Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **39** (2019) no. 1, 59–63. doi:10.1285/i15900932v39n1p59

A Bogomolov type property relative to a normalized height on $M_n(\overline{\mathbb{Q}})$

Masao Okazaki
Graduate School of Mathematics, Kyushu University, Motooka 744, Nishi-ku, Fukuoka 819-0395, Japan
m-okazaki@math.kyushu-u.ac.jp

Received: 31.1.2019; accepted: 28.2.2019.

Abstract. In [13], Talamanca introduced a normalized height on $M_n(\overline{\mathbb{Q}})$, which is an analogue of the canonical height on elliptic curves. In this paper, we examine whether $M_n(F)$ has a Bogomolov type property relative to this height if a subfield $F \subset \overline{\mathbb{Q}}$ has the Bogomolov property.

Keywords: normalized height on $M_n(\overline{\mathbb{Q}})$, Bogomolov property

MSC 2000 classification: Primary 11G50, Secondary 11C20

1 Introduction

Let K be an algebraic number field and $\mathcal{O}_K$ be the ring of integers of K. We denote the set of all field homomorphisms from K to $\mathbb{C}$ by $\mathcal{M}_K^\infty$, the set of all non-zero prime ideals of $\mathcal{O}_K$ by $\mathcal{M}_K^0$, and $\mathcal{M}_K^\infty \sqcup \mathcal{M}_K^0$ by $\mathcal{M}_K$. For $x \in K$, we set an Archimedean absolute value $|x|_\sigma := |\sigma(x)|$ for each $\sigma \in \mathcal{M}_K^\infty$ and a non-Archimedean absolute value $|x|_\mathfrak{p} := \#(\mathcal{O}_K/\mathfrak{p})^{-\mathrm{ord}_\mathfrak{p}(x)}$ for each $\mathfrak{p} \in \mathcal{M}_K^0$.

For $\vec{x} = {}^t(x_1, \cdots, x_n) \in K^n$, we set

$$H(\vec{x}) := \left(\prod_{v \in \mathcal{M}_K} \max\{|x_1|_v, \ldots, |x_n|_v\} \right)^{1/[K:\mathbb{Q}]}.$$

As usual, we set $H(\vec{0}) = 1$. Since the value of $H(\vec{x})$ is independent of the choice of K, we can consider H as a function on $\overline{\mathbb{Q}}^n$. The function H is called the *Weil height* on $\overline{\mathbb{Q}}^n$. To study the height function, the following property is of main interest.

Definition 1 ("Bogomolov property," [4], Section 1)**.** We say that a subfield $F \subset \overline{\mathbb{Q}}$ has the *Bogomolov property* if there exists a positive constant $C > 1$ such that for any $x \in F$, $H(1, x) < C$ implies that $H(1, x) = 1$.

It is known that any algebraic number field has the Bogomolov property and that there are several interesting examples of infinite extensions of $\mathbb{Q}$ which have the Bogomolov property (e.g. [1], [2], [4], [9], [10], [11]).

Now for each $v \in \mathcal{M}_K$, we set a norm N_v on K^n with respect to $|\cdot|_v$:

$$N_v(\vec{x}) := \begin{cases} \sqrt{|x_1|_v^2 + \cdots + |x_n|_v^2} & (v \in \mathcal{M}_K^\infty), \\ \max\{|x_1|_v, \ldots, |x_n|_v\} & (v \in \mathcal{M}_K^0). \end{cases}$$

Let K_v be the completion of K by the absolute value $|\cdot|_v$. We set

$$\mathcal{H}(A) := \left(\prod_{v \in \mathcal{M}_K} \|A\|_v \right)^{1/[K:\mathbb{Q}]},$$

where $A \in M_n(K)$ and $\|\cdot\|_v$ is the operator norm on $\mathrm{End}(K_v^n)$ induced by N_v. Since the value of $\mathcal{H}(A)$ is also independent of the choice of K, we can consider $\mathcal{H}$ as a function on $M_n(\overline{\mathbb{Q}})$. In [13], Talamanca introduced the *normalized height*

$$\mathcal{H}_s(A) := \lim_{k \to \infty} \mathcal{H}(A^k)^{1/k}.$$

As usual, we set $\mathcal{H}_s(A) = 1$ if A is a nilpotent matrix. This limit exists by submultiplicativity and is an analogue of the canonical height on elliptic curves.

When we know a height function $\mathfrak{H}$ on a set X, the following natural question arises: is there an interesting subset $S \subset X$ which has the Bogomolov property relative to $\mathfrak{H}$? Here we say that a subset $S \subset X$ has the *Bogomolov property relative to* $\mathfrak{H}$ if there exists a positive constant $C > 1$ such that for any $x \in S$, $\mathfrak{H}(x) < C$ implies that $\mathfrak{H}(x) = 1$. Indeed, in [5], Breuillard considered $\mathcal{H}_s$ as a function on $\mathrm{GL}_n(\overline{\mathbb{Q}})$ and found subsets with group-theoretic conditions which have the Bogomolov property relative to $\mathcal{H}_s$; see Theorem 1.2 in [5]. Whereas Breuillard group-theoretically studied $\mathcal{H}_s$ in [5], our interest in this paper is different: if a subfield $F \subset \overline{\mathbb{Q}}$ has the Bogomolov property, does $M_n(F)$ have the Bogomolov property relative to $\mathcal{H}_s$?

Theorem 1. *Let $\mathbb{Q}^{tr}$ be the field of all totally real numbers. Then there exists $A_k \in M_n(\mathbb{Q}^{tr})$ such that $\mathcal{H}_s(A_k) > 1$ for any $k \in \mathbb{N}$ and $\lim_{k \to \infty} \mathcal{H}_s(A_k) = 1$.*

Note that $\mathbb{Q}^{tr}$ has the Bogomolov property; see [10]. Therefore this theorem especially states that there exists a subfield $F \subset \overline{\mathbb{Q}}$ which has the Bogomolov property but $M_n(F)$ does not have the Bogomolov property relative to $\mathcal{H}_s$.

2 Proof of Theorem 1

To prove Theorem 1, we should refer to the work of Talamanca in [13].

Fact 1 ([13], Theorem 4.2)**.** Let $A \in M_n(\overline{\mathbb{Q}})$. Then

$$\mathcal{H}_s(A) = H(\lambda_1, \ldots, \lambda_n),$$

where ${}^t(\lambda_1, \ldots, \lambda_n)$ is the n-tuple formed by the eigenvalues of A.

Owing to this fact, we can explicitly compute the value of $\mathcal{H}_s(A)$.

Proof of Theorem 1. Note that $\mathbb{Q}^{tr}(\sqrt{-1})$ does not have the Bogomolov property. Explicitly, $\alpha_k := \big((2-\sqrt{-1})/(2+\sqrt{-1})\big)^{1/k} \in \mathbb{Q}^{tr}(\sqrt{-1})$ enjoys $H(1, \alpha_k) > 1$ and $\lim\limits_{k\to\infty} H(1, \alpha_k) = 1$; see Theorem 5.3 in [1]. Let $a_k + b_k\sqrt{-1} := \alpha_k$, where $a_k,\, b_k \in \mathbb{Q}^{tr}$. We set

$$A_k := \begin{pmatrix} a_k & -b_k & 0 & \cdots & 0 \\ b_k & a_k & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in M_n(\mathbb{Q}^{tr}).$$

Then the eigenvalues of A_k are α_k, $\overline{\alpha_k}$ and 0. Therefore, by Fact 1, we have

$$\mathcal{H}_s(A_k) = H(\alpha_k, \overline{\alpha_k}) = H(1, \overline{\alpha_k}/\alpha_k) = H(1, \overline{\alpha_1}/\alpha_1)^{1/k}.$$

Thus we have $\lim\limits_{k\to\infty} H(1, \overline{\alpha_1}/\alpha_1)^{1/k} = 1$. On the other hand, $H(1, \overline{\alpha_1}/\alpha_1)^{1/k} > 1$ since $\overline{\alpha_1}/\alpha_1$ is not a root of unity; see Theorem 1.5.9 in [3]. So A_k is what we want. $\boxed{QED}$

3 A supplemental remark on Theorem 1

By Theorem 1, we know that even though a subfield $F \subset \overline{\mathbb{Q}}$ has the Bogomolov property, $M_n(F)$ does not always have the Bogomolov property relative to $\mathcal{H}_s$. Then it is a natural problem to find a subfield $F \subset \overline{\mathbb{Q}}$ such that $M_n(F)$ has the Bogomolov property relative to $\mathcal{H}_s$.

Definition 2 ("Northcott property," [4], Section 1)**.** We say that a subfield $F \subset \overline{\mathbb{Q}}$ has the *Northcott property* if $\{\, x \in F \mid H(1, x) < C \,\}$ is a finite set for any positive constant $C > 1$.

We know that any subfield $F \subset \overline{\mathbb{Q}}$ which has the Northcott property also has the Bogomolov property. It is known that any algebraic number field has the Northcott property and that there are some infinite extensions of $\mathbb{Q}$ which have the Northcott property (e.g. [4], [6], [7], [14]).

Proposition 1. *Let F be a subfield of $\overline{\mathbb{Q}}$ with the Northcott property. Then $M_n(F)$ has the Bogomolov property relative to $\mathcal{H}_s$.*

Proof. It is known that if a subfield $F \subset \overline{\mathbb{Q}}$ has the Northcott property, then $\{ x \in \overline{\mathbb{Q}} \mid H(1,x) < C$ and $[F(x) : F] \leq d \}$ is a finite set for any positive constant $C > 1$ and $d \in \mathbb{N}$; see Theorem 2.1 in [7]. Therefore we have the proposition since the degree of each of eigenvalues of $A \in M_n(F)$ over F is at most n. $\boxed{QED}$

Acknowledgements. I am deeply grateful to my master's advisor Yuichiro Takeda for his invaluable support.

It is a great pleasure to express my gratitude to Valerio Talamanca, who was kind enough to send me a personal lecture note.

I would like to thank Shun-ichi Yokoyama. I got to know the height function during a personal seminar with him. I also thank Momonari Kudo, Akinori Morizono, Toshiki Matsusaka and Seiji Kuga for participating in the seminar.

Hyogo Shibahara improved my English sentences. I should thank him.

Finally, I thank the anonymous referee for reading my manuscript carefully and giving me many valuable comments.

References

[1] F. Amoroso, S. David, U. Zannier: *On fields with property* (B), Proc. Amer. Math. Soc. **142** (2014), no. 6, 1893–1910.

[2] F. Amoroso, R. Dvornicich: *A lower bound for the height in abelian extensions*, J. Number Theory **80** (2000), no. 2, 260–272.

[3] E. Bombieri, W. Gubler: Heights in Diophantine Geometry, New Mathematical Monographs, 4. Cambridge University Press, Cambridge, 2006.

[4] E. Bombieri, U. Zannier: *A note on heights in certain infinite extensions of* $\mathbb{Q}$, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **12** (2001), 5–14.

[5] E. Breuillard: *A height gap theorem for finite subsets of* $\mathrm{GL}_d(\overline{\mathbb{Q}})$ *and nonamenable subgroups*, Ann. of Math. (2) **174** (2011), no. 2, 1057–1110.

[6] S. Checcoli, M. Widmer: *On the Northcott property and other properties related to polynomial mappings*, Math. Proc. Cambridge Philos. Soc. **155** (2013), no. 1, 1–12.

[7] R. Dvornicich, U. Zannier: *On the properties of Northcott and Narkiewicz for fields of algebraic numbers*, Funct. Approx. Comment. Math. **39** (2008), part 1, 163–173.

[8] P. Habegger: *Small height and infinite nonabelian extensions*, Duke Math. J. **162** (2013), no. 11, 2027–2076.

[9] L. Pottmeyer: *Fields with the Bogomolov property*, Ph.D. thesis, Regensburg University, 2013.

[10] L. Pottmeyer: *A note on extensions of* $\mathbb{Q}^{tr}$, J. Théor. Nombres Bordeaux **28** (2016), no. 3, 735–742.

[11] A. Schinzel: *On the product of the conjugates outside the unit circle of an algebraic number*, Acta Arith. **24** (1973), 385–399.

[12] V. Talamanca: *Height preserving linear transformations on linear spaces*, Ph.D. thesis, Brandeis University, 1995.

[13] V. Talamanca: *A Gelfand-Beurling type formula for heights on endomorphism rings*, J. Number Theory **83** (2000), no. 1, 91–105.

[14] M. Widmer: *On certain infinite extensions of the rationals with Northcott property*, Monatsh. Math. **162** (2011), no. 3, 341–353.