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# On Defining the $(p, q, k)$-Generalized Gamma Function

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**Abstract.** In this paper, we introduce a $(p, q, k)$-generalization of the gamma function and investigate some basic identities and properties. Further, we define a $q$-integral representation for this function. As an application, we give some double inequalities concerning the $(p, q, k)$-generalized gamma function by using of its $q$-integral representation.

**Keywords:** Gamma function, $q$-integral, $(p, q, k)$-generalized gamma function, $(p, q, k)$-generalized Pocchammer symbol, inequality

**MSC 2000 classification:** 33B15, 33C47, 05A30, 26D07

## 1 Introduction

Euler's gamma function is defined by the integral representation

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx, \quad t > 0.$$

Since gamma function plays one of the most important roles in mathematics and physics, it is suggested to generalize of the ordinary special function can play similar roles in this area. There are many papers on giving an extension of the gamma function for which the usual properties and representations are naturally extended, see for example [1], [2], [4], [5], [11]. The main purpose of this paper is to give an integral representation for $q$-analogue of the two parameter gamma function ${}_p\Gamma_k$. For this, firstly we give a definition to the function ${}_p\Gamma_{q,k}(t)$. Note that the definition of ${}_p\Gamma_{q,k}$ given in [5] is slightly different from us. For example in [5] the function $\Gamma_{q,k}$ satisfies the condition that ${}_p\Gamma_{q,k}(k) = \left[\frac{p}{k}\right]$ but in our study ${}_p\Gamma_{q,k}(k) = \frac{[p]}{[k]}$. We need to give such a slightly different definition to give its integral representation to be consistent with the function $\Gamma_{q,k}$ given by Diaz and Teruel, see [2].

## 2 Notation and Preliminaries

In this section, we will give a summary of definitions and mathematical notations mentioned in this study for the convenience of the reader. The definitions related to $q$-calculus will be taken from the books in this field [3], [6]. Throughout of this work, we will fix $q \in (0,1)$. Further, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ indicates the set of complex numbers, real numbers, integers and natural numbers respectively. For any $t \in \mathbb{R}$, the number $[t]_q$ or simply $[t]$ and for any $n \in \mathbb{N}$ the $q$-factorial $[n]!$ are defined by

$$[t] = \frac{1-q^t}{1-q}, \quad [n]! = [n][n-1]\dots[2][1].$$

The $q$-derivative $D_q f$ of a function $f : \mathbb{R} \to \mathbb{R}$ is given by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x} \text{ if } x \neq 0 \text{ and } (D_q f)(0) = f'(0)$$

provided $f'(0)$ exists.

The $q$-integral and improper $q$-integral of a function $f : \mathbb{R} \to \mathbb{R}$ are defined respectively by

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^{\infty} q^n f(aq^n), \ \ a > 0,$$

$$\int_0^{\infty/a} f(x) d_q x = (1-q) \sum_{n\in\mathbb{Z}} \frac{q^n}{a} f(\frac{q^n}{a}).$$

The formula of $q$-integration by parts is given for suitable functions $f$ and $g$ from $\mathbb{R}$ to $\mathbb{R}$ by

$$\int_a^b f(x) d_q g(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(qx) d_q f(x). \tag{2.1}$$

Also, the Hölder's inequality for the $q$-integral is given by

$$\int_0^\infty |f(t)g(t)| d_q t \leq \left[\int_0^\infty |f(t)|^\alpha d_q t\right]^{1/\alpha} \left[\int_0^\infty |g(t)|^\beta d_q t\right]^{1/\beta} \tag{2.2}$$

where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $\alpha > 1$. More information about $q$-calculus can be found in [3], [6], [7], [8], [14], [15].

In [1], Diaz and Pariguan give the definition of Pochhemmer $k$-symbol $(t)_{n,k}$ and introduced the $k$-analogue of gamma function $\Gamma_k$ as the followings:

$$(t)_{n,k} = t(t+k)(t+2k)\dots(t+(n-1)k)$$

for $t \in \mathbb{C}$, $k \in \mathbb{R}$ ,$n \in \mathbb{Z}^+$ and

$$\Gamma_k(t) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{t}{k}-1}}{(t)_{n,k}}$$

for $t \in \mathbb{C} - k\mathbb{Z}^-$, $k > 0$. Also they give its integral representation as

$$\Gamma_k(t) = \int_0^\infty x^{t-1}e^{-\frac{x^k}{k}}\,dx \tag{2.3}$$

for $t \in \mathbb{C}$, $Re(t) > 0$, $k > 0$.

Two parameter gamma function and Pochhemmer symbol, namely $p, k$-gamma function and $p, k$-Pochhemmer symbol, denoted as ${}_p\Gamma_k$ and ${}_p(x)_{n,k}$ respectively are defined by

$${}_p\Gamma_k(t) = \frac{1}{k}\lim_{n\to\infty} \frac{n!p^{n+1}(np)^{\frac{t}{k}-1}}{{}_p(t)_{n,k}} \tag{2.4}$$

for $t \in \mathbb{C} - k\mathbb{Z}^-$, $k, p > 0$, $Re(t) > 0, n \in \mathbb{N}$ and

$${}_p(t)_{n,k} = \frac{tp}{k}\Big(\frac{tp}{k} + p\Big)\Big(\frac{tp}{k} + 2p\Big)\dots\Big(\frac{tp}{k} + (n-1)p\Big) \tag{2.5}$$

for $p, k, t \in \mathbb{R}$ and $n \in \mathbb{N}$, see [4], [5]. Also the integral representation of $p, k$-gamma function is given by

$${}_p\Gamma_k(t) = \int_0^\infty x^{t-1}e^{-\frac{x^k}{p}}\,dx \tag{2.6}$$

for $t \in \mathbb{C} - k\mathbb{Z}^-$, $k, p > 0$ and $Re(t) > 0$, see [4].

# 3 A New Version of the ${}_p\Gamma_{q,k}$ and Its Integral Representation

In this section, we define a $q$-analogue of ${}_p\Gamma_k$ for $p, k > 0$ which we will call the $(p, q, k)$-generalized gamma function and will be denoted by ${}_p\Gamma_{q,k}$. Our definition of ${}_p\Gamma_{q,k}$ is slightly different from the one given in [5]. The function ${}_p\Gamma_k$ introduced by Gehlot in [4] is satisfied the following properties:

$${}_p\Gamma_k(k) = \frac{p}{k},\ \ {}_p\Gamma_k(t + k) = \frac{tp}{k}\,{}_p\Gamma_k(t),\ \ t > 0. \tag{3.1}$$

As in [2], [5] our motivation for giving $(p, q, k)$-generalized gamma function ${}_p\Gamma_{q,k}$ is satisfying a $q$-analogue of properties (3.1) above. Thus we will assume ${}_p\Gamma_{q,k}$ such that;

$${}_p\Gamma_{q,k}(k) = \frac{[p]}{[k]},\ \ {}_p\Gamma_{q,k}(t + k) = \frac{[p]}{[k]}[t]\,{}_p\Gamma_{q,k}(t),\ \ t > 0. \tag{3.2}$$

Note that, in a rude way, the definition of a $q$-analogue $M_q$ of a mathematical subject $M$ is such that $M_q \to M$ as $q$ tends to 1. So one can find several $q$-analogues of the same object. For example in [5],

$$ {}_p\Gamma_{q,k}(k) = \left[\frac{p}{k}\right] \text{ and } {}_p\Gamma_{q,k}(t+k) = \left[\frac{tp}{k}\right] {}_p\Gamma_{q,k}(t). \tag{3.3}$$

since they give the $q$-analogue of $\frac{p}{k}$ as $\left[\frac{p}{k}\right]$. But in our study ${}_p\Gamma_{q,k}$ satisfies the condition that ${}_p\Gamma_{q,k}(k) = \frac{[p]}{[k]}$. We need such different analogues to give its integral representation to be consistent with the function $\Gamma_{q,k}(t)$ given by the following $q$-integral

$$\Gamma_{q,k}(t) = \int_0^{\left(\frac{[k]}{1-q^k}\right)^{\frac{1}{k}}} x^{t-1} E_{q,k}^{-\frac{q^k x^k}{[k]}} \, d_q x, \quad t > 0 \tag{3.4}$$

where

$$E_{q,k}^x = \sum_{n=0}^{\infty} \frac{q^{kn(n-1)/2} x^n}{[n]_{q^k}!} = (1 + (1-q^k)x)_{q,k}^{\infty} \tag{3.5}$$

is one of the $q,k$-analogues of the classical exponential function. Also note that, in general $[st] \neq [s][t]$. Instead of this, one has $[st] = [s]_{q^t}[t]$, for all $s, t \in \mathbb{R}$, see [2].

From (3.2) and the induction method we have

$$\begin{aligned} {}_p\Gamma_{q,k}(nk) &= \left(\frac{[p]}{[k]}\right)^n \prod_{j=1}^{n-1} [jk] = \left(\frac{[p]}{[k]}\right)^n \prod_{j=1}^{n-1} \frac{1-q^{jk}}{1-q} & (3.6)\\ &= \left(\frac{[p]}{[k]}\right)^n \frac{1}{(1-q)^{n-1}} \prod_{j=1}^{n-2} (1 - q^{jk} q^k). & (3.7) \end{aligned}$$

Since

$$(x+y)_{q,k}^n = \prod_{j=0}^{n-1} (x + q^{jk} y) \tag{3.8}$$

for $x, y \in \mathbb{R}$ and $n \in \mathbb{Z}^+$, see [2], we get

$$ {}_p\Gamma_{q,k}(nk) = \left(\frac{[p]}{[k]}\right)^n \frac{(1-q^k)_{q,k}^{n-1}}{(1-q)^{n-1}}. \tag{3.9}$$

Since $k > 0$, equation (3.9) helps us to give the definition of ${}_p\Gamma_{q,k}(t)$ for all positive real values of $t$ after the change of variable $t = nk$ as the following:

**Definition 3.1.** For any $t > 0$ the function ${}_p\Gamma_{q,k}$ is defined by

$$ {}_p\Gamma_{q,k}(t) = \left(\frac{[p]}{[k]}\right)^{\frac{t}{k}} \frac{(1-q^k)_{q,k}^{\frac{t}{k}-1}}{(1-q)^{\frac{t}{k}-1}}. \tag{3.10} $$

By using the equation

$$ (1+x)_{q,k}^{t} = \frac{(1+x)_{q,k}^{\infty}}{(1+q^{kt}x)_{q,k}^{\infty}} \tag{3.11} $$

for $x, y, t \in \mathbb{R}$, see [2], the infinite product expression for the function ${}_p\Gamma_{q,k}$ is given by

$$ {}_p\Gamma_{q,k}(t) = \left(\frac{[p]}{[k]}\right)^{\frac{t}{k}} \frac{(1-q^k)_{q,k}^{\infty}}{(1-q^t)_{q,k}^{\infty}(1-q)^{\frac{t}{k}-1}}, \ t > 0. \tag{3.12} $$

**Definition 3.2.** Let $p, k, t \in \mathbb{R}$ and $n \in \mathbb{N}$. The $(p, q, k)$-generalized Pochhammer symbol is given by

$$ {}_p[t]_{n,k} = \left(\frac{[p]}{[k]}\right)^{n} [t][t+k]\dots[t+(n-1)k]. \tag{3.13} $$

Note that ${}_p[t]_{n,k} \to_p (t)_{n,k}$ as $q \to 1$. With the next proposition we will prove that ${}_p\Gamma_{q,k}$ satisfies the properties (3.2) as promised. Also we will prove that our function ${}_p\Gamma_{q,k}$ is related to the $(p, q, k)$-generalized Pocchammer Symbol ${}_p[t]_{n,k}$ in the same way as the $q, k$-generalized gamma function $\Gamma_{q,k}$ and the $q, k$-generalized Pocchammer symbol $[t]_{n,k}$ are related to each other.

**Proposition 3.3.** The function ${}_p\Gamma_{q,k}$ satisfies the following identities for $t > 0$:

(i) ${}_p\Gamma_{q,k}(k) = \frac{[p]}{[k]}$,

(ii) ${}_p\Gamma_{q,k}(t+k) = \frac{[p]}{[k]}[t]\,{}_p\Gamma_{q,k}(t)$,

(iii) $\frac{{}_p\Gamma_{q,k}(t+nk)}{{}_p\Gamma_{q,k}(t)} = {}_p[t]_{n,k}$.

*Proof.* (i) is obvious from the definition 3.1. For (ii), we can write

$$ {}_p\Gamma_{q,k}(t+k) = \left(\frac{[p]}{[k]}\right)^{\frac{t}{k}+1} \frac{(1-q^k)_{q,k}^{\frac{t}{k}}}{(1-q)^{\frac{t}{k}}}. \tag{3.14} $$

Then since

$$(1+x)_{q,k}^{s+t} = (1+x)_{q,k}^{s}(1+q^{ks}x)_{q,k}^{t} \tag{3.15}$$

for $x, s, t \in \mathbb{R}$ see [2], we have

$${}_p\Gamma_{q,k}(t+k) = \left(\frac{[p]}{[k]}\right)^{\frac{t}{k}+1} \frac{(1-q^k)_{q,k}^{\frac{t}{k}-1}(1-q^t)_{q,k}}{(1-q)^{\frac{t}{k}-1}(1-q)} = \frac{[p]}{[k]}[t]\,{}_p\Gamma_{q,k}(t). \tag{3.16}$$

For proving (iii) we use induction method. Let $n = 1$. Then,

$$\frac{{}_p\Gamma_{q,k}(t+k)}{{}_p\Gamma_{q,k}(t)} = \frac{[p]}{[k]}[t].$$

Now suppose that the equation (iii) is true for $n-1$. Then we have

$$\frac{{}_p\Gamma_{q,k}(t+(n-1)k)}{{}_p\Gamma_{q,k}(t)} = {}_p[t]_{n-1,k}. \tag{3.17}$$

Then by using the equations (ii) and (3.17) we get

$$\begin{aligned} \frac{{}_p\Gamma_{q,k}(t+nk)}{{}_p\Gamma_{q,k}(t)} &= \frac{{}_p\Gamma_{q,k}(t+(n-1)k+k)}{{}_p\Gamma_{q,k}(t)} && (3.18)\\ &= \frac{[p]}{[k]}[t+(n-1)k]\frac{{}_p\Gamma_{q,k}(t+(n-1)k)}{{}_p\Gamma_{q,k}(t)} && (3.19)\\ &= \left(\frac{[p]}{[k]}\right)^{n}[t][t+k]\ldots[t+(n-1)k] = {}_p[t]_{n,k}, && (3.20) \end{aligned}$$

and the proof is completed. $\boxed{QED}$

We remark that the commutative diagram given in [5], which shows the relation between the different gamma functions, still holds. Now, we will give a $q$-integral representation for our function ${}_p\Gamma_{q,k}$.

**Definition 3.4.** The function ${}_p\Gamma_{q,k}$ is given by the following $q$-integral:

$${}_p\Gamma_{q,k}(t) = \int_0^{\left(\frac{[p]}{1-q^k}\right)^{\frac{1}{k}}} x^{t-1}E_{q,k}^{-\frac{q^k x^k}{[p]}}\,d_qx, \quad t>0. \tag{3.21}$$

${}_p\Gamma_{q,k}$ is a $q$-analogue of the $p,k$-generalized gamma function since it reduces to ${}_p\Gamma_k$ in the limit $q \to 1$. Also it satisfies properties analogous to (3.1). This is given in the following:

**Proposition 3.5.** The function ${}_p\Gamma_{q,k}$ given in (3.21) satisfies the conditions (i) and (ii) in the proposition 3.3 for $t > 0$.

*Proof.* For simplicity take $A = \left(\frac{[p]}{1-q^k}\right)^{\frac{1}{k}}$. Then since the $q$-derivative of $E^x_{q,k}$ is $D_q E^x_{q,k} = E^{qx}_{q,k}$, we have

$$\begin{aligned} {}_p\Gamma_{q,k}(k) &= \int_0^A -\frac{[p]}{[k]} D_q\Big(E_{q,k}^{-\frac{x^k}{[p]}}\Big) d_q x && (3.22)\\ &= -\frac{[p]}{[k]}\Big(E_{q,k}^{-\frac{1}{1-q^k}} - E^0_{q,k}\Big) = \frac{[p]}{[k]}. && (3.23)\end{aligned}$$

For proving (ii), we use the formula of $q$-integration by parts given by (2.1). Then

$$\begin{aligned} {}_p\Gamma_{q,k}(t+k) &= \int_0^A x^{t+k-1} E_{q,k}^{-\frac{q^k x^k}{[p]}} d_q x = -\frac{[p]}{[k]} \int_0^A x^t d_q\Big(E_{q,k}^{-\frac{x^k}{[p]}}\Big) && (3.24)\\ &= \frac{[p]}{[k]}[t] \int_0^A x^{t-1} E_{q,k}^{-\frac{q^k x^k}{[p]}} d_q x = \frac{[p]}{[k]}[t]\,{}_p\Gamma_{q,k}(k)(t), && (3.25)\end{aligned}$$

and the proof is completed. $\boxed{QED}$

# 4 Inequalities Involving the $(p, q, k)$-Generalized Gamma Function

In this section, we give some double inequalities as an application of the integral representation for ${}_p\Gamma_{q,k}$ by similar techniques as in [9], [10], [13]. Let us begin with the following lemma.

**Theorem 4.1.** Let $a \in (0, 1)$ and $t > 0$. Then the following double inequality is valid:

$$\left(\frac{[p]}{[k]}\right)^a \left(\frac{[t]}{[t+ak]}\right)^{1-a} \le \frac{{}_p\Gamma_{q,k}(t+ak)}{[t]^a\,{}_p\Gamma_{q,k}(t)} \le \left(\frac{[p]}{[k]}\right)^a. \quad (4.1)$$

*Proof.* We use the inequality (2.2). For this let $\alpha = \frac{1}{1-a}$, $\beta = \frac{1}{a}$, $f(t) = x^{(1-a)(t-1)} E_{q,k}^{-\frac{(1-a)q^k x^k}{[p]}}$, $g(t) = x^{a(t+k-1)} E_{q,k}^{-\frac{aq^k x^k}{[p]}}$ and

$A = \left(\frac{[p]}{1-q^k}\right)^{\frac{1}{k}}$. Then,

$$
\begin{aligned}
{}_p\Gamma_{q,k}(t+ak) &= \int_0^A x^{t+ak-1} E_{q,k}^{-\frac{q^k x^k}{[p]}} d_q x && (4.2)\\
&\le \left[\int_0^A \left(x^{(1-a)(t-1)} E_{q,k}^{-\frac{(1-a)q^k x^k}{[p]}}\right)^{\frac{1}{1-a}} d_q x\right]^{1-a}\\
&\quad \times \left[\int_0^A \left(x^{a(t+k-1)} E_{q,k}^{-\frac{a q^k x^k}{[p]}}\right)^{\frac{1}{a}} d_q x\right]^{a} && (4.3)\\
&= \left[\int_0^A x^{t-1} E_{q,k}^{-\frac{q^k x^k}{[p]}} d_q x\right]^{1-a} \left[\int_0^A x^{t+k-1} E_{q,k}^{-\frac{q^k x^k}{[p]}} d_q x\right]^{a} && (4.4)\\
&= \left[{}_p\Gamma_{q,k}(t)\right]^{1-a} \left[{}_p\Gamma_{q,k}(t+k)\right]^{a}. && (4.5)
\end{aligned}
$$

Using the equation (ii) in Proposition 3.3, we get

$$
{}_p\Gamma_{q,k}(t+ak) \le \left(\frac{[p]}{[k]}\right)^a [t]^a \, {}_p\Gamma_{q,k}(t) \tag{4.6}
$$

which proves the right side of the inequality (4.1). Since $a \in (0,1)$, the equation (4.6) is valid for $1-a$ instead of $a$. Then,

$$
{}_p\Gamma_{q,k}(t+k-ak) \le \left(\frac{[p]}{[k]}\right)^{1-a} [t]^{1-a} \, {}_p\Gamma_{q,k}(t). \tag{4.7}
$$

Since $t + ak > 0$, by writing $t + ak$ instead of $t$ in the equation (4.7) we get

$$
{}_p\Gamma_{q,k}(t+k) \le \left(\frac{[p]}{[k]}\right)^{1-a} [t+ak]^{1-a} \, {}_p\Gamma_{q,k}(t+ak). \tag{4.8}
$$

By using the equation (4.6), we get the proof of the lemma. $\boxed{QED}$

**Remark 4.2.** If we take $p = k$ in (4.1), we get the inequalities (6) given in [9].

**Remark 4.3.** If in (4.1), we allow $q \to 1^-$, one can get a double inequalities for the $p,k$-generalized gamma function presented in [11].

**Remark 4.4.** The inequalities in (4.1) imply that

$$
\lim_{t\to\infty} \frac{{}_p\Gamma_{q,k}(t+ak)}{[t]^a \, {}_p\Gamma_{q,k}(t)} = \left(\frac{[p]}{[k]}\right)^a. \tag{4.9}
$$

**Corollary 4.5.** The inequalities

$$\sqrt{\frac{[p][t]}{[k]}} \leq \frac{{}_p\Gamma_{q,k}(t+k)}{{}_p\Gamma_{q,k}(t+\frac{k}{2})} \leq \sqrt{\frac{[p][t+\frac{k}{2}]}{[k]}} \tag{4.10}$$

are valid for $t > 0$.

*Proof.* By writing $a = \frac{1}{2}$ in the inequalities (4.1) we get the result. $\boxed{QED}$

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