all $p \in \mathbb{N}$, where $D_n^{(m)}(x)$ and $\widetilde{D}_k^{(m)}(x)$ are the $m$-th derivative of Dirichlet's and conjugate Dirichlet's kernels, respectively.

Moreover, applying Abel's transformation in (3.3), with respect to $k$, we also get

$$S_n^s(x) = \begin{cases} -\sum_{k=1}^{n-2}(k+1)\Delta^2 b_{k,m} F_k^{(m)}(x) & \\ \qquad -n\Delta b_{n-1,m} F_{n-1}^{(m)}(x) - b_{n,m} D_n^{(m)}(x), & \text{if } m = 4p-3; \\ -\sum_{k=1}^{n-2}(k+1)\Delta^2 b_{k,m} \widetilde{F}_k^{(m)}(x) & \\ \qquad -n\Delta b_{n-1,m} \widetilde{F}_{n-1}^{(m)}(x) - b_{n,m} \widetilde{D}_n^{(m)}(x), & \text{if } m = 4p-2; \\ +\sum_{k=1}^{n-2}(k+1)\Delta^2 b_{k,m} F_k^{(m)}(x) & \\ \qquad +n\Delta b_{n-1,m} F_{n-1}^{(m)}(x) + b_{n,m} D_n^{(m)}(x), & \text{if } m = 4p-1; \\ +\sum_{k=1}^{n-2}(k+1)\Delta^2 b_{k,m} \widetilde{F}_k^{(m)}(x) & \\ \qquad +n\Delta b_{n-1,m} \widetilde{F}_{n-1}^{(m)}(x) + b_{n,m} \widetilde{D}_n^{(m)}(x), & \text{if } m = 4p, \end{cases} \tag{3.4}$$

for all $p \in \mathbb{N}$, where $F_n^{(m)}(x)$ and $\widetilde{F}_k^{(m)}(x)$ are the $m$-th derivative of Féjer's and conjugate Féjer's kernels, respectively.

By Lemma 1 and Lemma 2 we have

$$\begin{aligned} \left| n\Delta b_{n-1,m} K_{n-1}^{(m)}(x) \right| &\le \frac{Cn^{m+1}}{x} \left| \Delta b_{n-1,m} \right| \\ &\le \frac{Cn^{m+1}}{x} \left| \sum_{k=n-1}^{\infty} \Delta^2 \left(b_{k,m}\right) \right| \\ &\le \frac{C}{x} \sum_{k=n}^{\infty} k^{m+1} \left| \Delta^2 \left(b_{k,m}\right) \right| \to 0 \text{ as } n \to \infty, \end{aligned}$$

for all $x \neq 0(\mod 2\pi)$, where $K_{n-1}^{(m)}(x)$ denotes one of the Féjer's kernels $F_{n-1}^{(m)}(x)$ or $\widetilde{F}_{n-1}^{(m)}(x)$.

Also, by same Lemmas, we conclude that

$$\left|b_{n,m}\overline{D}_n^{(m)}(x)\right| \leq \frac{C}{x}|a_n| \to 0, \text{ as } n \to \infty,$$

for all $x \neq 0(\mod 2\pi)$, where $\overline{D}_n^{(m)}(x)$ denotes one of the Dirichlet's kernels $D_n^{(m)}(x)$ or $\widetilde{D}_n^{(m)}(x)$.

Whence from (3.4) the limit-function

$$g(x) = \lim_{n\to\infty} S_n^s(x) = \begin{cases} -\sum_{k=1}^{\infty}(k+1)\Delta^2 b_{k,m}F_k^{(m)}(x), & \text{if } m = 4p-3; \\ -\sum_{k=1}^{\infty}(k+1)\Delta^2 b_{k,m}\widetilde{F}_k^{(m)}(x), & \text{if } m = 4p-2; \\ +\sum_{k=1}^{\infty}(k+1)\Delta^2 b_{k,m}F_k^{(m)}(x), & \text{if } m = 4p-1; \\ +\sum_{k=1}^{\infty}(k+1)\Delta^2 b_{k,m}\widetilde{F}_k^{(m)}(x), & \text{if } m = 4p, \end{cases} \tag{3.5}$$

exists for all $x \neq 0(\mod 2\pi)$.

Now, let us show that $g \in L$. Namely, this is true since by Bernstein's inequality we have

$$\begin{aligned} \sum_{k=1}^{\infty}(k+1)\left|\Delta^2 b_{k,m}\right| \int_0^{\pi}\left|K_k^{(m)}(x)\right| dx &= \mathcal{O}\left(\sum_{k=1}^{\infty} k^{m+1}\left|\Delta^2 b_{k,m}\right| \int_0^{\pi} |K_k(x)|\, dx\right) \\ &= \mathcal{O}\left(\sum_{k=1}^{\infty} k^{m+1}\left|\Delta^2\left(\frac{a_k}{k^m}\right)\right|\right) < \infty, \end{aligned}$$

taking into account the well-known equality

$$\int_0^{\pi} F_k(x)dx = \pi,$$

and Lemma 4.

The proof of our theorem is implied now by generalized du Bois-Reymond theorem. The proof, pertaining to the integrability of cosine series, can be done with almost the same reasoning. For this reason, we skip this step entirely.

The proof is completed. $\boxed{QED}$

**Remark 1.** For $m = 1$, Theorem 3 reduces to Theorem 1.

Now we are ready to prove the following generalization of Theorem 2.

**Theorem 4.** *Let* $b_{k,m} := \frac{a_k}{k^m}$*;* $k \in \{1, 2, \dots\}$*,* $m \in \{1, 3, 5, \dots\}$*, and* $B_{k,m} := \max_{j \geq k} |\Delta b_{j,m}|$*. If*

$$\sum_{k=1}^{\infty} k^m B_{k,m} < \infty, \tag{3.6}$$

*then the series (1.2) converges for all* $x$*, its sum-function is itegrable one, and the inequality*

$$\int_0^{\pi} \left| \sum_{k=1}^{\infty} a_k \sin kx \right| dx \leq C k^m B_{k,m}, \tag{3.7}$$

*holds true, where* $C$ *is a positive constant.*

*Proof.* For the proof we shall use some parts of the proof of Theorem 3. Indeed, we have shown that for partial sums of the sine series hold

$$S_n^s(x) = \begin{cases} -\sum_{k=1}^{n-1} \Delta b_{k,m} D_k^{(m)}(x) - b_{n,m} D_n^{(m)}(x), & \text{if } m = 4p-3; \\ +\sum_{k=1}^{n-1} \Delta b_{k,m} D_k^{(m)}(x) + b_{n,m} D_n^{(m)}(x), & \text{if } m = 4p-1, \end{cases} \tag{3.8}$$

for all $p \in \mathbb{N}$.

Since by Lemma 2 $|D_n^{(m)}(x)| = \mathcal{O}\left(\frac{n^m}{x}\right)$ uniformly with respect to $x$ and $n$, the series

$$\sum_{k=1}^{\infty} k^m |\Delta b_{k,m}| \leq \sum_{k=1}^{\infty} k^m B_{k,m} < \infty,$$

by assumption, and $n^m b_{n,m} = a_n \to 0$ as $n \to \infty$, then the sine series converges.

Denoting its sum-function by $g(x)$, we write

$$g(x) = \begin{cases} -\sum_{k=1}^{\infty} \Delta b_{k,m} D_k^{(m)}(x), & \text{if } m = 4p-3; \\ +\sum_{k=1}^{\infty} \Delta b_{k,m} D_k^{(m)}(x), & \text{if } m = 4p-1; \end{cases} \tag{3.9}$$

for all $p \in \mathbb{N}$.

Let us prove (3.7). Under assumptions the number $B_{k,m}$ are monotone decreasing with respect to $k$, the series $\sum_{k=1}^{\infty} k^m B_{k,m} < \infty$, and that is why

$$k^{m+1} B_{k,m} \to 0 \quad \text{as} \quad n \to \infty,$$

for all $m \in \{1, 2, \dots\}$.

It is clear that (3.9) can be written as following

$$g(x) = \begin{cases} -\sum_{k=1}^{\infty} B_{k,m} \frac{\Delta b_{k,m}}{B_{k,m}} D_k^{(m)}(x), & \text{if } m = 4p-3; \\ +\sum_{k=1}^{\infty} B_{k,m} \frac{\Delta b_{k,m}}{B_{k,m}} D_k^{(m)}(x), & \text{if } m = 4p-1, \end{cases} \tag{3.10}$$

for all $p \in \mathbb{N}$.

Applying Abel's transformation in (3.10), with respect to $k$, we have

$$g(x)=\begin{cases}-\sum_{k=1}^{\infty}(B_{k,m}-B_{k+1,m})\sum_{j=1}^{k}\frac{\Delta b_{j,m}}{B_{j,m}}D_j^{(m)}(x), & \text{if } m=4p-3;\\ +\sum_{k=1}^{\infty}(B_{k,m}-B_{k+1,m})\sum_{j=1}^{k}\frac{\Delta b_{j,m}}{B_{j,m}}D_j^{(m)}(x), & \text{if } m=4p-1,\end{cases}\tag{3.11}$$

for all $p\in\mathbb{N}$.

From (3.11) we get

$$\int_0^{\pi}|g(x)|\,dx\le\sum_{k=1}^{\infty}(B_{k,m}-B_{k+1,m})\int_0^{\pi}\left|\sum_{j=1}^{k}\frac{\Delta b_{j,m}}{B_{j,m}}D_j^{(m)}(x)\right|dx,\tag{3.12}$$

if $m=4p-3$ or $m=4p-1$ for all $p\in\mathbb{N}$.

Using Bernsein's inequality in (3.12) we find that

$$\int_0^{\pi}|g(x)|\,dx\le\sum_{k=1}^{\infty}k^m(B_{k,m}-B_{k+1,m})\int_0^{\pi}\left|\sum_{j=1}^{k}\frac{\Delta b_{j,m}}{B_{j,m}}D_j(x)\right|dx,\tag{3.13}$$

if $m=4p-3$ or $m=4p-1$ for all $p\in\mathbb{N}$.

Subsequently, using Lemma 3 in (3.13) we have

$$\begin{aligned}\int_0^{\pi}|g(x)|\,dx &\le C\sum_{k=1}^{\infty}k^{m+1}(B_{k,m}-B_{k+1,m})\\ &= C\lim_{s\to\infty}\sum_{k=1}^{s}k^{m+1}(B_{k,m}-B_{k+1,m})\\ &= C\lim_{s\to\infty}\left\{\sum_{k=1}^{s}\left[k^{m+1}-(k-1)^{m+1}\right]B_{k,m}-s^{m+1}B_{s+1,m})\right\}\\ &= C\sum_{k=1}^{\infty}\left[k^{m+1}-(k-1)^{m+1}\right]B_{k,m}\\ &\le (m+1)C\sum_{k=1}^{\infty}k^mB_{k,m}<\infty,\end{aligned}$$

if $m=4p-3$ or $m=4p-1$ for all $p\in\mathbb{N}$.

The proof is completed. $\boxed{QED}$

**Remark 2.** For $m=1$, Theorem 4 reduces to Theorem 2.

We finalize this study with a proposition which gives some conditions under which condition (3.1) will be satisfied.

**Proposition 1.** *Condition (3.1) is satisfied if the conditions*

$$\sum_{k=1}^{\infty} k\left|\Delta^2 a_k\right| < \infty \quad \textit{and} \quad \sum_{k=1}^{\infty} k\left|2\left(\frac{k}{k+1}\right)^m - \left(\frac{k}{k+2}\right)^m - 1\right| |a_{k+2}| < \infty,$$

*are satisfied.*

*Proof.* First of all, under condition of our proposition, we have

$$\sum_{k=1}^{\infty} |\Delta a_k| \le \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} \left|\Delta^2 a_j\right| \le \sum_{j=1}^{\infty} j\left|\Delta^2 a_j\right| < \infty. \tag{3.14}$$

Then, we can write as follows

$$\begin{aligned} k^{m+1}\Delta^2\left(\frac{a_k}{k^m}\right) &= ka_k - 2k\left(\frac{k}{k+1}\right)^m a_{k+1} + k\left(\frac{k}{k+2}\right)^m a_{k+2} \\ &= k\Delta^2(a_k) + 2k\frac{(k+1)^m - k^m}{(k+1)^m}a_{k+1} + k\frac{k^m-(k+2)^m}{(k+2)^m}a_{k+2} \\ &= k\Delta^2(a_k) + 2k\frac{(k+1)^m - k^m}{(k+1)^m}\Delta a_{k+1} \\ &\quad + \left[2\frac{(k+1)^m - k^m}{(k+1)^m} - \frac{(k+2)^m - k^m}{(k+2)^m}\right] ka_{k+2} \\ &=: P_1(k) + P_2(k) + P_3(k). \end{aligned} \tag{3.15}$$

Since $(k+1)^m - k^m \le m(k+1)^{m-1}$, $m \ge 1$, then

$$|P_2| \le 2k\frac{m(k+1)^{m-1}}{(k+1)^m}|\Delta a_{k+1}| \le 2m|\Delta a_{k+1}|,$$

and using (3.14) we obtain

$$\sum_{k=1}^{\infty} |P_2(k)| \le 2m\sum_{k=1}^{\infty} |\Delta a_{k+1}| < \infty. \tag{3.16}$$

Also, by our assumptions we have

$$\begin{aligned} \sum_{k=1}^{\infty} |P_3(k)| &\le \sum_{k=1}^{\infty}\left|\frac{(k+2)^m - k^m}{(k+2)^m} - 2\frac{(k+1)^m - k^m}{(k+1)^m}\right| k|a_{k+2}| \\ &= \sum_{k=1}^{\infty}\left|2\left(\frac{k}{k+1}\right)^m - \left(\frac{k}{k+2}\right)^m - 1\right| k|a_{k+2}| < \infty. \end{aligned} \tag{3.17}$$

So, by (3.15), (3.16), and (3.17) we obtain

$$\sum_{k=1}^{\infty} k^{m+1}\left|\Delta^2\left(\frac{a_k}{k^m}\right)\right| < \infty, \quad m \in \{1, 2, \dots\}.$$

$\boxed{QED}$

**Remark 3.** For $m = 1$, Preposition 1 reduces to the sufficient part of Lemma 3 proved in [10].

**Remark 4.** For $m = 1$, the condition

$$\sum_{k=1}^{\infty} k\left|2\left(\frac{k}{k+1}\right)^m - \left(\frac{k}{k+2}\right)^m - 1\right| |a_{k+2}| < \infty,$$

ensures the well-known condition

$$\sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty.$$

However, the converse may not be true in general.

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