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Biharmonic Hermitian vector bundles over compact Kähler manifolds and compact Einstein Riemannian manifolds

Hajime Urakawa
Division of Mathematics, Graduate School of Information Sciences, Tohoku University, Aoba 6-3-09, Sendai, 980-8579, Japan urakawa@math.is.tohoku.ac.jp

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Abstract. We show, for every Hermitian vector bundle $\pi : (E, g) \to (M, h)$ over a compact Kähler Einstein manifold (M, h), if the projection π is biharmonic, then it is harmonic. On a biharmonic Hermitian vector bundle over a compact Riemannian manifold with positive Ricci curvature, we show a new estimate of the first eigenvalue of the Laplacian.

Keywords: biharmonic maps, harmonic maps, Kähler Einstein manifolds, Hermitian vector bundles

MSC 2000 classification: primary 53C43; secondary 58E20, 53C07

Introduction

Research of harmonic maps, which are critical points of the energy functional, is one of the central problems in differential geometry including minimal submanifolds. The Euler-Lagrange equation is given by the vanishing of the tension field. In 1983, Eells and Lemaire ([8]) proposed to study biharmonic maps, which are critical points of the bienergy functional, by definition, half of the integral of square of the norm of tension field $\tau(\varphi)$ for a smooth map φ of a Riemannian manifold (M, g) into another Riemannian manifold (N, h). After a work by G.Y. Jiang [21], several geometers have studied biharmonic maps (see [4], [12], [13], [20], [24], [26], [38], [39], etc.). Note that a harmonic maps is always biharmonic. One of central problems is to ask whether the converse is true. *B.-Y. Chen's conjecture* is to ask whether every biharmonic submanifold of the Euclidean space $\mathbb{R}^n$ must be harmonic, i.e., minimal ([5]). There are many works supporting this conjecture ([7], [10], [22], [1]). However, B.-Y. Chen's conjecture is still open. R. Caddeo, S. Montaldo, P. Piu ([4]) and C. Oniciuc ([36]) raised the *generalized B.-Y. Chen's conjecture* to ask whether each biharmonic submanifold in a Riemannian manifold (N, h) of non-positive sectional curvature must be harmonic (minimal). For the generalized Chen's conjecture, Ou and Tang gave ([37], [38]) a counter example in some Riemannian manifold of negative sectional curvature. But, it is also known (cf. [27], [28], [30]) that every biharmonic map of a complete Riemannian manifold into another Riemannian manifold of non-positive sectional curvature with finite energy and finite bienergy must be harmonic. For the target Riemannian manifold (N, h) of non-negative sectional curvature, theories of biharmonic maps and biharmonic immersions seems to be quite different from the case (N, h) of non-positive sectional curvature. There exit biharmonic submanifolds which are not harmonic in the unit sphere. S. Ohno, T. Sakai and myself [33], [34] determined (1) all the biharmonic hypersurfaces in irreducible symmetric spaces of compact type which are regular orbits of commutative Hermann actions of cohomogeneity one, and gave (2) a complete table of all the proper biharmonic singular orbits of commutative Hermann actions of cohomogeneity two, and (3) a complete list of all the proper biharmonic regular orbits of $(K_2 \times K_1)$-actions of cohomogeneity one on G for every commutative compact symmetric triad (G, K_1, K_2). We note that recently Inoguchi and Sasahara ([19]) investigated biharmonic homogeneous hypersurfaces in compact symmetric spaces. Sasahara ([40]) classified all biharmonic real hypersurfaces in a complex projective space, and Ohno studied biharmonic orbits of isotropy representations of symmetric spaces in the sphere (cf. [31], [32]).

In this paper, we treat with an Hermitian vector bundle $(E, g) \to (M, h)$ over a compact Riemannian manifold (M, h). We assume (M, h) is a compact Kähler Einstein Riemannian manifold, that is, the Ricci transform Ric^h of the Kähler metric h on M satisfies $\mathrm{Ric}^h = c\,\mathrm{Id}$, for some constant c. Then, we show the following (cf. Theorems 4 and 5):

Theorem 1. *Let $\pi : (E, g) \to (M, h)$ be an Hermitian vector bundle over a compact Kähler Einstein Riemannian manifold (M, h). If π is biharmonic, then it is harmonic.*

Theorem 2. *Let $\pi : (E, g) \to (M, h)$ be a biharmonic Hermitian vector bundle over a compact Einstein manifold (M, h) with Ricci curvature* $\mathrm{Ric}^h = c$ *for some positive constant $c > 0$. Then, either (i) π is harmonic, (ii) $f_0 = \langle \tau(\pi), \tau(\pi) \rangle$ is constant, or (iii) the first eigenvalue $\lambda_1(M, h)$ of (M, h) satisfies the following inequality:*

$$0 < \frac{n}{n-1}\, c \leq \lambda_1(M, h) \leq \frac{2c}{1 - X}, \tag{0.1}$$

where

$$0 < X := \frac{1}{\mathrm{Vol}(M,h)} \frac{\left(\int_M f_0\, v_h\right)^2}{\int_M {f_0}^2\, v_h} < 1, \tag{0.2}$$

and $f_0 := \langle \tau(\pi), \tau(\pi)\rangle \in C^\infty(M)$ *is the pointwise inner product of the tension field* $\tau(\pi)$*.*

The inequalities (1) and (2) can be rewritten as follows:

$$-1 < \frac{2-n}{n} = 1 - 2\frac{n-1}{n} \leq 1 - \frac{2c}{\lambda_1(M,h)} \leq \frac{1}{\mathrm{Vol}(M,h)} \frac{\left(\int_M f_0\, v_h\right)^2}{\int_M {f_0}^2\, v_h} < 1. \tag{0.3}$$

Theorem 1 shows the sharp contrasts on the biharmonicities between the case of vector bundles and the one of the principle G-bundles. Indeed, we treated with the biharmonicity of the projection of the principal G-bundle over a Riemannian manifold (M,h) with negative definite Ricci tensor field (cf. Theorem 2.3 in [45]). In Theorem 2, the behavior of the quantity $1-\frac{2c}{\lambda_1(M,h)}$ in (3) is very important. Indeed, $0 \leq 1 - \frac{2c}{\lambda_1(M,h)}$ if and only if $2c \leq \lambda_1(M,h)$ which is the theorem of M. Obata (cf [42] p.181, Theorem (3.23)), and $-\frac{n-1}{n} \leq 1 - \frac{2c}{\lambda_1(M,h)} < 0$ if and only if $\frac{n}{n-1}\,c \leq \lambda_1(M,h) < 2c$ under the condition $\mathrm{Ric}^h \geq c > 0$, which is the theorem of Lichnerowicz and Obata [42], p.182, Theorem (3.26).

We give an example of the projection of the principal G-bundle over a Riemannian manifold (M,h) which is biharmonic but not harmonic (cf. Example 1 in this paper, and also Theorem 5 in [46]). Finally, it should be mentioned that Oniciuc ([36]) gave examples of non-harmonic biharmonic projections of the tangent bundle over a compact Riemannian manifold, which has a sharp contrast our case of the Hermitian vector bundles over a compact Kähler-Einstein manifold.

1 Preliminaries

In this section, we prepare materials for the first variation formula for the bi-energy functional and bi-harmonic maps. Let us recall the definition of a harmonic map $\varphi : (M,g) \to (N,h)$, of a comoact Riemannian manifold (M,g) into another Riemannian manifold (N,h), which is an extremal of the *energy functional* defined by

$$E(\varphi) = \int_M e(\varphi)\, v_g,$$

where $e(\varphi) := \frac{1}{2}|d\varphi|^2$ is called the energy density of φ. That is, for all variation $\{\varphi_t\}$ of φ with $\varphi_0 = \varphi$,

$$\left.\frac{d}{dt}\right|_{t=0} E(\varphi_t) = -\int_M h(\tau(\varphi), V)v_g = 0, \tag{1.1}$$

where $V \in \Gamma(\varphi^{-1}TN)$ is a variation vector field along φ which is given by $V(x) = \frac{d}{dt}|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N$ ($x \in M$), and the *tension field* of φ is given by $\tau(\varphi) = \sum_{i=1}^m B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN)$, where $\{e_i\}_{i=1}^m$ is a locally defined frame field on (M, g). The second fundamental form $B(\varphi)$ of φ is defined by

$$\begin{aligned} B(\varphi)(X,Y) &= (\widetilde{\nabla}d\varphi)(X,Y) \\ &= (\widetilde{\nabla}_X d\varphi)(Y) \\ &= \overline{\nabla}_X(d\varphi(Y)) - d\varphi(\nabla_X Y) \\ &= \nabla^N_{d\varphi(X)}d\varphi(Y) - d\varphi(\nabla_X Y), \end{aligned} \tag{1.2}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Furthermore, ∇, and ∇^N, are connections on TM, TN of (M, g), (N, h), respectively, and $\overline{\nabla}$, and $\widetilde{\nabla}$ are the induced one on $\varphi^{-1}TN$, and $T^*M \otimes \varphi^{-1}TN$, respectively. By (4), φ is harmonic if and only if $\tau(\varphi) = 0$.

The second variation formula of the energy functional is also well known which is given as follows. Assume that φ is harmonic. Then,

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(\varphi_t) = \int_M h(J(V), V)v_g, \tag{1.3}$$

where J is an elliptic differential operator, called *Jacobi operator* acting on $\Gamma(\varphi^{-1}TN)$ given by

$$J(V) = \overline{\Delta}V - \mathcal{R}(V), \tag{1.4}$$

where $\overline{\Delta}V = \overline{\nabla}^*\overline{\nabla}V$ is the *rough Laplacian* and $\mathcal{R}$ is a linear operator on $\Gamma(\varphi^{-1}TN)$ given by $\mathcal{R}V = \sum_{i=1}^m R^N(V, d\varphi(e_i))d\varphi(e_i)$, and R^N is the curvature tensor of (N, h) given by $R^N(U, V) = {\nabla^N}_U{\nabla^N}_V - {\nabla^N}_V{\nabla^N}_U - {\nabla^N}_{[U,V]}$ for $U,\ V \in \mathfrak{X}(N)$.

J. Eells and L. Lemaire proposed ([8]) polyharmonic (k-harmonic) maps and Jiang studied ([21]) the first and second variation formulas of bi-harmonic maps. Let us consider the *bi-energy functional* defined by

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g, \tag{1.5}$$

where $|V|^2 = h(V, V)$, $V \in \Gamma(\varphi^{-1}TN)$.

Then, the first variation formula is given as follows.

Theorem 3. (*the first variation formula*)

$$\left.\frac{d}{dt}\right|_{t=0} E_2(\varphi_t) = -\int_M h(\tau_2(\varphi), V)v_g, \tag{1.6}$$

where

$$\tau_2(\varphi) = J(\tau(\varphi)) = \overline{\Delta}\tau(\varphi) - \mathcal{R}(\tau(\varphi)), \tag{1.7}$$

J is given in (7).

For the second variational formula, see [21] or [12].

Definition 1. A smooth map φ of M into N is called to be *bi-harmonic* if $\tau_2(\varphi) = 0$.

2 The case of compact Kähler manifolds.

To prove Theorem 1, we need the following:

Proposition 1. *Let $\pi : (E, g) \to (M, h)$ be an Hermitian vector bundle over a compact Kähler Einstein manifold (M, h). Assume that π is biharmonic. Then the following hold:*

(1) *The tension field $\tau(\pi)$ satisfies that*

$$\overline{\nabla}_{X'}\tau(\pi) = 0 \qquad (\forall\, X' \in \mathfrak{X}(M)). \tag{2.1}$$

(2) *The pointwise inner product $\langle\tau(\pi), \tau(\pi)\rangle = |\,\tau(\pi)\,|^2$ is constant on (M, g), say $d \geq 0$.*

(3) *The energy $E_2(\pi)$ satisfies that*

$$E_2(\pi) := \frac{1}{2}\int_M |\tau(\pi)|^2\, v_h = \frac{d}{2}\,\mathrm{Vol}(M, h). \tag{2.2}$$

By Proposition 1, Theorem 1 can be proved as follows. Assume that $\pi : (E, g) \to (M, h)$ is biharmonic. Due to (11) in Proposition 1, we have

$$\mathrm{div}(\tau(\pi)) := \sum_{i=1}^{n}(\overline{\nabla}_{e'_i}\tau(\pi))(e'_i) = 0, \tag{2.3}$$

where $\{e'_i\}_{i=1}^n$ is a locally defined orthonormal frame field on (M, h) and we put $n = \dim_{\mathbb{R}} M$. Then, for every $f \in C^\infty(M)$, it holds that, due to Proposition (3.29) in [42], p. 60, for example,

$$0 = \int_M f \,\mathrm{div}(\tau(\pi))\, v_h = -\int_M h(\nabla f, \tau(\pi))\, v_h. \tag{2.4}$$

Therefore, we obtain $\tau(\pi) \equiv 0$. QED

We will prove Proposition 1, later. Here, we give examples of the line bundles over some compact homogeneous Kähler Einstein manifolds (M, h):

Example 1. A generalized flag manifold G/H admits a unique Kähler Einstein metric h ([3] and [6]). Here, G is a compact semi-simple Lie group, and H is the centralizer of a torus S in G, i.e., $G^{\mathbb{C}}$ is the complexification of G, and B is its Borel subgroup. Then,

$$M = G/H = G^{\mathbb{C}}/B.$$

The Borel subgroup B is written as $B = TN$, where T is a maximal torus of B and N is a nilpotent Lie subgroup of B. Every character ξ_λ of a Borel subgroup B is given as a homomorphism $\xi_\lambda : B \to \mathbb{C}^* = \mathbb{C} - \{0\}$ which is written as

$$\xi_\lambda(tn) = \xi_\lambda(t) \qquad (t \in T,\ n \in N). \tag{2.5}$$

Here $\xi_\lambda : T \to U(1)$ is a character of T which is written as

$$\xi_\lambda(\exp(\theta_1 H_1 + \cdots + \theta_\ell H_\ell)) = e^{2\pi\sqrt{-1}(k_1\theta_1 + \cdots + k_\ell\theta_\ell)}, \qquad (\theta_1,\ \ldots,\ \theta_\ell \in \mathbb{R}), \tag{2.6}$$

where $k_1, \ldots, k_\ell$ are non-negative integers, and $\ell = \dim T$.

Note that every character ξ_λ of a nilpotent Lie group N must be $\xi_\lambda(n) = 1$ because $\xi_\lambda(n) = \xi_\lambda(\exp X) = e^{\xi_{\lambda'}(X)}$ where $n = \exp X$ $(X \in \mathfrak{n})$, and $\lambda' : \mathfrak{t} \to \mathbb{C}$ is a homomorphism, i.e., $\xi_{\lambda'}(X + Y) = \xi_{\lambda'}(X) + \xi_{\lambda'}(Y)$, $(X, Y \in \mathfrak{t})$. Then, there exists $k \in \mathbb{N}$ which satisfies that $\exp(k\,X) = n^k = e$. Then, $e^{k\,\xi_{\lambda'}(X)} = \xi_\lambda(n^k) = \xi_\lambda(e) = 1$. Thus, for every $a \in \mathbb{R}$,

$$e^{a\,k\,\xi_{\lambda'}(X)} = (e^{k\,\xi_{\lambda'}(X)})^a = 1.$$

This implies that $k\,\xi_{\lambda'}(X) = 0$. Thus, $\xi_{\lambda'}(X) = 0$ for all $X \in \mathfrak{n}$, i.e., $\xi_{\lambda'} \equiv 0$. Therefore, we have that $\xi_\lambda(n) = e$ $(n \in N)$. We have (15).

For every ξ_λ given by (15) and (16), we obtain the associated holomorphic vector bundle E_{ξ_λ} over $G^{\mathbb{C}}/B$ as $E_{\xi_\lambda} := \{[x, v] | (x, v) \in G^{\mathbb{C}} \times \mathbb{C}\}$, where the equivalence relation $(x, v) \sim (x', v')$ is $(x, v) = (x', v')$ if and only if there exists $b \in B$ such that $(x', v') = (xb^{-1}, \xi_\lambda(b)v)$, denoted by $[x, v]$, the equivalence class including $(x, v) \in G^{\mathbb{C}} \times \mathbb{C}$ (for example, [2], [41]).

3 Proof of Proposition 1.

For an Hermitian vector bundle $\pi : (E, g) \to (M, g)$ with $\dim_{\mathbb{R}} E = m$, and $\dim_{\mathbb{R}} M = n$, let us recall the definitions of the tension field $\tau(\pi)$ and the bitension field $\tau_2(\pi)$:

$$\begin{cases} \tau(\pi) = \sum_{j=1}^{m} \left\{ \overline{\nabla}^h_{e_j} \pi_* e_j - \pi_* \left(\nabla^g_{e_j} e_j \right) \right\}, \\ \tau_2(\pi) = \overline{\Delta} \tau(\pi) - \sum_{j=1}^{m} R^h(\tau(\pi), \pi_* e_j) \pi_* e_j. \end{cases} \quad (3.1)$$

Then, we have

$$\begin{aligned} \tau_2(\pi) &:= \overline{\Delta} \tau(\pi) - \sum_{j=1}^{m} R^h(\tau(\pi), \pi_* e_j) \pi_* e_j \\ &= \overline{\Delta} \tau(\pi) - \sum_{j=1}^{n} R^h(\tau(\pi), e'_j) e'_j \quad (3.2) \\ &= \overline{\Delta} \tau(\pi) - \mathrm{Ric}^h(\tau(\pi)). \quad (3.3) \end{aligned}$$

Here, recall that $\pi : (E, g) \to (M, h)$ is the Riemannian submersion and $\{e_i\}_{i=1}^m$ and $\{e'_j\}_{j=1}^n$ are locally defined orthonormal frame fields on (E, g) and (M, h), respectively, satisfying that $\pi_* e_j = e'_j$ $(j = 1, \cdots, n)$ and $\pi_*(e_j) = 0$ $(j = n + 1, \cdots, m)$. Therefore, we have (18) and (19) by means of the definition of the Ricci tensor field Ric^h of (M, h).

Assume that (M, h) is a real n-dimensional compact Kähler Einstein manifold with $\mathrm{Ric}^h = c\,\mathrm{Id}$, where n is even. Then, due to (19), we have that $\pi : (E, g) \to (M, h)$ is biharmonic if and only if

$$\overline{\Delta} \tau(\pi) = c\, \tau(\pi). \quad (3.4)$$

Since $\langle \tau(\pi), \tau(\pi) \rangle$ is a C^∞ function on a Riemannian manifold (M, h), we have, for each $j = 1, \cdots, n$,

$$\begin{aligned} e'_j \langle \tau(\pi), \tau(\pi) \rangle &= \langle \overline{\nabla}_{e'_j} \tau(\pi), \tau(\pi) \rangle + \langle \tau(\pi), \overline{\nabla}_{e'_j} \tau(\pi) \rangle \\ &= 2 \langle \overline{\nabla}_{e'_j} \tau(\pi), \tau(\pi) \rangle, \quad (3.5) \\ {e'_j}^2 \langle \tau(\pi), \tau(\pi) \rangle &= 2 e'_j \langle \overline{\nabla}_{e'_j} \tau(\pi), \tau(\pi) \rangle \\ &= 2 \langle \overline{\nabla}_{e'_j} (\overline{\nabla}_{e'_j} \tau(\pi)), \tau(\pi) \rangle + 2 \langle \overline{\nabla}_{e'_j} \tau(\pi), \overline{\nabla}_{e'_j} \tau(\pi) \rangle, \quad (3.6) \\ \nabla_{e'_j} e'_j \langle \tau(\pi), \tau(\pi) \rangle &= 2 \langle \overline{\nabla}_{\nabla_{e'_j} e'_j} \tau(\pi), \tau(\pi) \rangle. \quad (3.7) \end{aligned}$$

Therefore, the Laplacian $\Delta_h = -\sum_{j=1}^n ({e'_j}^2 - \nabla_{e'_j} e'_j)$ acting on $C^\infty(M)$, so that

$$
\begin{aligned}
\Delta_h \langle \tau(\pi), \tau(\pi)\rangle &= \qquad (3.8)\\
&= 2\sum_{j=1}^n \Big\{ -\langle \overline{\nabla}_{e'_j}(\overline{\nabla}_{e'_j}\tau(\pi)), \tau(\pi)\rangle - \langle \overline{\nabla}_{e'_j}\tau(\pi), \nabla_{e'_j}\tau(\pi)\rangle + \langle \overline{\nabla}_{\nabla_{e'_j}}\tau(\pi), \tau(\pi)\rangle \Big\}\\
&= 2\Big\langle -\sum_{j=1}^n \{\overline{\nabla}_{e'_j}\overline{\nabla}_{e'_j} - \overline{\nabla}_{\nabla_{e'_j}e'_j}\}\tau(\pi), \tau(\pi)\Big\rangle - 2\sum_{j=1}^n \big\langle \overline{\nabla}_{e'_j}\tau(\pi), \overline{\nabla}_{e'_j}\tau(\pi)\big\rangle\\
&= 2\big\langle \overline{\Delta}\tau(\pi), \tau(\pi)\big\rangle - 2\sum_{j=1}^n \langle \overline{\nabla}_{e'_j}\tau(\pi), \overline{\nabla}_{e'_j}\tau(\pi)\rangle \qquad (3.9)\\
&\leq 2\big\langle \overline{\Delta}\tau(\pi), \tau(\pi)\big\rangle, \qquad (3.10)
\end{aligned}
$$

because of $\langle \overline{\nabla}_{e'_j}\tau(\pi), \overline{\nabla}_{e'_j}\tau(\pi)\rangle \geq 0$, $(j = 1, \cdots, n)$.

If $\pi : (E, g) \to (M, h)$ is biharmonic, due to (20), $\overline{\Delta}\tau(\pi) = c\,\tau(\pi)$, the right hand side of (25) coincides with

$$
\begin{aligned}
(25) &= 2c\,\langle \tau(\pi), \tau(\pi)\rangle - 2\sum_{j=1}^n \langle \overline{\nabla}_{e'_j}\tau(\pi), \overline{\nabla}_{e'_j}\tau(\pi)\rangle \qquad (3.11)\\
&\leq 2c\,\langle \tau(\pi), \tau(\pi)\rangle. \qquad (3.12)
\end{aligned}
$$

Remember that due to M. Obata's theorem, (see Proposition 2 below),

$$\lambda_1(M, h) \geq 2c, \qquad (3.13)$$

since $\mathrm{Ric}_h = c\,\mathrm{Id}$, And the equation in (28) holds, i.e., $\lambda_1(M, h) = 2c$ and

$$\Delta_h \langle \tau(\pi), \tau(\pi)\rangle = 2c\,\langle \tau(\pi), \tau(\pi)\rangle \qquad (3.14)$$

holds. Then, (29) implies that the equality in the inequality (28) holds. We have that

$$\sum_{j=1}^n \langle \overline{\nabla}_{e'_j}\tau(\pi), \overline{\nabla}_{e'_j}\tau(\pi)\rangle = 0, \qquad (3.15)$$

which is equivalent to that

$$\overline{\nabla}_{X'}\tau(\pi) = 0 \qquad (\forall X' \in \mathfrak{X}(M)). \qquad (3.16)$$

Due to (32), for every $X' \in \mathfrak{X}(M)$,

$$X'\,\langle \tau(\pi), \tau(\pi)\rangle = 2\,\langle \overline{\nabla}_{X'}\tau(\pi), \tau(\pi)\rangle = 0. \qquad (3.17)$$

Therefore, the function $\langle \tau(\pi), \tau(\pi) \rangle$ on M is a constant function on M. Thus, it implies that the right hand side of (29) must vanish. Thus, $c = 0$ or $\tau(\pi) \equiv 0$. If we assume that $\tau(\pi) \not\equiv 0$, then by (29), it must hold that $2c = 0$. Then, $\overline{\Delta}\tau(\pi) = c\,\tau(\pi) = 0$, so that $\tau(\pi) \equiv 0$ due to (20).

Let $\lambda_1(M, g)$ be the first eigenvalue of the Laplacian Δ of a compact Riemannian manifold (M, g). Recall the theorem of M. Obata:

Proposition 2. *(cf. [42], pp. 180, 181) Assume that (M, h) is a compact Kähler manifold, and the Ricci transform ρ of (M, h) satisfies that*

$$h(\rho(u), u) \geq \alpha\, h(u, u), \qquad (\forall u \in T_xM), \tag{3.18}$$

for some positive constant $\alpha > 0$. Then, it holds that

$$\lambda_1(M, h) \geq 2\,\alpha. \tag{3.19}$$

If the equality holds, then M admits a non-zero holomorphic vector field.

Thus, we obtain Proposition 1, and the following theorem (cf. Theorem 1):

Theorem 4. *Let $\pi : (E, g) \to (M, h)$ be an Hermitian vector over a compact Kähler Einstein manifold (M, h). If π is biharmonic, then it is harmonic.*

4 Einstein manifolds and proof of Theorem 5.

Let $\pi : (E^m, g) \to (M^n, h)$ be an Hermitian vector bundle over a compact Riemannian manifold (M, h), and again let us recall the tension field and the bitension field

$$\tau(\pi) = \sum_{j=1}^{m} \left\{ \overline{\nabla}^h_{e_j} \pi_* e_j - \pi_* \left(\nabla^g_{e_j} e_j \right) \right\}, \tag{4.1}$$

$$\tau_2(\pi) = \overline{\Delta}\tau(\pi) - \sum_{j=1}^{m} R^h(\tau(\pi), \pi_* e_j)\pi_* e_j, \tag{4.2}$$

respectively. Then, we have

$$\tau_2(\pi) = \sum_{j=1}^{m} \left\{ \overline{\nabla}^h_{e_j} \pi_* e_j - \pi_* \left(\nabla^g_{e_j} e_j \right) \right\}, \tag{4.3}$$

$$\tau_2(\pi) = \overline{\Delta}\tau(\pi) - \sum_{j=1}^{m} R^h(\tau(\pi), \pi_* e_j)\pi_* e_j$$

$$
\begin{aligned}
&= \overline{\Delta}\tau(\pi) - \sum_{j=1}^{n} R^h(\tau(\pi), e'_j)e'_j \\
&= \overline{\Delta}\tau(\pi) - \mathrm{Ric}^h(\tau(\pi)) \\
&= \overline{\Delta}\tau(\pi) - c\,\tau(\pi) \qquad (4.4)
\end{aligned}
$$

since it holds that $\mathrm{Ric}^h = c\,h$ because of (M,h) is Einstein. Therefore, that $\pi : (E,g) \to (M,h)$ is biharmonic if and only if

$$\overline{\Delta}\tau(\pi) = c\,\tau(\pi). \qquad (4.5)$$

Since the Laplacian Δ_h of a Riemannian manifold (M,h) is expressed as

$$\Delta_h = -\sum_{j=1}^{n}({e'_j}^2 - \nabla^h_{e'_j}e'_j), \qquad (4.6)$$

and

$$
\begin{aligned}
e'_j\langle\tau(\pi),\tau(\pi)\rangle &= 2\,\langle\overline{\nabla}_{e'_j}\tau(\pi),\tau(\pi)\rangle, \\
{e'_j}^2\langle\tau(\pi),\tau(\pi)\rangle &= 2e'_j\,\langle\overline{\nabla}_{e'_j}\tau(\pi),\tau(\pi)\rangle, \\
&= 2\,\langle\overline{\nabla}_{e'_j}(\overline{\nabla}_{e'_j}\tau(\pi)),\tau(\pi)\rangle + 2\langle\overline{\nabla}_{e'_j}\tau(\pi),\overline{\nabla}_{e'_j}\tau(\pi)\rangle \\
\nabla_{e'_j}e'_j\,\langle\tau(\pi),\tau(\pi)\rangle &= 2\langle\overline{\nabla}_{\nabla_{e'_j}e'_j}\tau(\pi),\tau(\pi)\rangle,
\end{aligned}
$$

we have

$$
\begin{aligned}
\Delta_h\langle\tau(\pi),\tau(\pi)\rangle &= -\sum_{j=1}^{n}({e'_j}^2 - \nabla^h_{e'_j}e'_j)\,\langle\tau(\pi),\tau(\pi)\rangle \\
&= 2\langle\overline{\Delta}\tau(\pi),\tau(\pi)\rangle - 2\sum_{j=1}^{n}\langle\overline{\nabla}_{e'_j}\tau(\pi),\overline{\nabla}_{e'_j}\tau(\pi)\rangle \\
&\leq 2\langle\overline{\Delta}\tau(\pi),\tau(\pi)\rangle. \qquad (4.7)
\end{aligned}
$$

Assume that $\pi : (E,g) \to (M,h)$ is biharmonic. Then, we have

$$\overline{\Delta}\tau(\pi) = c\tau(\pi). \qquad (4.8)$$

Therefore, we have

$$\Delta_h\langle\tau(\pi),\tau(\pi)\rangle \leq 2c\langle\tau(\pi),\tau(\pi)\rangle. \qquad (4.9)$$

Then, we show the following theorem (cf. Theorem 2):

Theorem 5. *Let $\pi : (E, g) \to (M, h)$ be an Hermitian vector bundle over a compact Einstein manifold (M, h) with Ricci curvature $\mathrm{Ric}^h = c$ for some positive constant $c > 0$. Assume that $\pi : (E, g) \to (M, h)$ is biharmonic. Then, either (i) π is harmonic, or (ii) the first eigenvalue $\lambda_1(M, h)$ of (M, h) satisfies the following inequality:*

$$0 < \frac{n}{n-1}\,c \leq \lambda_1(M,h) \leq \frac{2c}{1-X} \tag{4.10}$$

where

$$0 < X := \frac{1}{\mathrm{Vol}(M,h)}\,\frac{\left(\int_M f_0\, v_h\right)^2}{\int_M {f_0}^2\, v_h} < 1, \tag{4.11}$$

and $f_0 := \langle \tau(\pi), \tau(\pi)\rangle \in C^\infty(M)$ is the pointwise inner product of the tension field $\tau(\pi)$.

The inequalities (1) and (2) can be rewritten as follows:

$$-1 < \frac{2-n}{n} = 1-2\frac{n-1}{n} \leq 1-\frac{2c}{\lambda_1(M,h)} \leq \frac{1}{\mathrm{Vol}(M,h)}\,\frac{\left(\int_M f_0\, v_h\right)^2}{\int_M {f_0}^2\, v_h} < 1. \tag{4.12}$$

First, let us recall the theorem of Lichinerowicz and Obata:

Theorem 6. *Assume that the Ricci curvature* Ric *of (M, h) is bounded below by a positive constant $c > 0$. Then, the first eigenvalue satisfies that*

$$\lambda_1(h) \geq \frac{n}{n-1}\,c, \tag{4.13}$$

and the equality in (45) holds if and only if (M, h) is isometric to the n-dimensional standard unit sphere (S^n, h_0).

The inequality (44) means that a C^∞ function f_0 on M defined by $f_0 = \langle \tau(\pi), \tau(\pi)\rangle \in C^\infty(M)$ satisfies that

$$\Delta_h f_0 \leq 2c\, f_0. \tag{4.14}$$

(The first step) We assume that $f_0 \not\equiv 0$ and not a constant. Then $\int_M {f_0}^2\, v_h > 0$, and we have by (45),

$$2c \geq \frac{\int_M f_0(\Delta_h f_0)\, v_h}{\int_M {f_0}^2\, v_h} = \frac{\int_M |\nabla f_0|^2\, v_h}{\int_M {f_0}^2\, v_h}. \tag{4.15}$$

(The second step) If we define $f_1 := f_0 - \frac{\int_M f_0\, v_h}{\mathrm{Vol}(M,h)} \in C^\infty(M)$, we have

$$\int_M f_1\, v_h = 0, \tag{4.16}$$

$$\nabla f_1 = \nabla f_0, \quad |\nabla f_1|^2 = |\nabla f_0|^2, \tag{4.17}$$

$$\int_M {f_1}^2\, v_h = \int_M {f_0}^2\, v_h - \frac{\left(\int_M f_0\, v_h\right)^2}{\mathrm{Vol}(M,h)}. \tag{4.18}$$

(The third step) Let us recall the well-known Schwarz inequality (M. Fujiwara, Differentiations and Integrations, Vol. I, page 434, 1934, 2015, ISBN978-4-7536-0163-9):

Lemma 1. *(Schwarz inequality) For every two continuous functions f and g on a compact Riemannian manifold (M,h), then it holds that*

$$\left(\int_M f(x)\, g(x)\, v_h(x)\right)^2 \leq \left(\int_M f(x)^2\, v_h(x)\right) \left(\int_M g(x)^2\, v_h(x)\right). \tag{4.19}$$

The equality holds if and only if there exist two real numbers λ and μ such that

$$\lambda\, f(x) + \mu\, g(x) \equiv 0 \qquad (\textit{everywhere on } M). \tag{4.20}$$

Then, we have

$$\left(\int_M f_0\, v_h\right)^2 \leq \mathrm{Vol}(M,h) \int_M {f_0}^2\, v_h. \tag{4.21}$$

Furthermore, we have

$$\int_M {f_0}^2\, v_h - \frac{\left(\int_M f_0\, v_h\right)^2}{\mathrm{Vol}(M,h)} > 0. \tag{4.22}$$

Because, if (57) does not occur, the equality holds for $f = f_0$ and $g \equiv 1$ in (54). Due to Lemma 1, there exist two real numbers λ and μ satisfying that

$$\lambda\, f_0(x) + \mu \cdot 1 \equiv 0 \qquad (\text{on } M) \tag{4.23}$$

which means that f_0 must be a constant on M and $\nabla f_0 \equiv 0$ which contradicts our assumption in Step 1. We have the inequality (57).

(The fourth step) The first eigenvalue $\lambda_1(M,h)$ of (M,h) satisfies that

$$\begin{aligned}\lambda_1(M,h) &\leq \frac{\int_M |\nabla f_1|^2\, v_h}{\int_M {f_1}^2\, v_h}\\ &= \frac{\int_M |\nabla f_0|^2\, v_h}{\int_M {f_0}^2\, v_h - \frac{\left(\int_M {f_0}^2\, v_h\right)^2}{\mathrm{Vol}(M,h)}}\\ &\leq 2c\, \frac{\int_M {f_0}^2\, v_h}{\int_M {f_0}^2\, v_h - \frac{\left(\int_M f_0\, v_h\right)^2}{\mathrm{Vol}(M,h)}}\\ &= 2c\, \frac{1}{1-X}, \end{aligned} \tag{4.24}$$

where we put $X := \frac{1}{\mathrm{Vol}(M,h)} \frac{\left(\int_M f_0\, v_h\right)^2}{\int_M {f_0}^2\, v_h}$, $(0 < X < 1)$. Indeed, $X < 1$ if and only if

$$\left(\int_M f_0\, v_h\right)^2 < \mathrm{Vol}(M,h) \int_M {f_0}^2\, v_h, \tag{4.25}$$

and

$$0 < X \quad \Longleftrightarrow \quad 0 < \int_M f_0\, v_h \quad \Longleftrightarrow \quad 0 \not\equiv f_0. \tag{4.26}$$

Furthermore, since $\lambda_1(M,h) \leq 2c\, \frac{1}{1-X}$ if and only if

$$1 - \frac{2c}{\lambda_1(M,h)} \leq X, \tag{4.27}$$

together with the inequality of Lichnerowicz-Obata, we have also the following inequalities:

$$-1 < \frac{2-n}{n} = 1 - 2\frac{n-1}{n} \leq 1 - \frac{2c}{\lambda_1(M,h)} \leq \frac{1}{\mathrm{Vol}(M,h)} \frac{\left(\int_M f_0\, v_h\right)^2}{\int_M {f_0}^2\, v_h} < 1. \tag{4.28}$$

We obtain Theorem 5. QED

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