Note di Matematica
Note Mat. **39** (2019) no. 2, 13–23.
ISSN 1123-2536, e-ISSN 1590-0932
doi:10.1285/i15900932v39n2p13

Evolutoids and pedaloids of plane curves

Shyuichi IZUMIYA
Department of Mathematics, Hokkaido University
izumiya@math.sci.hokudai.ac.jp

Nobuko TAKEUCHI
Department of Mathematics, Tokyo Gakugei University
nobuko@u-gakugei.ac.jp

Received: 28.2.2019; accepted: 28.6.2019.

Abstract. Evolutes and pedals of plane curves have been well investigated since the beginning of the history of differential geometry. However, there might be no direct relationships between the pedal and the evolute of a curve. We introduce families of relatives of pedals and evolutes and investigate some relationships between these families curves. Moreover, we generalize these notions to the category of frontal curves. Then the relation can be completely described in this category.

Keywords: plane curve, pedal, pedaloids, evolute, evolutoids

MSC 2000 classification: primary 53A04, secondary 53A05

1 Introduction

The notions of pedals and evolutes of regular curves in the Euclidean plane are classical subjects, whose singular points characterize the inflection points and the vertices of curves (cf. [2, 6]). In this paper we introduce relatives of pedals of plane curves and investigate some relationships between these curves.

Let $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^2$ be a unit speed plane curve, where $\mathbb{R}^2$ is the Euclidean plane with the canonical scaler product $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = a_1b_1 + a_2b_2$ for $\boldsymbol{a} = (a_1, a_2), \boldsymbol{b} = (b_1, b_2) \in \mathbb{R}^2$. The *norm* of $\boldsymbol{a}$ is defined to be $\|\boldsymbol{a}\| = \sqrt{\langle \boldsymbol{a}, \boldsymbol{a} \rangle} = \sqrt{a_1^2 + a_2^2}$. Then we have the Frenet formulae:

$$\begin{cases} \boldsymbol{t}'(s) = \kappa(s)\boldsymbol{n}(s), \\ \boldsymbol{n}'(s) = -\kappa(s)\boldsymbol{t}(s), \end{cases}$$

where $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ is the *unit tangent vector*, $\boldsymbol{n}(s) = J(\boldsymbol{t}(s))$ is the *unit normal vector*, J is the anti-clockwise rotation by $\pi/2$ on $\mathbb{R}^2$ and $\kappa(s) = x_1'(s)x_2''(s) - x_1''(s)x_2'(s)$ is the *curvature* of $\boldsymbol{\gamma}(s) = (x_1(s), x_2(s))$. Although the notion of pedals is defined for an arbitrary chosen point in $\mathbb{R}^2$, we choose the origin. A *pedal* of $\boldsymbol{\gamma}$ (with respect to $\boldsymbol{0} \in \mathbb{R}^2$) is defined to be $\mathrm{Pe}_{\boldsymbol{\gamma}}(s) = \langle \boldsymbol{\gamma}(s), \boldsymbol{n}(s) \rangle \boldsymbol{n}(s)$

(cf. [2, Page 36]). The pedal of a curve is know to be the locus of the projection image of the origin to the tangent line along the normal direction at each point of the curve. Since $\mathrm{Pe}'_{\boldsymbol{\gamma}}(s) = -\kappa(s)(\langle \boldsymbol{\gamma}(s), \boldsymbol{t}(s)\rangle \boldsymbol{n}(s) + \langle \boldsymbol{\gamma}(s), \boldsymbol{n}(s)\rangle \boldsymbol{t}(s))$, the singular points of the pedal of $\boldsymbol{\gamma}$ is the point s_0 where $\boldsymbol{\gamma}(s_0) = \mathbf{0}$ or $\kappa(s_0) = 0$ (i.e. the inflection point of $\boldsymbol{\gamma}$). If we assume that $\boldsymbol{\gamma}$ does not pass through the origin, the singular points of the pedal $\mathrm{Pe}_{\boldsymbol{\gamma}}$ are the inflection points of $\boldsymbol{\gamma}$. It is known that the pedal is given as the envelope of a family of circles as follows (cf. [2, Page 166]): Let $G : I \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function defined by

$$G(s, \boldsymbol{x}) = \left\| \boldsymbol{x} - \frac{1}{2}\boldsymbol{\gamma}(s) \right\|^2 - \frac{1}{4}\|\boldsymbol{\gamma}(s)\|^2 = \langle \boldsymbol{x}, \boldsymbol{x} - \boldsymbol{\gamma}(s)\rangle.$$

If we fix $s_0 \in I$, $G(s_0, \boldsymbol{x}) = 0$ is the equation of the circle with the center $\frac{1}{2}\boldsymbol{\gamma}(s_0)$ which passes through the origin. We have the following calculation: $(\partial G/\partial s)(s, \boldsymbol{x}) = \langle \boldsymbol{x}, -\boldsymbol{t}(s)\rangle$. Since $\{\boldsymbol{t}(s), \boldsymbol{n}(s)\}$ is an orthonormal basis of $\mathbb{R}^2$, we write $\boldsymbol{x} = \lambda \boldsymbol{t}(s) + \mu \boldsymbol{n}(s)$. Then $G(s, \boldsymbol{x}) = (\partial G/\partial s)(s, \boldsymbol{x}) = 0$ if and only if $\lambda = 0$ and $\mu(\mu - \langle \boldsymbol{n}(s), \boldsymbol{\gamma}(s)\rangle) = 0$. The last condition is equivalent to $\mu = 0$ or $\boldsymbol{x} = \langle \boldsymbol{\gamma}(s), \boldsymbol{n}(s)\rangle \boldsymbol{n}(s)$. This means that the pedal $\mathrm{Pe}_{\boldsymbol{\gamma}}(s)$ is the envelope of the above family of the circles. As an application of the unfolding theory of functions of one-variable, we have the following proposition (cf. [2, Page 166]).

Proposition 1.1. The pedal $\mathrm{Pe}_{\boldsymbol{\gamma}}$ of a curve $\boldsymbol{\gamma}$ around $\mathrm{Pe}_{\boldsymbol{\gamma}}(s_0)$ is locally diffeomorphic to the *ordinary cusp* $C = \{\ (x, y) \in \mathbb{R} \ \mid \ x = t^2, y = t^3\ \}$ if and only if $\kappa(s_0) = 0$ and $\kappa'(s_0) \neq 0$.

On the other hand, we define an *evolute* of $\boldsymbol{\gamma}$ by

$$\mathrm{Ev}_{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(s) + \frac{1}{\kappa(s)}\boldsymbol{n}(s),$$

for $\kappa(s) \neq 0$. The evolute is known to be the locus of the center of osculating circles. Since we have $\mathrm{Ev}'_{\gamma}(s) = -(\kappa'/\kappa^2)(s)\boldsymbol{n}(s)$, $s_0 \in I$ is a singular point of Ev_{γ} if and only if $\kappa'(s_0) = 0$ (i.e. the vertex of $\boldsymbol{\gamma}$). There might be no good relationships between the evolute and the pedal of a curve.

It is classically known that the evolute is the envelope of the family of normal lines along the curve $\boldsymbol{\gamma}$ (cf. [2, 6]). Moreover, the envelope of the family of tangent lines along $\boldsymbol{\gamma}$ is the original curve, together the tangent line at each inflection point of $\boldsymbol{\gamma}$. In [5] Giblin and Warder introduced the notion of evolutoids as a one-parameter family of associated curves of $\boldsymbol{\gamma}$, which fills in the gap between the evolute and the original curve. Each member of the evolutoids is defined as the envelope the family of lines such that each line has a constant angle with the tangent line at the same point of the original curve. Analogous to the evolutoids, we define the notion of pedaloids in this paper, which are a family of relatives of the pedal of a curve. By using these notions, we can clarify a relationship between the pedaloids and the evolutoids of a regular curve (cf. §2). Finally, we consider these notions for frontal curves and categorically complete those relationships (cf. §3).

We assume that all maps are class C^∞ throughout the whole paper unless the contrary is explicitly stated.

2 Evolutoids and pedaloids

In [5] the notion of *evolutoids* was defined as follows: For $\phi \in [0, 2\pi)$, define

$$\mathrm{Ev}[\phi]_{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(s) + \frac{\sin\phi}{\kappa(s)}(\cos\phi \boldsymbol{t}(s) + \sin\phi \boldsymbol{n}(s)),$$

where $\kappa(s) \neq 0$, which is called an *evolutoid* for each ϕ. In this paper we call it a *ϕ-evolutoid* of $\boldsymbol{\gamma}$. If we define a family of functions $F : I \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $F(s, \boldsymbol{x}) = \langle \boldsymbol{x} - \boldsymbol{\gamma}(s), \sin\phi \boldsymbol{t}(s) - \cos\phi \boldsymbol{n}(s)\rangle$. Then $f_s^{-1}(0)$ for $f_s(\boldsymbol{x}) = F(s, \boldsymbol{x})$ is a line through $\boldsymbol{\gamma}(s)$ such that the angle of $f_s^{-1}(0)$ with $\boldsymbol{t}(s)$ is ϕ. We can calculate that the envelope of the family $\{ f_s^{-1}(0) \}_{s\in I}$ is the ϕ-evolutoid (cf. [5]). If $\phi = \pi/2, 3\pi/2$, then $\mathrm{Ev}[\phi]_{\boldsymbol{\gamma}}(s) = \mathrm{Ev}_{\boldsymbol{\gamma}}(s)$. If $\phi = 0, \pi$, then $\mathrm{Ev}[\phi]_{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(s)$. For any ϕ, we can calculate that

$$\mathrm{Ev}[\phi]'_{\boldsymbol{\gamma}}(s) = \left(\cos\phi - \frac{\kappa'}{\kappa^2}(s)\sin\phi\right)(\cos\phi \boldsymbol{t}(s) + \sin\phi \boldsymbol{n}(s)),$$

so that $s_0 \in I$ is a singular point of $\mathrm{Ev}[\phi]_{\boldsymbol{\gamma}}$ if and only if $\kappa^2(s_0)\cos\phi - \kappa'(s_0)\sin\phi = 0$.

On the other hand, we introduce the notion of pedaloids. For $\psi \in [0, 2\pi)$, define

$$\mathrm{Pe}[\psi]_{\boldsymbol{\gamma}}(s) = \langle \boldsymbol{\gamma}(s), \cos\psi \boldsymbol{t}(s) + \sin\psi \boldsymbol{n}(s)\rangle(\cos\psi \boldsymbol{t}(s) + \sin\psi \boldsymbol{n}(s))$$

We call it a *ψ-pedaloid* of $\boldsymbol{\gamma}$. If $\psi = \pi/2, 3\pi/2$, then $\mathrm{Pe}[\psi]_{\boldsymbol{\gamma}}(s) = \mathrm{Pe}_{\boldsymbol{\gamma}}(s)$. If $\psi = 0, \pi$, then $\mathrm{Pe}[\psi]_{\boldsymbol{\gamma}}(s) = \langle \boldsymbol{\gamma}(s), \boldsymbol{t}(s)\rangle \boldsymbol{t}(s)$, which is known as a *contrapedal* (i.e. a *C-pedal* for short) (cf. [8]). We write the C-pedal by $\mathrm{CPe}_{\boldsymbol{\gamma}}(s)$. We calculate that

$$\begin{aligned}\mathrm{Pe}[\psi]'_{\boldsymbol{\gamma}}(s) &= \{\cos^2\psi + \kappa(s)(\cos 2\psi\langle \boldsymbol{\gamma}(s), \boldsymbol{n}(s)\rangle - \sin 2\psi\langle \boldsymbol{\gamma}(s), \boldsymbol{t}(s)\rangle)\}\boldsymbol{t}(s) \\ &+ \{\cos\psi\sin\psi + \kappa(s)(\sin 2\psi\langle \boldsymbol{\gamma}(s), \boldsymbol{n}(s)\rangle + \cos 2\psi\langle \boldsymbol{\gamma}(s), \boldsymbol{t}(s)\rangle)\}\boldsymbol{n}(s).\end{aligned}$$

We can show that $s_0 \in I$ is a singular point of $\mathrm{Pe}[\psi]_{\boldsymbol{\gamma}}$ if and only if

$$\kappa(s_0)\langle \boldsymbol{\gamma}(s_0), \boldsymbol{t}(s_0)\rangle = \cos\psi\sin\psi,\ \ \kappa(s_0)\langle \boldsymbol{\gamma}(s_0), \boldsymbol{n}(s_0)\rangle = -\cos^2\psi.$$

Then we have the following beautiful relation between evolutoids and pedaloids.

Theorem 2.1. Suppose that $\kappa(s) \neq 0$, $\kappa^2(s)\sin\psi - \kappa'(s)\cos\psi \neq 0$ (i.e. the $(\psi+\pi/2)$-evolutoid is non-singular). Then we have

$$\mathrm{Pe}_{\mathrm{Ev}[\psi+\pi/2]_{\boldsymbol{\gamma}}}(s) = \mathrm{Pe}[\psi]_{\boldsymbol{\gamma}}(s).$$

Proof. We define a family of functions $G[\psi] : I \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$\begin{aligned} G[\psi](s,\boldsymbol{x}) &= \left\| \boldsymbol{x} - \frac{1}{2}\mathrm{Ev}[\psi+\pi/2]_{\boldsymbol{\gamma}}(s) \right\|^2 - \frac{1}{4}\left\|\mathrm{Ev}[\psi+\pi/2]_{\boldsymbol{\gamma}}(s)\right\|^2 \\ &= \langle \boldsymbol{x}, \boldsymbol{x} - \mathrm{Ev}[\psi+\pi/2]_{\boldsymbol{\gamma}}(s)\rangle. \end{aligned}$$

By the assumption, $\mathrm{Ev}[\psi+\pi/2]_{\boldsymbol{\gamma}}(s)$ is a regular curve, so that the envelope given by the family $G[\psi](s,\boldsymbol{x}) = 0$ is the pedal $\mathrm{Pe}_{\mathrm{Ev}[\psi+\pi/2]_{\boldsymbol{\gamma}}}$ of $\mathrm{Ev}[\psi+\pi/2]_{\boldsymbol{\gamma}}(s)$. For any ϕ, since

$$\mathrm{Ev}[\phi]'_{\boldsymbol{\gamma}}(s) = \left(\cos\phi - \frac{\kappa'}{\kappa^2}(s)\sin\phi\right)(\cos\phi\boldsymbol{t}(s) + \sin\phi\boldsymbol{n}(s)),$$

we have

$$\frac{\partial G[\psi]}{\partial s}(s,\boldsymbol{x}) = \left(\cos\phi - \frac{\kappa'}{\kappa^2}(s)\sin\phi\right)\langle \boldsymbol{x}, \cos\phi\boldsymbol{t}(s) + \sin\phi\boldsymbol{n}(s)\rangle,$$

where $\phi = \psi + \pi/2$. Since $\{\boldsymbol{t}(s), \boldsymbol{n}(s)\}$ is a orthonormal frame, there exist $\lambda, \mu \in \mathbb{R}$ such that $\boldsymbol{x} = \lambda\boldsymbol{t}(s) + \mu\boldsymbol{n}(s)$. By the assumption, $\frac{\partial G[\psi]}{\partial s}(s,\boldsymbol{x}) = 0$ if and only if $\lambda\cos(\psi+\pi/2) + \mu\sin(\psi+\pi/2) = 0$, which is equivalent to $\lambda\sin\psi - \mu\cos\psi = 0$. For $\phi = \psi + \pi/2$, we have

$$\begin{aligned} G[\phi](s,\boldsymbol{x}) &= \lambda^2 + \mu^2 - \lambda\langle \boldsymbol{t}(s), \boldsymbol{\gamma}(s)\rangle - \mu\langle \boldsymbol{n}(s), \boldsymbol{\gamma}(s)\rangle + \frac{\sin\phi}{\kappa(s)}(\lambda\cos\phi + \mu\sin\phi) \\ &= \lambda^2 + \mu^2 - \lambda\langle \boldsymbol{t}(s), \boldsymbol{\gamma}(s)\rangle - \mu\langle \boldsymbol{n}(s), \boldsymbol{\gamma}(s)\rangle. \end{aligned}$$

Suppose that $\psi \neq 0, \pi$ (i.e. $\sin\psi = \cos\phi \neq 0$). Then $G[\phi](s,\boldsymbol{x}) = 0$ if and only if $\mu^2 + \mu\cos\phi(\sin\phi\langle \boldsymbol{t}, \boldsymbol{\gamma}\rangle - \cos\phi\langle \boldsymbol{n}, \boldsymbol{\gamma}\rangle) = 0$. The last condition is equivalent to $\lambda = \mu = 0$ or $\mu \neq 0$. Suppose that $\mu \neq 0$. Then

$$\mu = \cos^2\phi\langle \boldsymbol{n}, \boldsymbol{\gamma}\rangle - \cos\phi\sin\phi\langle \boldsymbol{t}, \boldsymbol{\gamma}\rangle,\ \ \lambda = -\sin\phi\cos\phi\langle \boldsymbol{n}, \boldsymbol{\gamma}\rangle + \sin^2\phi\langle \boldsymbol{t}, \boldsymbol{\gamma}\rangle.$$

It follows that

$$\begin{aligned} \boldsymbol{x} &= -(\cos\phi\langle \boldsymbol{n}(s), \boldsymbol{\gamma}(s)\rangle + \sin\phi\langle \boldsymbol{t}(s), \boldsymbol{\gamma}(s)\rangle)\sin\phi\boldsymbol{t}(s) \\ &\quad +(\cos\phi\langle \boldsymbol{n}(s), \boldsymbol{\gamma}(s)\rangle - \sin\phi\langle \boldsymbol{t}(s), \boldsymbol{\gamma}(s)\rangle)\cos\phi\boldsymbol{n}(s) \\ &= \langle \cos\psi\boldsymbol{t}(s) + \sin\psi\boldsymbol{n}(s), \boldsymbol{\gamma}(s)\rangle(\cos\psi\boldsymbol{t}(s) + \sin\psi\boldsymbol{n}(s)) = \mathrm{Pe}[\psi]_{\boldsymbol{\gamma}}(s). \end{aligned}$$

If $\lambda = \mu = 0$, then $\boldsymbol{x} = \boldsymbol{0}$, so that $\boldsymbol{x}$ satisfies the condition $G[\phi](s_0, \boldsymbol{x}) = \frac{\partial G[\phi]}{\partial s}(s_0, \boldsymbol{x}) = 0$. Moreover, we have

$$\langle \sin\phi \boldsymbol{t}(s_0) - \cos\phi \boldsymbol{n}(s_0), \boldsymbol{\gamma}(s_0)\rangle = \sin\phi\langle \boldsymbol{t}(s_0), \boldsymbol{\gamma}(s_0)\rangle - \cos\phi\langle \boldsymbol{n}(s_0), \boldsymbol{\gamma}(s_0)\rangle = 0.$$

Therefore, we have $\mathrm{Pe}[\psi]_{\boldsymbol{\gamma}}(s_0) = \boldsymbol{0} = \boldsymbol{x}$. This means that the envelope of the family defined by the family $G[\psi](s, \boldsymbol{x}) = 0$ is the ψ-pedaloid $\mathrm{Pe}[\psi]_{\boldsymbol{\gamma}}(s)$. For $\psi = 0, \pi/2$, it is a trivial equality. □

We have the following simple corollary ([8, pp. 151]).

Corollary 2.2. Suppose that $\kappa(s) \neq 0$, $\kappa'(s) \neq 0$ and the evolute of $\boldsymbol{\gamma}$ does not pass through the origin of $\mathbb{R}^2$. Then the C-pedal of $\boldsymbol{\gamma}$ is the pedal of the evolute of $\boldsymbol{\gamma}$:

$$\mathrm{CPe}_{\boldsymbol{\gamma}}(s) = \mathrm{Pe}_{\mathrm{Ev}_{\boldsymbol{\gamma}}}(s).$$

3 Pedaloids and evolutoids of frontals

In the previous sections we investigated the pedaloids and the evolutoids of regular curves. However, the pedaloids and the evolutoids generally have singularities even for regular curves. In the last section we consider the pedal of the evolutoid of a curve. Therefore, we need to generalize the notions of pedals of certain singular curves. One of the natural singular curves in the Euclidean plane for which we can develop the differential geometry is the notion of frontals [3, 4].

We say that $(\boldsymbol{\gamma}, \boldsymbol{\nu}) : I \longrightarrow \mathbb{R}^2 \times S^1$ is a *Legendrian curve* if $(\boldsymbol{\gamma}, \boldsymbol{\nu})^*\theta = 0$, where θ is the canonical contact 1-form on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ (cf. [1]). The last condition is equivalent to $\langle \dot{\boldsymbol{\gamma}}(t), \boldsymbol{\nu}(t)\rangle = 0$ for any $t \in I$. We say that $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^2$ is a *frontal* if there exists $\boldsymbol{\nu} : I \longrightarrow S^1$ such that $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ is a Legendrian curve. If $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ is an immersion, $\boldsymbol{\gamma}$ is said to be a *front*. A differential geometry on frontals was constructed in [4]. For a Legendrian curve $(\boldsymbol{\gamma}, \boldsymbol{\nu}) : I \longrightarrow \mathbb{R}^2 \times S^1$, we define a unit vector field $\boldsymbol{\mu}(t) = J(\boldsymbol{\nu}(t))$ along $\boldsymbol{\gamma}$. Then we have the following Frenet type formulae [4]:

$$\begin{cases} \dot{\boldsymbol{\nu}}(t) = \ell(t)\boldsymbol{\mu}(t), \\ \dot{\boldsymbol{\mu}}(t) = -\ell(t)\boldsymbol{\nu}(t), \end{cases}$$

where $\ell(t) = \langle \dot{\boldsymbol{\nu}}(t), \boldsymbol{\mu}(t)\rangle$. Moreover, there exists $\beta(t)$ such that $\dot{\boldsymbol{\gamma}}(t) = \beta(t)\boldsymbol{\mu}(t)$ for any $t \in I$. The pair (ℓ, β) is called a *curvature of the Legendrian curve* $(\boldsymbol{\gamma}, \boldsymbol{\nu})$. By definition, $t_0 \in I$ is a singular point of $\boldsymbol{\gamma}$ if and only if $\beta(t_0) = 0$. Moreover, for a regular curve $\boldsymbol{\gamma}$, $\boldsymbol{\mu}(t) = \boldsymbol{t}(t)$ and $\ell(t) = \|\dot{\boldsymbol{\gamma}}(t)\|\kappa(t)$. The Legendrian curve $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ is immersive (i.e. $\boldsymbol{\gamma}$ is a front) if and only if $(\ell(t), \beta(t)) \neq (0, 0)$ for any

$t \in I$. So the inflection point $t_0 \in I$ of the frontal $\boldsymbol{\gamma}$ is a point $\ell(t_0) = 0$. For more detailed properties of Legendrian curves, see [3, 4].

In [4], the *evolute* of a frontal $\boldsymbol{\gamma}$ is defined by

$$\mathcal{E}v_{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}(t) - \alpha(t)\boldsymbol{\nu}(t),$$

with the assumption that there exists $\alpha(t)$ such that $\beta(t) = \alpha(t)\ell(t)$ for any $t \in I$. Moreover, $\mathcal{E}v_{\boldsymbol{\gamma}}(t)$ is a frontal and it is a front if $\boldsymbol{\gamma}$ is a front [3, 4].

Let $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ be a Legendrian curve. We assume that there exists $\alpha(t)$ such that $\beta(t) = \alpha(t)\ell(t)$ for any $t \in I$. Then we define a ϕ-*evolutoid* of $\boldsymbol{\gamma}$ by

$$\mathcal{E}v_{\boldsymbol{\gamma}}[\phi](t) = \boldsymbol{\gamma}(t) - \alpha(t)\sin\phi(\cos\phi\boldsymbol{\mu}(t) + \sin\phi\boldsymbol{\nu}(t)).$$

We have $\mathcal{E}v_{\boldsymbol{\gamma}}[0](t) = \mathcal{E}v_{\boldsymbol{\gamma}}[\pi](t) = \boldsymbol{\gamma}(t)$ and $\mathcal{E}v_{\boldsymbol{\gamma}}[\pi/2](t) = \mathcal{E}v_{\boldsymbol{\gamma}}[3\pi/2](t) = \mathcal{E}v_{\boldsymbol{\gamma}}(t)$. Then we have the following proposition.

Proposition 3.1. Let $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ be a Legendrian curve with the curvatue (ℓ, β). Suppose that there exists $\alpha(t)$ such that $\beta(t) = \alpha(t)\ell(t)$ for any $t \in I$. Then the ϕ-evolutoid $\mathcal{E}v_{\boldsymbol{\gamma}}[\phi]$ is a frontal with the curvature (ℓ, ℓ). If $\ell(t) \neq 0$ for any $t \in I$, then the ϕ-evolutoid $\mathcal{E}v_{\boldsymbol{\gamma}}[\phi]$ is a front.

Proof. By a straightforward calculation, we have

$$\frac{d}{dt}\mathcal{E}v_{\boldsymbol{\gamma}}[\phi](t) = (\beta(t)\cos\phi - \dot{\alpha}(t)\sin\phi)(\cos\phi\boldsymbol{\mu}(t) + \sin\phi\boldsymbol{\nu}(t)).$$

Then we define

$$\boldsymbol{\nu}[\phi](t) = J(\cos\phi\boldsymbol{\mu}(t) + \sin\phi\boldsymbol{\nu}(t)) = -\cos\phi\boldsymbol{\nu}(t) + \sin\phi\boldsymbol{\mu}(t).$$

If we consider $(\mathcal{E}v_{\boldsymbol{\gamma}}[\phi], \boldsymbol{\nu}[\phi]) : I \longrightarrow \mathbb{R}^2 \times S^1$, then we have

$$\left\langle \frac{d}{dt}\mathcal{E}v_{\boldsymbol{\gamma}}[\phi](t), \boldsymbol{\nu}[\phi](t) \right\rangle = 0,$$

so that $(\mathcal{E}v_{\boldsymbol{\gamma}}[\phi], \boldsymbol{\nu}[\phi])$ is a Legendrian curve. This means that $\mathcal{E}v_{\boldsymbol{\gamma}}[\phi]$ is a frontal.

On the other hand, we have

$$\boldsymbol{\mu}[\phi](t) = J\boldsymbol{\nu}[\phi](t) = -(\sin\phi\boldsymbol{\nu}(t) + \cos\phi\boldsymbol{\mu}(t)).$$

Moreover, we have

$$\frac{d}{dt}\boldsymbol{\nu}[\phi](t) = -\ell(t)(\cos\phi\boldsymbol{\mu}(t) + \sin\phi\boldsymbol{\nu}(t)),$$

so that

$$\ell[\phi](t) = \left\langle \frac{d}{dt}\boldsymbol{\nu}[\phi](t), \boldsymbol{\mu}[\phi](t) \right\rangle = \ell(t),\ \frac{d}{dt}\boldsymbol{\nu}[\phi](t) = \ell(t)\boldsymbol{\mu}[\phi](t).$$

Thus, the curvature of $\mathcal{E}v_{\boldsymbol{\gamma}}[\phi]$ is (ℓ, ℓ). If $\ell(t) \neq 0$, then $(\mathcal{E}v_{\boldsymbol{\gamma}}[\phi], \boldsymbol{\nu}[\phi])$ is immersive. This competes the proof. $\square$

We also define a *ψ-pedaloid* of a frontal $\boldsymbol{\gamma}$ by

$$\mathcal{P}e[\psi]_{\boldsymbol{\gamma}}(t) = \langle \boldsymbol{\gamma}(t), \cos\psi\boldsymbol{\mu}(t) + \sin\psi\boldsymbol{\nu}(t)\rangle(\cos\psi\boldsymbol{\mu}(t) + \sin\psi\boldsymbol{\nu}(t)).$$

Here, we can define the ψ-pedaloid for a Legendrian curve $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ without any assumptions. We denote that

$$\mathcal{P}e_{\boldsymbol{\gamma}}(t) = \mathcal{P}e[\pi/2]_{\boldsymbol{\gamma}}(t) = \mathcal{P}e[3\pi/2]_{\boldsymbol{\gamma}}(t), \mathcal{CP}e_{\boldsymbol{\gamma}}(t) = \mathcal{P}e[0]_{\boldsymbol{\gamma}}(t) = \mathcal{P}e[\pi]_{\boldsymbol{\gamma}}(t).$$

We call $\mathcal{P}e_{\boldsymbol{\gamma}}(t)$ a *pedal* of $\boldsymbol{\gamma}$ and $\mathcal{CP}e_{\boldsymbol{\gamma}}(t)$ a *contrapedal* of $\boldsymbol{\gamma}$. It is rather hard to show that the ψ-pedaloid for a Legendrian curve is a frontal. Here, we only show the following proposition, (see also [7]).

Proposition 3.2. Let $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ be a Legendrian curve with the curvatue (ℓ, β). Suppose that there exist $\delta(t)$ and $\boldsymbol{\sigma} : I \longrightarrow S^1$such that $\boldsymbol{\gamma}(t) = \delta(t)\boldsymbol{\sigma}(t)$ for any $t \in I$. Then the pedal $\mathcal{P}e_{\boldsymbol{\gamma}}(t)$ of $\boldsymbol{\gamma}$ is a frontal.

Proof. By a straightforward calculation, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{P}e_{\boldsymbol{\gamma}}(t) &= \ell(t)(\langle\boldsymbol{\gamma}(t), \boldsymbol{\nu}(t)\rangle\boldsymbol{\mu}(t) + \langle\boldsymbol{\gamma}(t), \boldsymbol{\mu}(t)\rangle\boldsymbol{\nu}(t)) \\ &= \ell(t)\delta(t)(\langle\boldsymbol{\sigma}(t), \boldsymbol{\nu}(t)\rangle\boldsymbol{\mu}(t) + \langle\boldsymbol{\sigma}(t), \boldsymbol{\mu}(t)\rangle\boldsymbol{\nu}(t)).\end{aligned}$$

We define a unit vector field $\overline{\boldsymbol{\nu}}(t)$ along $\boldsymbol{\gamma}$ by

$$\overline{\boldsymbol{\nu}}(t) = \frac{1}{\sqrt{\langle\boldsymbol{\sigma}(t), \boldsymbol{\mu}(t)\rangle^2 + \langle\boldsymbol{\sigma}(t), \boldsymbol{\nu}(t)\rangle^2}}(\langle\boldsymbol{\sigma}(t), \boldsymbol{\mu}(t)\rangle\boldsymbol{\mu}(t) - \langle\boldsymbol{\sigma}(t), \boldsymbol{\nu}(t)\rangle\boldsymbol{\nu}(t)).$$

We have $\langle \frac{d}{dt}\mathcal{P}e_{\boldsymbol{\gamma}}(t), \overline{\boldsymbol{\nu}}(t)\rangle = 0$ for any $t \in I$. This completes the proof. $\square$

We remark that the assumption of the above proposition is satisfied if $\boldsymbol{\gamma}(t) \neq 0$. The curvature of $\mathcal{P}e_{\boldsymbol{\gamma}}(t)$ is rather complicated even for regular curve $\boldsymbol{\gamma}$. Then we have the following generalization of Theorem 2.1.

Theorem 3.3. Let $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ be a Legendrian curve with the curvature (ℓ, β). Suppose that there exists $\alpha(t)$ such that $\beta(t) = \alpha(t)\ell(t)$ for any $t \in I$. Then we have

$$\mathcal{P}e[\psi]_{\boldsymbol{\gamma}}(t) = \mathcal{P}e_{\mathcal{E}v_{\boldsymbol{\gamma}}[\psi+\pi/2]}(t).$$

Proof. By definition and a straightforward calculation, we have

$$\begin{aligned}\mathcal{P}e_{\mathcal{E}v_{\boldsymbol{\gamma}}[\phi]}(t) &= \langle\mathcal{E}v_{\boldsymbol{\gamma}}[\phi](t), \boldsymbol{\nu}[\phi](t)\rangle\boldsymbol{\nu}[\phi](t) \\ &= \langle\boldsymbol{\gamma}(t), -\cos\phi\boldsymbol{\nu}(t) + \sin\phi\boldsymbol{\mu}(t)\rangle(-\cos\phi\boldsymbol{\nu}(t) + \sin\phi\boldsymbol{\mu}(t)).\end{aligned}$$

Here, we substitute $\psi + \pi/2$ for ϕ. Then we have

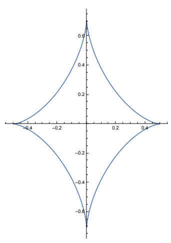
$$\begin{aligned} \mathcal{P}e_{\mathcal{E}v_{\boldsymbol{\gamma}}[\psi+\pi/2]}(t) &= \langle \boldsymbol{\gamma}(t), -\sin\psi\boldsymbol{\nu}(t) - \cos\psi\boldsymbol{\mu}(t)\rangle(-\sin\psi\boldsymbol{\nu}(t) - \cos\psi\boldsymbol{\mu}(t)) \\ &= \mathcal{P}e[\psi]_{\boldsymbol{\gamma}}(t). \end{aligned}$$

This completes the proof. □

We have the following corollary of Proposition 3.2 and Theorem 3.3.

Corollary 3.4. Let $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ be a Legendrian curve with the curvatue (ℓ, β). Suppose that there exist $\alpha(t)$, $\delta(t)$ and $\boldsymbol{\sigma} : I \longrightarrow S^1$ such that $\beta(t) = \alpha(t)\ell(t)$ and $\mathcal{E}v_{\boldsymbol{\gamma}}[\psi + \pi/2](t) = \delta(t)\boldsymbol{\sigma}(t)$ for any $t \in I$. Then the ψ-pedaloid $\mathcal{P}e[\psi]_{\boldsymbol{\gamma}}$ is a frontal.

Example 3.5. We consider a curve $\boldsymbol{\gamma}_1 : [-\pi, \pi] \longrightarrow \mathbb{R}^2$ defined by $\boldsymbol{\gamma}_1(t) = \left(\frac{\cos^3 t}{2}, \frac{\sin^3 t}{\sqrt{2}}\right)$. We can easily show that $\boldsymbol{\gamma}$ is a front. We can draw the picture of the image of $\boldsymbol{\gamma}_1$ as Fig. 1. The 0-pedaloid $\mathcal{P}e[0]_{\boldsymbol{\gamma}_1}$ is also depicted in Fig. 2.



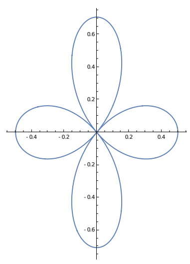
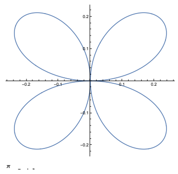


Fig. 1 : $\boldsymbol{\gamma}_1(t) = \left(\frac{\cos^3 t}{2}, \frac{\sin^3 t}{\sqrt{2}}\right)$. Fig. 2 : The 0-pedaloid of $\boldsymbol{\gamma}_1$.

We draw the $\pi/2$-pedaloid $\mathcal{P}e[\pi/2]_{\boldsymbol{\gamma}_1}$ in Fig. 3. Moreover, the $\pi/4$-pedaloid $\mathcal{P}e[\pi/4]_{\boldsymbol{\gamma}_1}$ is drawn in Fig. 4. Finally, we draw the pictures of pedaloids altogether in Fig. 5.



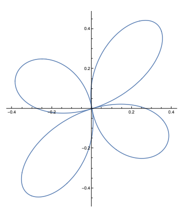


Fig. 3 : The $\pi/2$-pedaloid of $\boldsymbol{\gamma}_1$. Fig. 4 : The $\pi/4$-pedaloid of $\boldsymbol{\gamma}_1$.

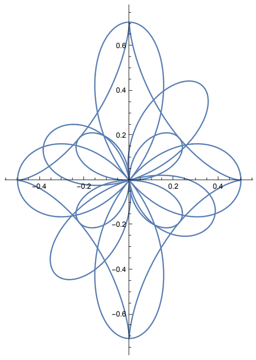
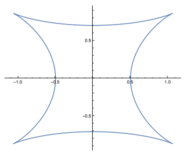


Fig. 5 : $\boldsymbol{\gamma}_1$ and pedaloids.

On the other hand, we draw the pictures of $\pi/2$-evolutoid $\mathcal{E}v_{\boldsymbol{\gamma}_1}[\pi/2]$ in Fig. 6 and $\pi/2+\pi/2$-evolutoid $\mathcal{E}v_{\boldsymbol{\gamma}_1}[\pi/2+\pi/2]$ in Fig. 7, respectively



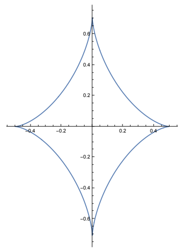
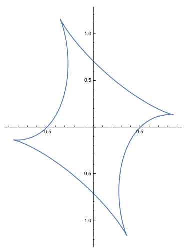


Fig. 6 : The $\pi/2$-evolutoid of $\boldsymbol{\gamma}_1$. Fig. 7 : The $\pi/2+\pi/2$-evolutoid of $\boldsymbol{\gamma}_1$.

Moreover, the $\pi/4+\pi/2$-evolutoid $\mathcal{E}v_{\boldsymbol{\gamma}_1}[\pi/4+\pi/2]$ is drawn in Fig. 8. Finally, we draw the pictures of evolutoids altogether in Fig. 9.



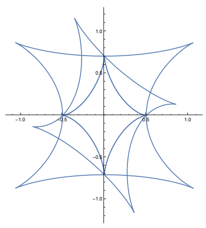
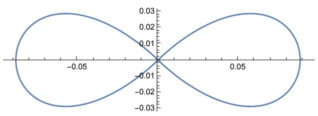


Fig. 8 : The $\pi/4+\pi/2$-evolutoid of $\boldsymbol{\gamma}_1$. Fig. 9 : Evolutoids.

Example 3.6. We consider a curve $\gamma_2 : [-\pi, \pi] \longrightarrow \mathbb{R}^2$ defined by $\gamma_2(t) = \left(\frac{\cos^3 t \sin^3 t}{\sqrt{2}}, \frac{\cos^3 t \cos 2t \sin^3 t}{\sqrt{2}}\right)$. We can easily show that γ_2 is a frontal and not a front. We can draw the picture of the image of γ_2 as Fig. 10. The 0-pedaloid $\mathcal{P}e[0]_{\gamma_2}$ is also depicted in Fig. 11.



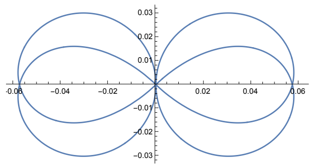
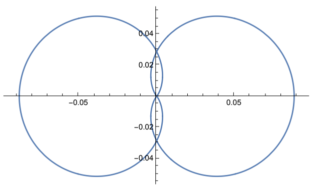


Fig. 10 : $\gamma_2(t) = \left(\frac{\cos^3 t \sin^3 t}{\sqrt{2}}, \frac{\cos^3 t \cos 2t \sin^3 t}{\sqrt{2}}\right)$. Fig. 11 : The 0-pedaloid of γ_2.

We draw the $\pi/2$-pedaloid $\mathcal{P}e[\pi/2]_{\gamma_2}$ in Fig. 12. Moreover, the $\pi/4$-pedaloid $\mathcal{P}e[\pi/4]_{\gamma_2}$ is drawn in Fig. 13. Finally, we draw the pictures of pedaloids altogether in Fig. 14.



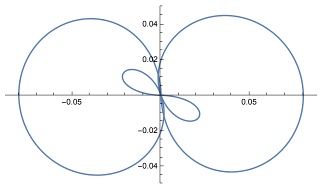


Fig. 12 : The $\pi/2$-pedaloid of γ_2. Fig. 13 : The $\pi/4$-pedaloid of γ_2.

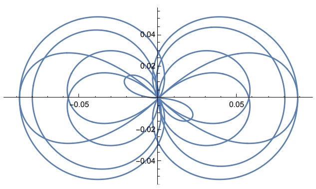


Fig. 14 : γ_2 and pedaloids.

On the other hand, we draw the pictures of $\pi/2$-evolutoid $\mathcal{E}v_{\gamma_2}[\pi/2]$ in Fig. 15 and $\pi/2+\pi/2$-evolutoid $\mathcal{E}v_{\gamma_2}[\pi/2+\pi/2]$ in Fig. 16, respectively

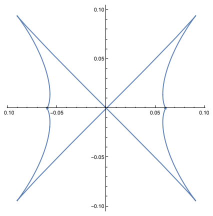


Fig. 15 : The $\pi/2$-evolutoid of $\boldsymbol{\gamma}_2$.

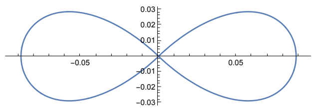


Fig. 16 : The $\pi/2+\pi/2$-evolutoid of $\boldsymbol{\gamma}_2$.

Moreover, the $\pi/4+\pi/2$-evolutoid $\mathcal{E}v_{\boldsymbol{\gamma}_2}[\pi/4+\pi/2]$ is drawn in Fig. 17. Finally, we draw the pictures of evolutoids altogether in Fig. 18.

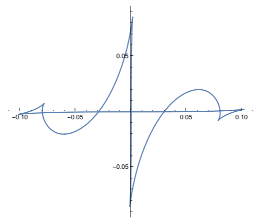


Fig. 17 : The $\pi/4+\pi/2$-evolutoid of $\boldsymbol{\gamma}_2$.

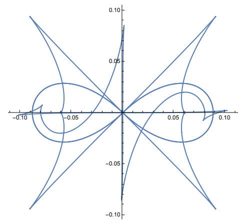


Fig. 18 : Evolutoids.

References

[1] V. I. Arnol'd: Singularities of Caustics and Wave Fronts, Mathematics and its Applications, **62**, Kluwere Academic Publications, Dordrecht 1990.

[2] J. W. Bruce and P. J. Giblin: Curves and singularities (second edition), Cambridge University press, Cambridge 1992.

[3] T. Fukunaga and M. Takahashi: *Evolutes of fronts in the Euclidean plane*, Journal of Singularities, **10**, 92–107, 2014.

[4] T. Fukunaga and M. Takahashi: *Evolutes and involutes of frontals in the Euclidean plane*, Demonstratio Mathematica, **XLVIII**, 147–166, 2015.

[5] P. J. Giblin and J. P. Warder: *Evolving Evolutoids*, American Mathematical Monthly, **121**, 871–889, 2014.

[6] S. Izumiya, M. C. Romero Fuster, M. A. Soares Ruas and F. Tari: Differential Geometry from a singularity theory viewpoint, World Scientific Publishing, Singapore 2015.

[7] Y. Li and D-H. Pei: *Pedal curves of frontals in the Euclidean plane*, Math. Methods Appl. Sci., **41**, 1988–1997, 2018.

[8] C. Zwikker: The Advanced Geometry of Plane Curves and Their Applications, Dover, New York 1963.