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A new subclass of meromorphic functions with positive coefficients defined by Bessel function

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Abstract. In this paper, we introduce and study a new subclass of meromorphic univalent functions defined by Bessel function. We obtain coefficient inequalities, extreme points, radius of starlikeness and convexity. Finally we obtain partial sums and neighborhood properties for the class $\sigma_p^*(\eta, k, \lambda, \tau, c)$.

Keywords: meromorphic; extreme point; partial sums; neighborhood.

MSC 2000 classification: 30C45 .

1 Introduction

Let Σ denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the punctured disc $U^* = \{z : z \in C \text{ and } 0 < |z| < 1\} = U \setminus \{0\}$.

A function $f \in \Sigma$ is said to be meromorphically starlike of order α if it satisfies

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha$$

for some $\alpha (0 \leq \alpha < 1)$ and for all $z \in U$. Further, a function $f \in \Sigma$ is said to be meromorphically convex of order α if it satisfies

$$Re\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \alpha,$$

for some $\alpha (0 \leq \alpha < 1)$ and for all $z \in U$.

Some subclasses of Σ were introduced and studied by Pommerenke [12], Miller [10], Mogra et al. [11], Cho [6], Cho et al. [7], Aouf [1, 2] and see also [17, 18].

For functions $f \in \Sigma$ given by (1.1) and $g \in \Sigma$ given by

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n,$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n.$$

We recall here the generalized Bessel function of first kind of order γ (see [8]), denoted by

$$W(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\Gamma(\gamma + n + \frac{b+1}{2})} \left(\frac{z}{2}\right)^{2n+\gamma} \ (z \in U)$$

(where Γ stands for the Gamma Euler function) which is the particular solution of the second order linear homogeneous differential equation (see, for details, [19])

$$z^2 w''(z) + bzw'(z) + [cz^2 - \gamma^2 + (1-b)\gamma]w(z) = 0$$

where $c, \gamma, b \in C$.

We introduce the function φ defined, in terms of the generalized Bessel function w by

$$\varphi(z) = 2^{\gamma}\Gamma\left(\gamma + \frac{b+1}{2}\right) z^{-\left(1+\frac{\gamma}{2}\right)} w(\sqrt{z}).$$

By using the well-known Pochhammer symbol $(x)_\mu$ defined, for $x \in C$ and in terms of the Euler gamma function, by

$$(x)_\mu = \frac{\Gamma(x+\mu)}{\Gamma(x)} = \begin{cases} 1, & (\mu = 0); \\ x(x+1)\cdots(x+n-1), & (\mu = n \in N = \{1,2,3\cdots\}). \end{cases}$$

We obtain the following series representation for the function $\varphi(z)$

$$\varphi(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(-c)^{n+1}}{4^{n+1}(n+1)!(\tau)_{n+1}} z^n$$
$$\left(\text{where } \tau = \gamma + \frac{b+1}{2} \notin Z_0^- = \{0,-1,-2,\cdots\}\right).$$

Corresponding to the function φ define the Bessel operator $S_\tau{}^c$ by the following Hadamard product

$$\begin{aligned} S_\tau{}^c f(z) = (\varphi * f)(z) &= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{\left(\frac{-c}{4}\right)^{n+1} a_n}{(n+1)!(\tau)_{n+1}} z^n \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \phi(n,\tau,c) a_n z^n, \end{aligned} \tag{1.2}$$

where $\phi(n,\tau,c) = \frac{\left(\frac{-c}{4}\right)^n}{(n)!(\tau)_n}$.

It easy to verify from the definition (1.2) that

$$z[S_{\tau+1}^{\ c} f(z)] = \tau S_\tau{}^c f(z) - (\tau+1) S_{\tau+1}^{\ c} f(z). \tag{1.3}$$

Motivated by Sivaprasad Kumar et al. [16], Atshan et al. [5] and Venkateswarlu et al. [17, 18], now we define a new subclass $\sigma_p^*(\eta,k,\lambda,\tau,\ c)$ of $\sum$.

Definition 1. For $0 \le \eta < 1, k \ge 0, 0 \le \lambda < \frac{1}{2}$, we let $\sigma_p^*(\eta,k,\lambda,\tau,\ c)$ be the subclass of $\sum_p$ consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\begin{aligned} -Re\left(\frac{z(S_\tau{}^c f(z))'}{S_\tau{}^c f(z)} + \lambda z^2 \frac{(S_\tau{}^c f(z))''}{S_\tau{}^c f(z)} + \eta\right) \\ > k\left|\frac{z(S_\tau{}^c f(z))'}{S_\tau{}^c f(z)} + \lambda z^2 \frac{(S_\tau{}^c f(z))''}{S_\tau{}^c f(z)} + 1\right|. \end{aligned} \tag{1.4}$$

In order to prove our results wee need the following lemmas [4].

Lemma 1. *If η is a real number and $\omega = -(u+iv)$ is a complex number then*

$$Re(\omega) \ge \eta \Leftrightarrow |\omega + (1-\eta)| - |\omega - (1-\eta)| \ge 0.$$

Lemma 2. *If $\omega = u + iv$ is a complex number and η is a real number then*

$$-Re(\omega) \geq k|\omega + 1| + \eta \Leftrightarrow -Re\left(\omega(1 + ke^{i\theta}) + ke^{i\theta}\right) \geq \eta, -\pi \leq \ \theta \leq \pi.$$

The main object of this paper is to study some usual properties of the geometric function theory such as the coefficient bounds, extreme points, radii of meromorphic starlikeness and convexity for the class $\sigma_p^*(\eta, k, \lambda, \tau,\ c)$. Further, we obtain partial sums and neighborhood properties for the class also.

2 Coefficient estimates

In this section we obtain necessary and sufficient condition for a function f to be in the class $\sigma_p^*(\eta, k, \lambda, \tau,\ c)$.

Theorem 1. *Let $f \in \sum$ be given by* (1.1)*. Then $f \in \sigma_p^*(\eta, k, \lambda, \tau,\ c)$ if and only if*

$$\sum_{n=1}^{\infty}[n(k+1)(1+(n-1)\lambda) + (k+\eta)]\phi(n, \tau, c)a_n \leq (1-\eta) - 2\lambda(1+k). \quad (2.1)$$

Proof. Let $f \in \sigma_p^*(\eta, k, \lambda, \tau,\ c)$. Then by Definition 1 and using Lemma 2, it is enough to show that

$$-Re\left\{\left(\frac{z(S_\tau{}^c f(z))'}{S_\tau{}^c f(z)} + \lambda z^2 \frac{(S_\tau{}^c f(z))''}{S_\tau{}^c f(z)}\right)(1 + ke^{i\theta}) + ke^{i\theta}\right\} > \eta, -\pi \leq \ \theta \leq \pi. \quad (2.2)$$

For convenience

$$C(z) = -\left[z(S_\tau{}^c f(z))' + \lambda z^2 (S_\tau{}^c f(z))''\right](1 + ke^{i\theta}) - ke^{i\theta} S_\tau{}^c f(z)$$
$$D(z) = S_\tau{}^c f(z)$$

that is, the equation (2.2) is equivalent to

$$-Re\left(\frac{C(z)}{D(z)}\right) \geq \eta.$$

In view of Lemma 1, we only need to prove that

$$|C(z) + (1-\eta)D(z)| - |C(z) - (1-\eta)D(z)| \geq 0.$$

Therefore

$$
\begin{aligned}
|C(z)+(1-\eta)D(z)| &\\
&\geq (2-\eta-2\lambda(k+1))\frac{1}{|z|}\\
&\quad -\sum_{n=1}^{\infty}[n(k+1)(1+(n-1)\lambda)+(k+\eta-1)]\phi(n,\tau,c)a_n|z|^n
\end{aligned}
$$

and

$$
\begin{aligned}
|C(z)-(1-\eta)D(z)| &\\
&\leq (\eta+2\lambda(k+1)))\frac{1}{|z|}\\
&\quad +\sum_{n=1}^{\infty}[n(k+1)(1+(n-1)\lambda)+(k+\eta+1)]\phi(n,\tau,c)a_n|z|^n.
\end{aligned}
$$

It is to show that

$$
\begin{aligned}
&|C(z)+(1-\eta)D(z)|-|C(z)-(1+\eta)D(z)|\\
&\geq(2(1-\eta)-4\lambda(k+1))\frac{1}{|z|}\\
&\quad -2\sum_{n=1}^{\infty}[n(k+1)(1+(n-1)\lambda)+(k+\eta)]\phi(n,\tau,c)a_n|z|^n\\
&\geq 0,\ \text{by the given condition (2.1).}
\end{aligned}
$$

Conversely, suppose $f\in\sigma_p^*(\eta,k,\lambda,\tau,\ c)$. Then by Lemma 1, we have (2.2).

Choosing the values of z on the positive real axis the inequality (2.2) reduces to

$$
Re\left\{\frac{[1-\eta-2\lambda(1+ke^{i\theta})]\frac{1}{z^2}+\sum\limits_{n=1}^{\infty}[n(1+(n-1)\lambda)(1+ke^{i\theta})+(\eta+ke^{i\theta})]\phi(n,\tau,c)z^{n-1}}{\frac{1}{z^2}+\sum\limits_{n=1}^{\infty}\phi(n,\tau,c)a_nz^{n-1}}\right\}\geq 0.
$$

Since $Re(-e^{i\theta})\geq-|e^{i\theta}|=-1$, the above inequality reduces to

$$
Re\left\{\frac{[1-\eta-2\lambda(1+k)]\frac{1}{r^2}+\sum\limits_{n=1}^{\infty}[n(1+k)(1+(n-1)\lambda)+(\eta+k)]\phi(n,\tau,c)a_nr^{n-1}}{\frac{1}{r^2}+\sum\limits_{n=1}^{\infty}\phi(n,\tau,c)r^{n-1}}\right\}\geq 0.
$$

Letting $r \to 1^-$ and by the mean value theorem, we have obtained the inequality (2.1). $\boxed{QED}$

Corollary 1. *If* $f \in \sigma_p^*(\eta, k, \lambda, \tau,\ c)$ *then*

$$a_n \leq \frac{(1-\eta) - 2\lambda(k+1)}{[n(1+k)(1+(n-1)\lambda) + (\eta+k)]\phi(n,\tau,c)}. \tag{2.3}$$

By taking $\lambda = 0$ in Theorem 1, we get the following corollary.

Corollary 2. *If* $f \in \sigma_p^*(\eta, k, \lambda, \tau,\ c)$ *then*

$$a_n \leq \frac{2-\eta}{[n(1+k) + (\eta+k)]\phi(n,\tau,c)}. \tag{2.4}$$

Theorem 2. *If* $f \in \sigma_p^*(\eta, k, \lambda, \tau,\ c)$ *then for* $0 < |z| = r < 1$,

$$\frac{1}{r} - \frac{(1-\eta) - 2\lambda(k+1)}{(2k+\eta+1)\phi(1,\tau,c)}\ r \leq |f(z)| \leq \frac{1}{r} + \frac{(1-\eta) - 2\lambda(k+1)}{(2k+\eta+1)\phi(1,\tau,c)} r. \tag{2.5}$$

This result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{(1-\eta) - 2\lambda(k+1)}{(2k+\eta+1)\phi(1,\tau,c)}\ z. \tag{2.6}$$

Proof. Since $f(z) = \frac{1}{z} + \sum\limits_{n=1}^{\infty} a_n z^n$, we have

$$|f(z)| = \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \leq \frac{1}{r} + r\sum_{n=1}^{\infty} a_n. \tag{2.7}$$

Since $n \geq 1,\ (2k+\eta+1) \leq n(k+1)(k+\eta)\phi(n,\tau,c)$, using Theorem 1, we have

$$\begin{aligned}
(2k+\eta+1)\sum_{n=1}^{\infty} a_n &\leq \sum_{n=1}^{\infty} n(k+1)(k+\eta)\phi(n,\tau,c) \\
&\leq (1-\eta) - 2\lambda(k+1) \\
\Rightarrow \sum_{n=1}^{\infty} a_n &\leq \frac{(1-\eta) - 2\lambda(k+1)}{(2k+\eta+1)\phi(1,\tau,c)}.
\end{aligned}$$

Using the above inequality in (2.7), we have

$$|f(z)| \leq \frac{1}{r} + \frac{(1-\eta) - 2\lambda(k+1)}{(2k+\eta+1)\phi(1,\tau,c)} r$$

$$\text{and}\quad |f(z)| \geq \frac{1}{r} - \frac{(1-\eta) - 2\lambda(k+1)}{(2k+\eta+1)\phi(1,\tau,c)} r.$$

The result is sharp for the function $f(z) = \frac{1}{z} + \frac{(1-\eta)-2\lambda(k+1)}{(2k+\eta+1)\phi(1,\tau,c)} z$. $\boxed{QED}$

Corollary 3. *If* $f \in \sigma_p^*(\eta, k, \lambda, \tau,\ c)$ *then*

$$\frac{1}{r^2} - \frac{(1-\eta) - 2\lambda(k+1)}{(2k+\eta+1)\phi(1,\tau,c)} \le |f'(z)| \le \frac{1}{r^2} + \frac{(1-\eta) - 2\lambda(k+1)}{(2k+\eta+1)\phi(1,\tau,c)}.$$

The result is sharp for the function given by (2.6).

3 Extreme points

Theorem 3. *Let* $f_0(z) = \frac{1}{z}$ *and*

$$f_n(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(1-\eta) - 2\lambda(k+1)}{[n(1+k)(1+(n-1)\lambda) + (\lambda+k)]\phi(n,\tau,c)} z^n, \quad n \ge 1. \quad (3.1)$$

Then $f \in \sigma_p^*(\eta, k, \lambda, \tau,\ c)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=0}^{\infty} u_n f_n(z), u_n \ge 0 \text{ and } \sum_{n=1}^{\infty} u_n = 1. \quad (3.2)$$

Proof. Suppose $f(z)$ can be expressed as in (3.2). Then

$$\begin{aligned} f(z) =& \sum_{n=0}^{\infty} u_n f_n(z) = u_0 f_0(z) + \sum_{n=1}^{\infty} u_n f_n(z) \\ =& \frac{1}{z} + \sum_{n=1}^{\infty} u_n \frac{(1-\eta) - 2\lambda(k+1)}{[n(1+k)(1+(n-1)\lambda) + (\lambda+k)]\phi(n,\tau,c)} z^n. \end{aligned}$$

Therefore

$$\begin{aligned} &\sum_{n=1}^{\infty} u_n \frac{\{(1-\eta) - 2\lambda(k+1)\}\{[n(1+k)(1+(n-1)\lambda) + (\lambda+k)]\phi(n,\tau,c)\}}{\{[n(1+k)(1+(n-1)\lambda) + (\lambda+k)]\phi(n,\tau,c)\}\{(1-\eta) - 2\lambda(k+1)\}} \\ &= \sum_{n=1}^{\infty} u_n = 1 - u_0 \le 1. \end{aligned}$$

So by Theorem 1, $f \in \sigma_p^*(\eta, k, \lambda, \tau,\ c)$.

Conversely, suppose that $f \in \sigma_p^*(\eta, k, \lambda, \tau,\ c)$. Since

$$a_n \le \frac{(1-\eta) - 2\lambda(k+1)}{[n(1+k)(1+(n-1)\lambda) + (\lambda+k)]\phi(n,\tau,c)} \; n \ge 1.$$

We set $u_n = \frac{[n(1+k)(1+(n-1)\lambda)+(\eta+k)]\phi(n,\tau,c)}{(1-\eta)-2\lambda(k+1)} a_n,\ n \ge 1$ and $u_0 = 1 - \sum_{n=1}^{\infty} u_n$.

Then we have $f(z) = \sum_{n=0}^{\infty} u_n f_n(z) = u_0 f_0(z) + \sum_{n=1}^{\infty} u_n f_n(z)$.

Hence the result follows. $\boxed{QED}$

4 Radii of meromorphically starlike and meromorphically convexity

Theorem 4. *Let $f \in \sigma_p^*(\eta, k, \lambda, \tau,\ c)$. Then f is meromorphically starlike of order δ, $(0 \le \delta \le 1)$ in the unit disc $|z| < r_1$, where*

$$r_1 = \inf_n \left[\frac{(1-\delta)}{(n+2-\delta)} \frac{[n(1+k)(1+(n-1)\lambda) + (\lambda+k)]\phi(n,\tau,c)}{(1-\eta) - 2\lambda(k+1)} \right]^{\frac{1}{n+1}}, \quad n \ge 1.$$

The result is sharp for the extremal function $f(z)$ given by (2.6).

Proof. The function $f \in \sigma_p^*(\eta, k, \lambda, \tau,\ c)$ of the form (1.1) is meromorphically starlike of order δ in the disc $|z| < r_1$ if and only if it satisfies the condition

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| < (1-\delta). \tag{4.1}$$

Since

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \le \left| \frac{\sum\limits_{n=1}^{\infty} (n+1) a_n z^{n+1}}{1 + \sum\limits_{n=1}^{\infty} a_n z^{n+1}} \right| \le \frac{\sum\limits_{n=1}^{\infty} (n+1)|a_n||z|^{n+1}}{1 - \sum\limits_{n=1}^{\infty} |a_n||z|^{n+1}}.$$

The above expression is less than $(1-\delta)$ if $\sum\limits_{n=1}^{\infty} \frac{(n+2-\delta)}{(1-\delta)} a_n |z|^{n+1} < 1$.
Using the fact that $f(z) \in \sigma_p^*(\eta, k, \lambda, \tau,\ c)$ if and only if

$$\sum_{n=1}^{\infty} \frac{[n(1+k)(1+(n-1)\lambda) + (\lambda+k)]\phi(n,\tau,c)}{(1-\eta) - 2\lambda(k+1)} a_n \le 1.$$

Thus, (4.1) will be true if $\frac{(n+2-\delta)}{(1-\delta)} |z|^{n+1} < \frac{[n(1+k)(1+(n-1)\lambda)+(\eta+k)]\phi(n,\tau,c)}{(1-\eta)-2\lambda(k+1)}$
or equivalently $|z|^{n+1} < \frac{(1-\delta)}{(n+2-\delta)} \frac{[n(1+k)(1+(n-1)\lambda)+(\eta+k)]\phi(n,\tau,c)}{(1-\eta)-2\lambda(k+1)}$
which yields the starlikeness of the family. QED

The proof of the following theorem is analogous to that of Theorem 4, hence we omit the proof.

Theorem 5. *Let $f \in \sigma_p^*(\eta, k, \lambda, \tau,\ c)$. Then f is meromorphically convex of order δ, $(0 \le \delta \le 1)$ in the unit disc $|z| < r_2$, where*

$$r_2 = \inf_n \left[\frac{(1-\delta)}{n(n+2-\delta)} \frac{[n(1+k)(1+(n-1)\lambda) + (\eta+k)]\phi(n,\tau,c)}{(1-\eta) - 2\lambda(k+1)} \right]^{\frac{1}{n+1}}, \quad n \ge 1.$$

The result is sharp for the extremal function $f(z)$ given by (2.6).

5 Partial Sums

Let $f \in \sum$ be a function of the form (1.1). Motivated by Silverman [14] and Silvia [15] and also see [3], we define the partial sums f_m defined by

$$f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} a_n z^n, (m \in N). \tag{5.1}$$

In this section we consider partial sums of function from the class $\sigma_p^*(\eta, k, \lambda, \tau, c)$ and obtain sharp lower bounds for the real part of the ratios of f to f_m and f' to f'_m.

Theorem 6. *Let $f \in \sigma_p^*(\eta, k, \lambda, \tau,\ c)$ be given by* (1.1) *and define the partial sums $f_1(z)$ and $f_m(z)$ by*

$$f_1(z) = \frac{1}{z} \text{ and } f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} |a_n| z^n, (m \in N \setminus \{1\}). \tag{5.2}$$

Suppose also that $\sum\limits_{n=1}^{\infty} d_n |a_n| \leq 1,$ *where*

$$d_n \geq \begin{cases} 1\ , & \text{if } n = 1, 2, \cdots, m \\ \frac{[n(1+k)(1+(n-1)\lambda)+(\eta+k)]\phi(n,\tau,c)}{(1-\eta)-2\lambda(k+1)}, & \text{if } n = m+1, m+2, \cdots \end{cases}. \tag{5.3}$$

Then $f \in \sigma_p^*(\eta, k, \lambda, \tau,\ c)$. Furthermore

$$Re\left(\frac{f(z)}{f_m(z)}\right) > 1 - \frac{1}{d_{m+1}} \tag{5.4}$$

$$\text{and } \ Re\left(\frac{f_m(z)}{f(z)}\right) > \frac{d_{m+1}}{1 + d_{m+1}}. \tag{5.5}$$

Proof. For the coefficient d_n given by (5.3) it is not difficult to verify that

$$d_{m+1} > d_m > 1. \tag{5.6}$$

Therefore we have

$$\sum_{n=1}^{m} |a_n| + d_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} |a_n| d_m \leq 1 \tag{5.7}$$

by using the hypothesis (5.3). By setting

$$g_1(z) = d_{m+1}\left(\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_{m+1}}\right)\right) = 1 + \frac{d_{m+1} \sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=1}^{\infty} |a_n| z^{n-1}}$$

then it sufficient to show that

$$Re\big(g_1(z)\big) \geq 0, (z \in U) \quad \text{or} \quad \left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq 1, \ (z \in U)$$

and applying (5.7), we find that

$$\begin{aligned}\left|\frac{g_1(z)-1}{g_1(z)+1}\right| &\leq \frac{d_{m+1}\sum\limits_{n=m+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{m}|a_n| - d_{m+1}\sum\limits_{n=m+1}^{\infty}|a_n|}\\ &\leq 1, \quad (z \in U)\end{aligned}$$

which ready yields the assertion (5.4) of Theorem 6. In order to see that

$$f(z) = \frac{1}{z} + \frac{z^{m+1}}{d_{m+1}} \tag{5.8}$$

gives sharp result, we observe that for

$$z = re^{\frac{i\pi}{m}} \text{ that } \frac{f(z)}{f_m(z)} = 1 - \frac{r^{m+2}}{d_{m+1}} \to 1 - \frac{1}{d_{m+1}} \quad \text{as} \quad r \to 1^-.$$

Similarly, if we takes $g_2(z) = (1+d_{m+1})\left(\frac{f_m(z)}{f(z)} - \frac{d_{m+1}}{1+d_{m+1}}\right)$
and making use of (5.7), we denote that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| < \frac{(1+d_{m+1})\sum\limits_{n=m+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{m}|a_n| - (1-d_{m+1})\sum\limits_{n=m+1}^{\infty}|a_n|}$$

which leads us immediately to the assertion (5.5) of Theorem 6.
The bound in (5.5) is sharp for each $m \in N$ with extremal function $f(z)$ given by (5.8) . $\boxed{QED}$

The proof of the following theorem is analogous to that of Theorem 6, so we omit the proof.

Theorem 7. *If* $f \in \sigma_p^*(\eta, k, \lambda, \tau,\ c)$ *be given by* (1.1) *and satisfies the condition* (2.1) *then*

$$\begin{aligned} Re\left(\frac{f'(z)}{f'_m(z)}\right) &> 1 - \frac{m+1}{d_{m+1}}\\ \textit{and} \quad Re\left(\frac{f'_m(z)}{f'(z)}\right) &> \frac{d_{m+1}}{m+1+d_{m+1}},\end{aligned}$$

$$where\ \ d_n \geq \begin{cases} n\ , & if\ n = 2,3,\cdots,m \\ \frac{[n(1+k)(1+(n-1)\lambda)+(\eta+k)]\phi(n,\tau,c)}{(1-\eta)-2\lambda(k+1)}, & if\ n = m+1, m+2, \cdots \end{cases}.$$

The bounds are sharp with the extremal function $f(z)$ *of the form* (2.6).

6 Neighbourhoods for the class $\sigma_p^{*\xi}(\eta, k, \lambda, \tau,\ c)$

In this section, we determine the neighborhood for the class $\sigma_p^{*\xi}(\eta, k, \lambda, \tau,\ c)$ which we define as follows:

Definition 2. A function $f \in \sum$ is said to be in the class $\sigma_p^{*\xi}(\eta, k, \lambda, \tau,\ c)$ if there exits a function $g \in \sigma_p^{*}(\eta, k, \lambda, \tau,\ c)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \xi, \quad (z \in E, 0 \leq \xi < 1). \tag{6.1}$$

Following the earlier works on neighbourhoods of analytic functions by Goodman [9] and Ruscheweyh [13], we define the $\delta-$neighbourhoods of function $f \in \sum$ by

$$N_\delta(f) = \Big\{g \in \sum : g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \ \ and \ \ \sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta\Big\}. \tag{6.2}$$

Theorem 8. *If* $g \in \sigma_p^{*}(\eta, k, \lambda, \tau,\ c)$ *and*

$$\xi = 1 - \frac{\delta(2k+\eta+1)\phi(1,\tau,c)}{(2k+\eta+1)\phi(1,\tau,c) - (1-\eta) + 2\lambda(k+1)} \tag{6.3}$$

then $\quad N_\delta(g) \subset \sigma_p^{*\xi}(\eta, k, \lambda, \tau,\ c)$.

Proof. Let $f \in N_\delta(g)$. Then we find from (6.2) that

$$\sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta \tag{6.4}$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta \ \ (n \in N). \tag{6.5}$$

Since $g \in \sigma_p^{*}(\eta, k, \lambda, \tau,\ c)$, we have

$$\sum_{n=1}^{\infty} b_n \leq \frac{(1-\eta) - 2\lambda(k+1)}{(2k+\eta+1)\phi(1,c,\tau)}. \tag{6.6}$$

So that

$$\begin{aligned}\left|\frac{f(z)}{g(z)}-1\right| &< \frac{\sum\limits_{n=1}^{\infty}|a_n-b_n|}{1-\sum\limits_{n=1}^{\infty}b_n}\\ &= \frac{\delta(2k+\eta+1)L(1,c,\tau)}{(2k+\eta+1)\phi(1,\tau,c)-(1-\eta)+2\lambda(k+1)}\\ &= 1-\xi\end{aligned}$$

provided ξ is given by (6.3). Hence by definition, $f \in \sigma_p^{*\xi}(\eta, k, \lambda, \tau,\ c)$ for ξ given by which completes the proof. $\boxed{QED}$

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