Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **39** (2019) no. 2, 111–119. doi:10.1285/i15900932v39n2p111

Lacunary Statistically ϕ- Convergence

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Received: 17.10.2019; accepted: 11.12.2019.

Abstract. In this paper by using the lacunary sequence and Orlicz function ϕ, we introduce a new concept of lacunary statistically ϕ-convergence, as a generalization of the statistically $\phi-$convergence and $\phi-$convergence. Based on this concepts, introduce a new sequence space $S_\theta - \phi$ and investigate some of its basic properties. Also studied some inclusion relations.

Keywords: Orlicz function, lacunary sequence, statistical convergence, $\phi-$convergence.

MSC 2000 classification: 40A05, 40C05, 40D25.

1 Introduction

The idea of *convergence* of a real sequence was extended to *statistical convergence* by Fast [5] (see also Steinhaus [18]) as follows:

A real number sequence $x = (x_n)$ is said to be *statistically convergent* to the number L if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \mid \{k \leq n :\mid x_k - L \mid \geq \varepsilon\} \mid = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. L is called the statistical limit of the sequence (x_n) and we write $S - \lim, x = L$ or $x_k \to L(S)$. We shall also use S to denote the set of all statistically convergent sequences. Statistical convergence turned out to be one of the most active areas of research in summability theory after the works of Fridy [9], Salat [15]. Some applications of statistical convergence in number theory and mathematical analysis can be found in [1, 2, 8, 11, 12, 13, 16, 17, 19]. There is a natural relationship [3] between statistical convergence and strong Cesàro summability:

$$\mid \sigma_1 \mid = \{x = (x_n) : \text{for some } L, \underset{n}{lim} \left(\frac{1}{n}\sum_{k=1}^{n} \mid x_k - L \mid\right) = 0\}.$$

In another direction, a new type of convergence called *lacunary statistical convergence* was introduced in [10] as follows (for details one may refer [4]):

A *lacunary sequence* is an increasing integer sequence $\theta = (k_r)_{r\in N\cup\{0\}}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. Let $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$.

A real number sequence $x = (x_n)$ is said to be *lacunary statistically convergent* to the number L if for each $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} \mid \{k \in I_r :\mid x_k - L \mid\geq \varepsilon\} \mid= 0;$$

L is called the lacunary statistical limit of the sequence (x_n) and we write $S_\theta - \lim x = L$ or $x_k \to L(S_\theta)$. We shall also use S_θ to denote the set of all lacunary statistically convergent sequences with respect to the lacunary sequence θ. The relation between lacunary statistical convergence and statistical convergence was established among other related things in[10]. There is a strong connection between $\mid \sigma_1 \mid$ and the sequence space N_θ [6], which is defined as

$$N_\theta = \{x = (x_n) : \text{for some } L, \lim_r (\frac{1}{h_r} \sum_{k\in I_r} \mid x_k - L \mid) = 0\}.$$

In the literature, statistical convergence of any real sequence is defined relatively to absolute value. While, we know that the absolute value of real numbers is a special case of an *Orlicz function* [14] i.e. a function $\phi : \mathbb{R} \to \mathbb{R}$ such that it is even, non-decreasing on $\mathbb{R}^+$, continuous on $\mathbb{R}$, and satisfying

$$\phi(x) = 0 \Longleftrightarrow x = 0 \text{ and } \phi(x) \to \infty \text{ as } x \to \infty.$$

Rao and Ren [14] describe the important roles and applications that Orlicz functions have in many areas such as economics, stochastic problems etc.

An Orlicz function $\phi : \mathbb{R} \to \mathbb{R}$ is said to satisfy the $\triangle_2$ condition, if there exists an $M > 0$ such that $\phi(2x) \leq M .\phi(x)$, for every $x \in \mathbb{R}^+$.

Example 1. (i) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined $\phi(x) =\mid x \mid$ is an Orlicz function.

(ii) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = x^3$ is not an Orlicz function.

(iii) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = x^2$ is an Orlicz function satisfying the $\triangle_2$ condition.

(iv) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = e^{|x|} - \mid x \mid -1$ is an Orlicz function not satisfying the $\triangle_2$ condition.

In this paper by using the lacunary sequence θ and Orlicz function ϕ, we introduce a new concept of *lacunary statistically $\phi-$convergence*, as a generalization of the statistically convergence [5] and lacunary statistically convergence[10] and based on this concepts, introduce a new sequence space $S_\theta - \phi$. We investigate some of its basic properties. Also we study some inclusion relations.

2 Definitions and Preliminaries

Definition 1. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A sequence $x = (x_n)$ is said to be $\phi-$*convergent* to L if $\lim_n \phi(x_n - L) = 0$.

In this case, L is called the $\phi-$limit of (x_n) and denoted by $\phi - \lim x = L$.

Note 1. If we take $\phi(x) =| x |$, then $\phi-$convergent concepts coincide with usual convergence. Also it is easy to check, if $x = (x_n)$ is $\phi-$convergent to L, then any of its subsequence is $\phi-$convergent to L as well.

Definition 2. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A sequence $x = (x_n)$ is said to be *statistically* $\phi-$*convergent* to L if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \mid \{k \leq n : \phi(x_k - L) \geq \varepsilon\} \mid = 0.$$

L is called the statistical $\phi-$ limit of the sequence (x_n) and we write $S - \phi \ \lim x = L$ or $x_k \to L(S - \phi)$. We shall also use $S - \phi$ to denote the set of all statistically $\phi-$convergent sequences.

Note 2. If we take $\phi(x) =| x |$, then $S - \phi$ convergence concepts coincide with statistically convergence.

Definition 3. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. We define new sequence spaces $| \sigma_1 |_\phi$ and $N_\theta - \phi$ are as follows:

$$| \sigma_1 |_\phi = \{x = (x_n) : \text{for some } L, \lim_n (\tfrac{1}{n} \textstyle\sum_{k=1}^{n} \phi(x_k - L)) = 0\},$$

$$N_\theta - \phi = \{x = (x_n) : \text{for some } L, \lim_r (\tfrac{1}{h_r} \sum_{k \in I_r} \phi(x_k - L)) = 0\}.$$

Note 3. If we take $\phi(x) =| x |$, then the spaces $| \sigma_1 |_\phi$ and $N_\theta - \phi$ coincides with $| \sigma_1 |$ and N_θ respectively.

3 Main Results

Definition 4. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and θ be a lacunary sequence. A sequence $x = (x_n)$ is said to be *lacunary statistically* $\phi-$*convergent* to L if for each $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} \mid \{k \in I_r : \phi(x_k - L) \geq \varepsilon\} \mid = 0.$$

In this case, L is called the lacunary statistical $\phi-$ limit of the sequence (x_n) and we write $S_\theta - \phi \lim x = L$ or $x_k \to L(S_\theta - \phi)$. We shall also use $S_\theta - \phi$ to denote the set of all lacunary statistically $\phi-$convergent sequences.

Note 4. If we take $\phi(x) =| x |$, then $S_\theta - \phi$ convergence coincide with $S_\theta-$convergence; which was studied by Fridy and Orhan [10]. Thus $S_\theta - \phi$ convergence is a generalization of $S_\theta-$convergence.

Example 2. Let $\phi(x) = x^2$ and $\theta = (2^r)$. It is obvious that ϕ, satisfies the $\triangle_2$ condition. Let us consider the sequence (x_n), defined by

$$x_n = \begin{cases} \sqrt{n}, & n = k^2, k \in N \\ \frac{1}{\sqrt{n}}, & otherwise \end{cases}$$

then the sequence (x_n) is $S_\theta - \phi$ convergent to 0, although (x_n) is not convergent.

Justification: We have

$$\lim_r \frac{1}{h_r} \mid \{k \in I_r : \phi(x_k - L) \geq \varepsilon\} \mid = \lim_r \frac{1}{2^{r-1}} \mid \left\{k \in (2^{r-1}, 2^r] : \phi(x_k - 0) \geq \varepsilon\right\} \mid$$

$$= 2 \lim_r \frac{1}{2^r} \mid \left\{k \in (2^{r-1}, 2^r] : x_k^2 \geq \varepsilon\right\} \mid \leq 2 \lim_r \frac{1}{2^r} \mid \left\{k \leq 2^r : x_k^2 \geq \varepsilon\right\} \mid$$

$$= 2 \lim_r \frac{1}{n} \mid \left\{k \leq n : x_k^2 \geq \varepsilon\right\} \mid = 0.$$

This shows that the sequence (x_n) is $S_\theta - \phi$ convergent to 0, although (x_n) is not convergent.

Example 3. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function with $\phi(x) =| x |$, θ be any lacunary sequence, then the sequence (x_n) defined by $x_n = n^2$, for every $n \in N$ is not $S_\theta - \phi$ convergent.

Justification: Take any $x \in \mathbb{R}$. Then $x \leq 0$ or $x > 0$. If $x \leq 0$, choose $\epsilon = \frac{1}{2}$, then for every $n \in N$,

$$K(\varepsilon) = \{k \in I_r :\mid x_k - x \mid \geq \varepsilon\} = I_r.$$

Therefore for $x \leq 0$, $\lim_r \frac{1}{h_r} \mid \{k \in I_r :\mid x_k - x \mid \geq \varepsilon\} \mid = \lim_r \frac{1}{h_r} \mid I_r \mid = 1$.

If $x > 0$, then there exists $n_0 \in N$ such that $x_{n_0-1} \leq x \leq x_{n_0}$.

In this case, if $x < 1$, by taking $\epsilon = \frac{1}{2} \min\{x, 1 - x\}$, we get

$$K(\varepsilon) = \{k \in I_r :\mid x_k - x \mid \geq \varepsilon\} = I_r.$$

Again, if $x \geq 1$, by taking $\epsilon = \frac{1}{2} \min\{x - x_{n_0-1}, x_{n_0} - x\}$, we get

$$K(\varepsilon) = \{k \in I_r :\mid x_k - x \mid \geq \varepsilon\} = I_r.$$

Thus for $x > 0$, $\lim_r \frac{1}{h_r} \mid \{k \in I_r :\mid x_k - x \mid \geq \varepsilon\} \mid = \lim_r \frac{1}{h_r} \mid I_r \mid = 1$.

Hence the result.

Definition 5. A sequence $x = (x_n)$ is said to be $\phi-$ *bounded* with respect to the Orlicz function ϕ, if there exists $M > 0$ such that $\phi(x_n) \leq M$, for every $n \in \mathbb{N}$.

In the following theorem we give some inclusion relations between the spaces $N_\theta - \phi$ and $S_\theta - \phi$ and show that they are equivalent for $\phi-$ bounded sequences.

Theorem 1. *Let $\theta = (k_r)$ be a lacunary sequence, then*
(i) $x_k \to L(N_\theta - \phi)$ implies $x_k \to L(S_\theta - \phi)$, and reverse is not true.
(ii) If x is $\phi-$ bounded and $x_k \to L(S_\theta - \phi)$ then $x_k \to L(N_\theta - \phi)$.

Proof. (i) If $\varepsilon > 0$ and $x_k \to L(N_\theta - \phi)$, we may write

$$\sum_{k \in I_r} \phi(x_k - L) \geq \sum_{\substack{k \in I_r \\ \phi(x_k - L) \geq \varepsilon}} \phi(x_k - L) \geq \varepsilon \mid \{k \in I_r : \phi(x_k - L) \geq \varepsilon\} \mid$$

from which the first result follows.

In order to establish the 2nd part, we will construct a sequence which is in $S_\theta - \phi$ but not in $N_\theta - \phi$. For this, let $\phi(x) = \mid x \mid$, proceeding as in [10], page-45, θ be given and define x_k to be 1,2,.......,$[\sqrt{h_r}]$ at the first $[\sqrt{h_r}]$ integers in I_r and $x_k = 0$, otherwise. Note that x is not bounded. It was shown in [10] that $x_k \to 0(S_\theta)$, but x_k is not convergent to $0(N_\theta)$. By Note 3 and 4, we conclude that $x_k \to 0(S_\theta - \phi)$ but x_k is not convergent to $0(N_\theta - \phi)$. Hence we may write $(N_\theta - \phi) \subseteq (S_\theta - \phi)$.

(ii) Let $x_k \to L(S_\theta - \phi)$ and x is $\phi-$ bounded, i.e $\phi(x_k) \leq M$ for every $k \in N$. Given $\varepsilon > 0$, we get

$$\frac{1}{h_r} \sum_{k \in I_r} \phi(x_k - L) = \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \phi(x_k - L) \geq \varepsilon}} \phi(x_k - L) + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \phi(x_k - L) < \varepsilon}} \phi(x_k - L)$$

$$\leq \frac{M + \phi(L)}{h_r} \mid \{k \in I_r : \phi(x_k - L) \geq \varepsilon\} \mid + \varepsilon,$$ which yields the result.

QED

Note 5. As a consequence of the result (i) and (ii) of the above theorem, we can conclude that, if x is $\phi-$ bounded then $S_\theta - \phi = N_\theta - \phi$.

In the following lemmas we study the inclusions $S - \phi \subseteq S_\theta - \phi$ and $S_\theta - \phi \subseteq S - \phi$ under certain restrictions on $\theta = (k_r)$.

Lemma 1. *For any lacunary sequence θ and any Orlicz function ϕ, $S - \phi \lim x = L$ implies $S_\theta - \phi \lim x = L$ if and only if* $\liminf_r q_r > 1$. *If* $\liminf_r q_r = 1$, *then there exists a bounded $S - \phi$ summable sequence that is not $S_\theta - \phi$ summable (to any limit).*

Proof. (Sufficiency) Suppose that $\liminf_r q_r > 1$, then there exists a $\delta > 0$ such that $q_r > 1 + \delta$, for sufficiently large r, which implies that $\frac{h_r}{k_r} > \frac{\delta}{1+\delta}$.

If $x_k \to L(S-\phi)$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\frac{1}{h_r} \mid \{k \in I_r : \phi(x_k - L) \geq \varepsilon\} \mid = \frac{k_r}{h_r}\frac{1}{k_r} \mid \{k \in I_r : \phi(x_k - L) \geq \varepsilon\} \mid$$
$$\leq \frac{1+\delta}{\delta}\frac{1}{k_r} \mid \{n \leq k_r : \phi(x_n - L) \geq \varepsilon\} \mid .$$

Thus $x_k \to L(S_\theta - \phi)$.

(Necessity) Assume that $\liminf_r q_r = 1$ and construct a sequence which is $S-\phi$ convergent but not $S_\theta - \phi$ convergent. For this, let $\phi(x) = \mid x \mid$, proceeding as in ([7], page-510 and [10], page-46), we can select a subsequence (k_{r_j}) of the lacunary sequence θ such that $\frac{k_{r_j}}{k_{r_{j-1}}} < 1 + \frac{1}{j}$ and $\frac{k_{r_j-1}}{k_{r_{j-1}}} > j$, where $r_j \geq r_{j-1} + 2$.

Now we define a bounded sequence $x = (x_i)$ by $x_i = \begin{cases} 1, & i \in I_{r_j}, j = 1,2,3... \\ 0, & otherwise \end{cases}$

Then for any real L, we have $\frac{1}{h_{r_j}} \sum_{I_{r_j}} \mid x_i - L \mid = \mid 1 - L \mid$, $j = 1,2,3,...$

and $\frac{1}{h_{r_j}} \sum_{I_r} \mid x_i - L \mid = \mid L \mid$, for $r \neq r_j$

i.e $\lim_r \frac{1}{h_r} \mid \{k \in I_r : \phi(x_k - L) \geq \varepsilon\} \mid \neq 0$

Thus x is not $S_\theta - \phi$ convergent to L.

However x is $S-\phi$ convergent, since if t is any sufficiently large integer we can find the unique j for which $k_{r_j-1} < t \leq k_{r_{j+1}-1}$ and write

$$\frac{1}{t}\sum_{i=1}^{t} \phi(x_i) = \frac{1}{t}\sum_{i=1}^{t} \mid x_i \mid \leq \frac{k_{r_{j-1}} + h_{r_j}}{k_{r_j-1}} \leq \frac{1}{j} + \frac{1}{j} = \frac{2}{j}$$

as $t \to \infty$, it follows that $j \to \infty$. Hence $x \in \mid \sigma_1 \mid^0$. It follows from Theorem 2.1 of [3] that x is statistically convergent. The above Note 2 implies x is $S-\phi$ convergent. QED

The following example shows that there exists a $S_\theta - \phi$ convergent sequence which has a subsequence that is not $S_\theta - \phi-$convergent.

Example 4. Let $\theta = (2^r)$ be the lacunary sequence, $\phi(x) = \mid x \mid$ be an Orlicz function and (x_n) be a sequence defined by $x_n = \begin{cases} n, & n = k^2, k \in N \\ \frac{1}{n}, & otherwise \end{cases}$.

Then the sequence (x_n) is $S_\theta - \phi$ convergent to 0. However, (x_n) has a subsequence, which is not $S_\theta - \phi$ convergent.

Lemma 2. *For any lacunary sequence θ and any Orlicz function ϕ, $S_\theta - \phi$ $\lim x = L$ implies $S-\phi$ $\lim x = L$ if and only if* $\limsup_r q_r < \infty$. *If* $\limsup_r q_r = \infty$*, then there exists a bounded $S_\theta - \phi$ summable sequence that is not $S-\phi$ summable (to any limit).*

Proof. If $\limsup_r q_r < \infty$ then there is an $H > 0$ such that $q_r < H$ for all r. Suppose that $x_k \to L(S_\theta - \phi)$, and let $N_r = \mid \{k \in I_r : \phi(x_k - L) \geq \varepsilon\} \mid$.

By the definition of $S_\theta - \phi$ convergence we have, given any $\varepsilon' > 0$, there is an $r_0 \in N$ such that

$$\frac{N_r}{h_r} < \varepsilon' \text{ for all } r > r_0.$$

Now let $M = \max\{N_r : 1 \leq r \leq r_0\}$ and let n be any integer satisfying $k_{r-1} < n \leq k_r$; the we can write

$$\frac{1}{n} \mid \{k \leq n : \phi(x_k - L) \geq \varepsilon\} \mid \leq \frac{1}{k_{r-1}} \mid \{k \leq k_r : \phi(x_k - L) \geq \varepsilon\} \mid$$
$$= \frac{1}{k_{r-1}}\{N_1 + N_2 + \cdots\cdots + N_{r_0} + N_{r_0+1} + \cdots\cdots + N_r\}$$
$$\leq \frac{M}{k_{r-1}} r_0 + \frac{1}{k_{r-1}}\{h_{r_0+1}\frac{N_{r_0+1}}{h_{r_0+1}} + \cdots\cdots + h_r \frac{N_r}{h_r}\}$$
$$\leq \frac{r_0 M}{k_{r-1}} + \frac{1}{k_{r-1}}(\sup_{r>r_0} \frac{N_r}{h_r})\{h_{r_0+1} + \cdots\cdots + h_r\}$$
$$\leq \frac{r_0 M}{k_{r-1}} + \varepsilon' \frac{k_r - k_{r_0}}{k_{r-1}} \leq \frac{r_0 M}{k_{r-1}} + \varepsilon' q_r \leq \frac{r_0 M}{k_{r-1}} + \varepsilon' H$$

and the sufficiency follows immediately.

Conversely, suppose that $\limsup_r q_r = \infty$ and construct a sequence which is $S_\theta - \phi$ convergent but not $S - \phi$ convergent. For this, let $\phi(x) = \mid x \mid$. Following the idea in ([6], page-511 and [10], page-47), we can select a subsequence (k_{r_j}) of the lacunary sequence $\theta = (k_r)$ such that $q_{r_j} > j$, and defined a bounded sequence $x = (x_i)$ by

$$x_i = \begin{cases} 1, & k_{r_j-1} < i \leq 2k_{r_j-1},\ j = 1,2,3... \\ 0, & otherwise \end{cases}$$

It is shown in ([7], page-511) that $x \in N_\theta$ but $x \notin \mid \sigma_1 \mid$. By Theorem 1(i) of ([10], page-44), we have x is S_θ-convergent. The above Note 5 implies x is $S_\theta - \phi$ convergent, but it follows from Theorem 2.1 of [3] that x is not $S-$convergent. By above Note 2 implies x is not $S - \phi$ convergent. QED

Combining the above two lemmas we get

Theorem 2. *Let θ be any lacunary sequence; then $S - \phi = S_\theta - \phi$ if and only if* $1 \leq \liminf_r q_r \leq \limsup_r q_r < \infty$.

Theorem 3. *Let θ be a lacunary sequence and ϕ be a convex Orlicz function. If the sequence (x_n) is $S_\theta - \phi$ convergent, then $S_\theta - \phi$ limit of (x_n) is unique.*

Proof. If possible, let $S_\theta - \phi\ \lim x_n = x_0$ and $S_\theta - \phi\ \lim x_n = y_0$. Then

$$\lim_r \frac{1}{h_r} \mid \{k \in I_r : \phi(x_k - x_0) \geq \varepsilon\} \mid = 0$$

and $\lim_r \frac{1}{h_r} \mid \{k \in I_r : \phi(x_k - y_0) \geq \varepsilon\} \mid = 0$

i.e, $\lim_r \frac{1}{h_r} \mid \{k \in I_r : \phi(x_k - x_0) < \varepsilon\} \mid = 1$

and $\lim_r \frac{1}{h_r} \mid \{k \in I_r : \phi(x_k - y_0) < \varepsilon\} \mid= 1$

Let us consider such $k \in I_r$ for which both of $\phi(x_k - x_0) < \varepsilon$ and $\phi(x_k - y_0) < \varepsilon$ are true. For such $k \in I_r$ we have

$\phi(\frac{1}{2}(x_0 - y_0)) = \phi(\frac{1}{2}(x_0 - x_k + x_k - y_0)) \leq \frac{1}{2}\phi(x_k - x_0) + \frac{1}{2}\phi(x_k - y_0) = \epsilon$

Hence the theorem. QED

For the next result we assume the convex Orlicz function which satisfies $\triangle_2-$condition.

Theorem 4. *If* (x_n) *and* (y_n) *are* $S_\theta - \phi$ *convergent and* α *is any real constant, then*

(i) $(x_n + y_n)$ *is* $S_\theta - \phi$ *convergent and* $S_\theta - \phi\ \lim(x_n + y_n) = S_\theta - \phi\ \lim x_n + S_\theta - \phi\ \lim y_n$.

(ii) (αx_n) *is* $S_\theta - \phi$ *convergent and* $S_\theta - \phi\ \lim(\alpha x_n) = \alpha.\, S_\theta - \phi\ \lim x_n$.

Proof. Since ϕ satisfies the $\triangle_2-$ condition, then there exists $M > 0$ such that $\phi(2x) \leq M.\phi(x)$, for every $x \in R$.

(i) Let $S_\theta - \phi\ \lim x_n = x$ and $S_\theta - \phi\ \lim y_n = y$

i.e $\lim_r \frac{1}{h_r} \mid \{k \in I_r : \phi(x_k - x) \geq \varepsilon\} \mid= 0 = \lim_r \frac{1}{h_r} \mid \{k \in I_r : \phi(y_k - y) \geq \varepsilon\} \mid$

i.e $\lim_r \frac{1}{h_r} \mid \{k \in I_r : \phi(x_k - x) < \varepsilon\} \mid= 1 = \lim_r \frac{1}{h_r} \mid \{k \in I_r : \phi(y_k - y) < \varepsilon\} \mid$

Let us consider such $k \in I_r$ for which both of $\phi(x_k - x) < \frac{\varepsilon}{2M}$ and $\phi(y_k - y) < \frac{\varepsilon}{2M}$ are true.

Then for such $k \in I_r$, we have

$\phi((x_k + y_k) - (x + y)) = \phi((x_k - x) + (y_k - y)) \leq \phi(2(x_k - x) + 2(y_k - y))$

$\leq M(\phi(x_k - x) + \phi(y_k - y)) = M.(\frac{\varepsilon}{2M} + \frac{\varepsilon}{2M}) = \varepsilon.$

Thus $\lim_r \frac{1}{h_r} \mid \{k \in I_r : \phi(x_k + y_k - x - y) < \varepsilon\} \mid= 1$

i.e $\lim_r \frac{1}{h_r} \mid \{k \in I_r : \phi(x_k + y_k - x - y) \geq \varepsilon\} \mid= 0$

i.e $S_\theta - \phi\ \lim(x_n + y_n) = x + y = S_\theta - \phi\ \lim x_n + S_\theta - \phi\ \lim y_n$.

(ii) Let $p \in N$ such that $\mid \alpha \mid\leq 2^p$ and $S_\theta - \phi\ \lim x_n = x$

then $\lim_r \frac{1}{h_r} \mid \{k \in I_r : \phi(x_k - x) < \varepsilon\} \mid= 1$.

Let us consider such $k \in I_r$ for which $\phi(x_k - x) < \frac{\varepsilon}{2^p}$, then

$\phi(\alpha(x_k - x)) = \phi(\mid \alpha \mid (x_k - x)) \leq \phi(2^p(x_k - x)) \leq 2^p\phi(x_k - x) \leq 2^p.\frac{\varepsilon}{2^p} = \varepsilon.$

Thus $\lim_r \frac{1}{h_r} \mid \{k \in I_r : \phi(\alpha(x_k - x)) < \varepsilon\} \mid= 1$

i.e $\lim_r \frac{1}{h_r} \mid \{k \in I_r : \phi(\alpha x_k - \alpha x) \geq \varepsilon\} \mid= 0.$

Hence $S_\theta - \phi\ \lim(\alpha x_n) = \alpha.x = \alpha.\, S_\theta - \phi\ \lim x_n$. QED

References

[1] R. C. Buck: *Generalized asymptotic density.* Amer. J. Math. **75** (1953) 335-346.

[2] H. Cakalli: *A study on statistical convergence. Functional Analysis, Approximation and Computation.* **1-2** (2009) 19-24.

[3] J.Connor: *The statistical and strong p-Cesàro convergence of sequences.* Analysis **8** (1988) 47-63.

[4] G.Das and B.K.Patel: *Lacunary distribution of sequences.* Indian J. Pure Appl. Math. **20(1)** (1989) 64-74.

[5] H.Fast: *Sur la convergence statistique.* Colloq. Math. **2** (1951) 241-244.

[6] A.R.Freedman and J.J.Sember: *Densities and summability.Pacific J. Math.* ***95*** *(1981) 293-305.*

[7] A. R. Freedman, J. J. Sember and M. Raphael: *Some Cesàro type summability space.* Proc. London Math. Soc. **37** (1978) 508-520.

[8] J. A. Fridy: *Statistical limit points.* Proc. Amer. Math. Soc. **4** (1993) 1187-1192.

[9] J. A. Fridy: *On statistical convergence.* Analysis **5** (1985) 301-313.

[10] J. A. Fridy and C. Orhan: *Lacunary statistical convergence.* Pacific J. Math. **160(1)** (1993) 43-51.

[11] M. Mamedov and S. Pehlivan: *Statistical cluster points and turnpike theorem in non convex problems.* J. Math. Anal. Appl. **256** (2001) 686-693.

[12] D. S. Mitrinovic, J. Sandor and B. Crstici: Handbook of Number Theory. Kluwer Acad. Publ. Dordrecht-Boston-London, 1996.

[13] M. Mursaleen, S. Debnath and D. Rakshit: *I-statistical limit superior and I-statistical limit inferior.* Filomat **31(7)** (2017) 2103-2108.

[14] M. M. Rao and Z. D. Ren: Applications of Orlicz spaces. Marcel Dekker Inc. 2002.

[15] T. Salat: *On statistically convergent sequences of real numbers.* Math. Slova. **30** (1980) 139-150.

[16] E. Savas and S. Borgohain: *On strongly almost lacunary statistical A-convergence defined by a Musielak-Orlicz function.* Filomat **30(3)** (2016) 689-697.

[17] E. Savas and P Das: *A generalized statistical convergence via ideals.* Appl. Math. Lett. **24** (2011) 826-830.

[18] H. Steinhaus: *Sur la convergence ordinaire et la convergence asymptotique.* Colloq. Math. **2** (1951) 73-74.

[19] B. C. Tripathy: *On statistically convergent and statistically bounded sequences*, Bull. Malaysian. Math. Soc. **20** (1997) 31-33.