Note di Matematica
Note Mat. **39** (2019) no. 2, 87–93.
ISSN 1123-2536, e-ISSN 1590-0932
doi:10.1285/i15900932v39n2p87

Normal subgroups of finite non-abelian metacyclic p-groups of class two of odd order

Pradeep Kumar
Department of Mathematics, Central University of South Bihar, India
14p.shaoran@gmail.com

Received: 23.2.2019; accepted: 8.10.2019.

Abstract. In this paper, we determine the normal subgroups of a finite non-abelian metacyclic p-group of class two for odd prime p.

Keywords: Metacyclic p-groups, Subgroups, Normal subgroups

MSC 2000 classification: primary 20D15, secondary 20D60

Introduction

Determining the subgroups of finite groups is one of the important problems in finite group theory. In last century, the problem was completely solved for finite abelian groups (see [2, 5]). In [3], Calhoun determined the subgroups of ZM-groups (finite group with all Sylow subgroups are cyclic). Motivated by the paper of Calhoun [3], M. Tărnăuceanu determined the normal subgroups of ZM-groups [9]. In [9], Tărnăuceanu also proposed the following problem:

"Describe the normal subgroups of an arbitrary metacyclic group".

In this paper, we partially answer the above problem by determining the normal subgroups of finite non-abelian metacyclic p-groups of class two (p odd). A group G is said to be metacyclic if it contains a normal cyclic subgroup C with cyclic quotient group G/C. Throughout the paper groups will always be finite and $\mathbb{N}$ denotes the set of positive integers.

1 Basic Results

In this section, we give some results that will be needed later. First, we state a theorem that gives a presentation for a non-abelian metacyclic p-group of class two, p odd.

Theorem 1 ([1, Theorem 1.1]). *Let M be a non-abelian metacyclic p-group of class two, $p > 2$. Then M is isomorphic to the following group:*

$$M \cong \langle x, y \mid x^{p^r} = y^{p^s} = 1, [x, y] = x^{p^{r-\delta}} \rangle,$$

where $r, s, \delta \in \mathbb{N} = \{ 1, 2, \dots \}, r \geq 2\delta, s \geq \delta \geq 1$.

Lemma 1 ([8]). *Let G be a p-group of class two, and let $x, y, z \in G$. Then*

(i) $[xy, z] = [x, z][y, z]$,

(ii) $[x^m, y] = [x, y^m] = [x, y]^m$, *for any integer m.*

Now, we state the Goursat's Lemma related to the subgroups of the direct product of two groups.

Proposition 1 ([4, Goursat's Lemma]). *Let X and Y be arbitrary groups. Then there is a bijection between the set of all subgroups of $X \times Y$ and the set T of all 5-tuples (A, A', B, B', ϕ), where $A' \trianglelefteq A \leq X$, $B' \trianglelefteq B \leq Y$ and $\phi : A/A' \longrightarrow B/B'$ is an isomorphism. More precisely, the subgroup corresponding to (A, A', B, B', ϕ) is*

$$H = \{ (x, y) \in A \times B \mid \phi(xA') = yB' \}. \tag{1.1}$$

1.1 Subgroups of $\mathbb{Z}_{p^r} \times \mathbb{Z}_{p^s}$

In this subsection, we give a representation of subgroups of $\mathbb{Z}_{p^r} \times \mathbb{Z}_{p^s}$ given in [10]. With out loss of generality, let $r \geq s \geq 1$. With the notations used in Proposition 1, let $X = \mathbb{Z}_{p^r} = \langle x \rangle$ and $Y = \mathbb{Z}_{p^s} = \langle y \rangle$. Let $|A| = p^u, |A'| = p^v$, $|B| = p^q$, $|B'| = p^t$.

Further, let $A \leq X, A = \langle x^{p^{r-u}} \rangle$, where $0 \leq u \leq r$, $A' \leq A$, and $A' = \langle x^{p^{r-v}} \rangle$, where $0 \leq v \leq u$. Then $A/A' = \langle x^{p^{r-u}} A' \rangle$. Similarly, let $B \leq Y, B = \langle y^{p^{s-q}} \rangle$, where $0 \leq q \leq s$, $B' \leq B$, and $B' = \langle y^{p^{s-t}} \rangle$, where $0 \leq t \leq q$. Then $B/B' = \langle y^{p^{s-q}} B' \rangle$.

Again, for $|A/A'| = |B/B'|$ that is, $u - v = q - t$, the isomorphisms $\phi_l : A/A' \to B/B'$ are given by

$$\phi_l(x^{ip^{r-u}} A') = y^{ilp^{s-q}} B',$$

where $1 \leq l \leq p^{u-v}$ with $\gcd(l, p) = 1$.

Using Proposition 1, one can deduce that the subgroups H of $\mathbb{Z}_{p^r} \times \mathbb{Z}_{p^s}$ are of the form $H = \{ x^{ip^{r-u}} y^{jp^{s-q}} \in A \times B \mid (il - j)p^{s-q} \equiv 0 \mod p^{s-t} \}$ (for details see [7]).

Now, let

$$S_{p^r,p^s} := \{ (u,v,q,t,l) \mid 0 \le u \le r, 0 \le v \le u, 0 \le q \le s, 0 \le t \le q,\\ u-v = q-t, 1 \le l \le p^{u-v}, \text{ and } \gcd(l,p) = 1 \}.$$

For $(u,v,q,t,l) \in S_{p^r,p^s}$, define

$$H_{u,v,q,t,l} = \{ x^{ip^{r-u}} y^{jp^{s-q}} \mid (il-j)p^{s-q} \equiv 0 \mod p^{s-t}, 1 \le i \le p^u, \text{ and}\\ 1 \le j \le p^q \}. \tag{1.2}$$

Theorem 2 ([10, Theorem 3.1])**.** *The map $(u,v,q,t,l) \mapsto H_{u,v,q,t,l}$ is a bijection between the set S_{p^r,p^s} and the set of subgroups of $\mathbb{Z}_{p^r} \times \mathbb{Z}_{p^s}$, where $r,s \in \mathbb{N}$.*

2 Representation of Normal Subgroups of Finite Non-Abelian Metacyclic p-Groups

In this section, first we determine the subgroups of non-abelian metacyclic p-groups M of class two (p odd). For this, we use Baer's trick to construct an abelian group M_w corresponding to M. Then we show that there is a one-one correspondence between subgroups of M and M_w.

Let G be a group. If we can define a binary operation $\circ$ on G by

$$x \circ y = w(x,y)$$

where w is some fixed word in $x, y \in G$ such that the set G forms a group with operation $\circ$, then we say w to be a group-word for G, and we write the corresponding group by G_w, that is, as a set $G_w = G$ and operation of G_w is $\circ$.

Now if G is a p-group of class two, p odd, then we can define a group-word w for G as follows; for $x, y \in G$, $w(x,y) = x \circ y := xy[x,y]^{\frac{m-1}{2}}$ (where $[x,y] = x^{-1}y^{-1}xy$ and m is the exponent of $\gamma_2(G)$, the commutator subgroup of group G). Moreover, $x \circ y = xy[x,y]^{\frac{m-1}{2}} = yx[x,y]^{\frac{m+1}{2}} = yx[y,x]^{\frac{m-1}{2}} = y \circ x$. Thus the corresponding group G_w is abelian (for more details see [6, p. 142] and [7]). Now onward by w, we mean the group word defined as above, M denotes a non-abelian metacyclic p-group of class two (p odd), $n = \frac{m-1}{2}$ where m is the exponent of $\gamma_2(M)$, and M_w is the corresponding abelian group of M defined as above.

Proposition 2. *The corresponding abelian group of M is given by*

$$M_w \cong \langle x, y \mid x^{p^r} = y^{p^s} = 1,\ xy = yx \rangle.$$

Proof. Let $K = \langle x, y \mid x^{p^r} = y^{p^s} = 1,\ xy = yx \rangle$. As a set $M_w = M$. Now, take an element $g = x^i y^j \in M$. So $g = x^i \circ y^j \circ [x^i, y^j]^{-n} = x^{i - p^{r-\delta} ijn} \circ y^j$. Therefore, $M_w = \langle x, y \rangle$. Since powers of each element in M and M_w are same, so $x^{p^r} = y^{p^s} = 1$. Also, $x \circ y = y \circ x$. Thus, the generators of M_w satisfy the relations of K, so by Von Dyck's Theorem [8, p. 51], there is a surjective homomorphism $\phi : K \longrightarrow M_w$ with $x \to x$ and $y \to y$. Moreover, $|M_w| = |M| = p^{r+s}$. So, $|M_w| = |K|$. Thus, $M_w \cong K$. This completes the proof. QED

Note that to avoid ambiguity of operations, we write

$$M_w = \langle x, y \mid x^{p^r} = y^{p^s} = 1,\ x \circ y = y \circ x \rangle.$$

It is clear that $M_w \cong \mathbb{Z}_{p^r} \times \mathbb{Z}_{p^s}$.

Lemma 2. *A subset of M is a subgroup of M if and only if it is a subgroup of M_w.*

Proof. It is not hard to see that subgroups of M are subgroups of M_w. For converse, consider an arbitrary subgroup H of M_w. Using equation (1.2), the subgroup H is of the form

$$H = \{\, x^{ip^{r-u}} \circ y^{jp^{s-q}} \mid il \equiv j \mod p^{q-t}, 1 \le i \le p^u, \text{ and } 1 \le j \le p^q \,\},$$

where $q > t$ and for $q = t$,

$$H = \{\, x^{ip^{r-u}} \circ y^{jp^{s-q}} \mid 1 \le i \le p^u \text{ and } 1 \le j \le p^q \,\}.$$

Now, take $g_1, g_2 \in H$, where $g_1 = x^{ip^{r-u}} \circ y^{jp^{s-q}}$, and $g_2 = x^{i'p^{r-u}} \circ y^{j'p^{s-q}}$. To show that H is also a subgroup of M, it is sufficient to show that H is closed with the operation of M, that is, $g_1 g_2 \in H$.

We have $g_1 g_2 = g_1 \circ g_2 \circ [g_1, g_2]^{-n}$, where $n = \frac{m-1}{2}$, m is the exponent of $\gamma_2(M)$. Further, $[g_1, g_2] = [x^{ip^{r-u}} \circ y^{jp^{s-q}}, x^{i'p^{r-u}} \circ y^{j'p^{s-q}}]$, that is, in turn, equivalent to

$$[g_1, g_2] = [x, y]^{(ij' - ji')p^{r+s-u-q}} \qquad \text{(Lemma 1)}.$$

Now,

$$\begin{aligned} g_1 g_2 &= x^{ip^{r-u}} \circ y^{jp^{s-q}} \circ x^{i'p^{r-u}} \circ y^{j'p^{s-q}} \circ [x, y]^{-n(ij' - ji')p^{r+s-u-q}} \\ &= x^{\{\, i + i' - n(ij' - ji')p^{r-\delta+s-q} \,\}p^{r-u}} \circ y^{\{\, j + j' \,\}p^{s-q}} \qquad ([x, y] = x^{p^{r-\delta}}). \end{aligned}$$

For $q = t$, it is evident that $g_1 g_2 \in H$. Now, assume that $q > t$. Since $g_1, g_2 \in H$, the following equations hold

$$il \equiv j \mod p^{q-t}, \tag{2.1}$$

$$i'l \equiv j' \mod p^{q-t}. \tag{2.2}$$

Using (2.1) & (2.2), we deduce that $il+i'l \equiv j+j' \mod p^{q-t}$ and $(ij'-ji')l \equiv 0 \mod p^{q-t}$. Since $\gcd(l,p)=1$, we get $(ij'-ji') \equiv 0 \mod p^{q-t}$. Thus, we conclude that

$$\{\, i+i'-n(ij'-ji')p^{r-\delta+s-q}\,\}l \equiv j+j' \mod p^{q-t}.$$

Thus, $g_1g_2 \in H$. This completes the proof. QED

Now, we determine the normal subgroups of non-abelian metacyclic group M of class two (p odd).

By Theorem 1, we have

$$M \cong \langle x,y \mid x^{p^r} = y^{p^s} = 1, [x,y] = x^{p^{r-\delta}} \rangle, \qquad (*)$$

where $r,s,\delta \in \mathbb{N}, r \geq 2\delta, s \geq \delta \geq 1$. By subsection 1.1 and Lemma 2, the subgroups of M are of the form

$$H_{u,v,q,t,l} \cong \{\, x^{ip^{r-u}} \circ y^{jp^{s-q}} \mid (il-j)p^{s-q} \equiv 0 \mod p^{s-t}, 1 \leq i \leq p^u, \text{ and } 1 \leq j \leq p^q \,\},$$

where, $(u,v,q,t,l) \in S_{p^r,p^s}$.

Lemma 3. *For $(u,v,q,t,l) \in S_{p^r,p^s}$, the subgroup $H_{u,v,q,t,l}$ of M is a normal subgroup if and only if $u-\delta+s-q \geq q-t$ and $r-\delta \geq q-t$.*

Proof. The subgroup $H_{u,v,q,t,l}$ is a normal subgroup if and only if $g^{-1}hg \in H_{u,v,q,t,l}$ for every $g \in M$ and $h \in H_{u,v,q,t,l}$. Take an element $g = x^ay^b \in M$ and $h = x^{ip^{r-u}} \circ y^{jp^{s-q}} \in H_{u,v,q,t,l}$. Now, we have $g^{-1}hg = h[h,g] = h \circ [h,g]$. Thus

$$\begin{aligned} g^{-1}hg &= h \circ [h,g] \\ &= x^{ip^{r-u}} \circ y^{jp^{s-q}} \circ [x^{ip^{r-u}} \circ y^{jp^{s-q}}, x^ay^b] \\ &= x^{ip^{r-u}} \circ y^{jp^{s-q}} \circ [x,y]^{ibp^{r-u}-ajp^{s-q}} && \text{(Lemma 1)} \\ &= x^{ip^{r-u}+p^{r-\delta}(ibp^{r-u}-ajp^{s-q})} \circ y^{jp^{s-q}} && ([x,y] = x^{p^{r-\delta}}). \end{aligned}$$

Let $H_{u,v,q,t,l}$ be a normal subgroup of M. Now, if $g^{-1}hg \in H_{u,v,q,t,l}$, then

$$ip^{r-u} + p^{r-\delta}(ibp^{r-u} - ajp^{s-q}) \equiv 0 \mod p^{r-u},$$

and that is equivalent to $ajp^{r-\delta+s-q} \equiv 0 \mod p^{r-u}$. The latter equation must hold for every possible a,j. Thus $p^{r-u} \mid p^{r-\delta+s-q}$. So, $r-\delta+s-q \geq r-u$. This implies $u-\delta+s-q \geq 0$. Now, assume that $u-\delta+s-q \geq 0$, then $g^{-1}hg = x^{(i+ibp^{r-\delta}-ajp^{u-\delta+s-q})p^{r-u}} \circ y^{jp^{s-q}}$. Now, if $g^{-1}hg \in H$, then $[(i+ibp^{r-\delta}-$

$ajp^{u-\delta+s-q})l - j]p^{s-q} \equiv 0 \mod p^{s-t}$. For $q = t$, the latter equation always holds. Now, suppose $q > t$, then $(i + ibp^{r-\delta} - ajp^{u-\delta+s-q})l \equiv j \mod p^{q-t}$. Since $h \in H$, so $il \equiv j \mod p^{q-t}$. Thus we have that $(ibp^{r-\delta} - ajp^{u-\delta+s-q})l \equiv 0 \mod p^{q-t}$. Since $\gcd(l, p) = 1$, $(ibp^{r-\delta} - ajp^{u-\delta+s-q}) \equiv 0 \mod p^{q-t}$. The latter equation must hold for every a, b and i, j such that $h \in H, g \in M$. This implies $p^{u-\delta+s-q} \equiv 0 \mod p^{q-t}$ and $p^{r-\delta} \equiv 0 \mod p^{q-t}$. Thus $u - \delta + s - q \geq q - t$ and $r - \delta \geq q - t$. It is not hard to see that converse part holds. This completes the proof. QED

Now, for every $r, s, \delta \in \mathbb{N}$ such that $r \geq 2\delta, s \geq \delta \geq 1$, let

$$J'_{p^r,p^s} := \{\, (u, v, q, t, l) \mid 0 \leq u \leq r, 0 \leq v \leq u, 0 \leq q \leq s, 0 \leq t \leq q, u - v = q - t, \\ 1 \leq l \leq p^{u-v},\ \gcd(l, p) = 1, u - \delta + s - q \geq q - t,\ \text{and}\ r - \delta \geq q - t \,\}.$$

For $(u, v, q, t, l) \in J'_{p^r,p^s}$, define

$$N_{u,v,q,t,l} := \{\, x^{ip^{r-u}} y^{jp^{s-q}} [x, y]^{nijp^{r-u+s-q}} \mid (il - j)p^{s-q} \equiv 0 \mod p^{s-t}, \\ 1 \leq i \leq p^u,\ \text{and}\ 1 \leq j \leq p^q \,\}.$$

Theorem 3. *The map $(u, v, q, t, l) \mapsto N_{u,v,q,t,l}$ is a bijection between the set J'_{p^r,p^s} and the set of normal subgroups of non-abelian metacyclic p-group M of class two (p odd) as in $(*)$.*

Proof. This follows from Lemmas 2, 3 and Theorem 2. QED

Acknowledgements. The author would like to thank the referee for his very valuable comments and suggestions.

References

[1] J. R. Beuerle: *An elementary classification of finite metacyclic p-groups of class at least three*, Algebr. Colloq. **12** (2005), 553–562.

[2] G. Bhowmik: *Evaluation of divisor functions of matrices*, Acta Arith. **74** (1996), 155–154.

[3] W. C. Calhoun: *Counting the subgroups of some finite groups*, Amer. Math. Monthly **94** (1987), 54–59.

[4] R. R. Crawford, K. D. Wallace: *On the number of subgroups of index two-an application of Goursat's theorem for groups*, Math. Mag. **48** (1975), 172–174.

[5] S. Delsarte: *Fonctions de möbius sur les groupes abeliens finis*, Ann. of Math. **49** (1948), 600–609.

[6] I. M. Isaacs: Finite Group Theory, Graduate Studies in Mathematics, Vol. 92, Amer. Math. Soc. Providence, 2008.

[7] P. Kumar, V. K. Jain: *On the number of subgroups of non-metacyclic minimal non-abelian p-groups*, Asian-Eur. J. Math, doi: 10.1142/S1793557120500941.

[8] D. J. S. Robinson: A Course in the Theory of Groups, 2nd Edition, Springer, New York, 1996.

[9] M. Tărnăuceanu, *The normal subgroup structure of ZM-groups*, Ann. Mat. Pura Appl. **193** (2014), 1085–1088.

[10] L. Tóth, *Subgroups of finite abelian groups having rank two via Goursat's lemma*, Tatra Mt. Math. Publ. **59** (2014), 93–103.