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Adjoint symmetries for graded vector fields

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Abstract. Suppose that $\mathcal{M} = (M, \mathcal{A}_M)$ is a graded manifold and consider a direct subsheaf $\mathcal{D}$ of $Der\mathcal{A}_M$ and a graded vector field Γ on $\mathcal{M}$, both satisfying certain conditions. $\mathcal{D}$ is used to characterize the local expression of Γ. Thus we review some of the basic definitions, properties, and geometric structures related to the theory of adjoint symmetries on a graded manifold and discuss some ideas from Lagrangian supermechanics in an informal fashion. In the special case where $\mathcal{M}$ is the tangent supermanifold, we are able to find a generalization of the adjoint symmetry method for time-dependent second-order equations to the graded case. Finally, the relationship between adjoint symmetries of Γ and Lagrangians is studied.

Keywords: supermanifold, involutive distribution, second-order differential equation field, Lagrangian systems, adjoint symmetry.

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1 Introduction

This paper is a continuation of the previous paper [1], dealing with adjoint symmetries for super second order differential equations. Adjoint symmetries are 1-forms that are the dual objects of the symmetry vector fields of a second order differential equation field on TM. A similar situation already arises for adjoint symmetries of time-dependent second order equations, see for example [2], [7], [8], [13, 16].

Naturally, adjoint symmetry is a key concept for studying the geometry of the systems of super second order differential equations. Thus it is interesting to generalize this concept to the graded geometry and apply it to the study of Lagrangian supermechanics. In this process, a geometrical object play an important role, and that is the concept of a pseudo almost tangent structure. It is essential in the Lagrangian description of analytical supermechanics.

There are two approaches to provide supermechanics with a geometrical base. The configuration of each case is a graded manifold $\mathcal{M} = (M, \mathcal{A}_M)$ of dimension (m, n), but the difference in the approaches is related to a generalization of the tangent bundle in the graded case. In the first approach, supermechanical systems can be considered on the tangent supermanifold such that its dimension is $(2m, 2n)$, see [9]. In the second approach supermechanical systems considered on the tangent superbundle with dimension $(2m+n, 2n+m)$. Related references are [3, 6]. Because each of the tangent superbundle or the tangent supermanifold is a supermanifold, we want to bring some geometric structures related to the concept of adjoint symmetries to the configuration space $\mathcal{M}$. To achieve this, we first consider a graded vector field Γ on a $(m, n)-$dimensional graded manifold $\mathcal{M} = (M, \mathcal{A}_M)$ and a direct subsheaf $\mathcal{D}$ of $Der\mathcal{A}_M$ of rank (r, s), both satisfying certain conditions. When we studied the the transformed dynamics of Γ, we saw that the local representation of Γ is similar to a superspray. Thus we review some of the basic definitions, properties, and geometric structures related to the theory of adjoint symmetries on a graded manifold and discuss some ideas from Lagrangian supermechanics in an informal fashion. This is intended to give context to apply a similar discussion on tangential structures.

We associate to $\mathcal{D}$, a pseudo almost tangent structure such that in certain manifolds like the tangent supermanifold, it is often called an almost tangent structure. By using the Lie bracket of Γ and graded vector fields of $\mathcal{D}$, we construct another direct subsheaf $\mathcal{C} := \mathcal{D} + [\Gamma, \mathcal{D}]$ of $Der\mathcal{A}_M$ such that $rank_p(\mathcal{C}) = (r + rank[\Gamma, \mathcal{D}]_0, s + rank[\Gamma, \mathcal{D}]_1)$. As shown in [1], if $|\Gamma| = 0$, $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ have maximal ranks respectively r and s, and if $|\Gamma| = 1$, depending on the dimension of the graded manifold $\mathcal{M}$, there are several cases for introducing the maximal rank of $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ (which in this article, we only consider one of them).

In this paper, we only consider the situation $m = 2r + 1$ and $n = 2s$. As shown in [1], for each $p \in \mathcal{M}$, there is a coordinate neighborhood $\mathcal{U}$ of p and coordinates $(t, x_i, y_i; \eta_\mu, \zeta_\mu)$, for $i = 1, 2, \cdots, r$ and $\mu = 1, 2, \cdots, s$ such that $\mathcal{D}|_{\mathcal{U}} = \langle \frac{\partial}{\partial y_i}; \frac{\partial}{\partial \zeta_\mu} \rangle$, and the local expression of the graded vector field $\Gamma \in Der\mathcal{A}_M$ is

$$\Gamma = \frac{\partial}{\partial t} + y_i \frac{\partial}{\partial x_i} + \Gamma_i(t, x_i, y_i; \eta_\mu, \zeta_\mu) \frac{\partial}{\partial y_i} + \zeta_\mu \frac{\partial}{\partial \eta_\mu} + \Gamma'_\mu(x_i, y_i; \eta_\mu, \zeta_\mu) \frac{\partial}{\partial \zeta_\mu}.$$

In section 3, we introduce a new graded tensor field $\tilde{J}$ of type (1,1) on $\mathcal{M}$, defined using pseudo almost tangent structure J on $\mathcal{C}$. $\tilde{J}$ allows a useful characterization of $(Der\mathcal{A}_M)_\Gamma$. We show that if $X \in (Der\mathcal{A}_M)_\Gamma$ and $\mathcal{L}_\Gamma X \in (Der\mathcal{A}_M)_\Gamma$, then X is a pseudo-dynamical symmetry of Γ.

In section 4, we associate to Γ a subsheaf $(Der\mathcal{A}_M)^*_\Gamma$ of $(Der\mathcal{A}_M)^*$ which consists of those 1-forms ψ for which $\mathcal{L}_\Gamma(\tilde{J}^*(\psi)) = \psi$ (see Section 3 page 12 for the definition of $\tilde{J}^*$). We show that, for such a form ψ, $\mathcal{L}_\Gamma\psi$ is a section of $(Der\mathcal{A}_M)^*_\Gamma$ if and only if we have for $i \in \{1,\ldots,r\}$ and $\mu \in \{1,\ldots,s\}$

$$\Gamma\Gamma(a^i) + \Gamma(a^j\frac{\partial\Gamma_j}{\partial y_i}) + \Gamma(b^\nu\frac{\partial\Gamma'_\nu}{\partial y_i}) - a^j\frac{\partial\Gamma_j}{\partial x_i} - b^\nu\frac{\partial\Gamma'_\nu}{\partial x_i} = 0, \quad (1.1)$$

$$\Gamma\Gamma(b^\mu) - \Gamma(a^j\frac{\partial\Gamma_j}{\partial\zeta_\mu}) + \Gamma(b^\nu\frac{\partial\Gamma'_\nu}{\partial\zeta_\mu}) + a^j\frac{\partial\Gamma_j}{\partial\eta_\mu} - b^\nu\frac{\partial\Gamma'_\nu}{\partial\eta_\mu} = 0. \quad (1.2)$$

Such a 1- form is called a pseudo-adjoint symmetry of Γ. Thus pseudo-adjoint symmetries correspond to solutions a^i and b^μ of the equations (1.1) and (1.2). We show that if ψ is a pseudo-adjoint symmetry of Γ such that $\psi = \mathcal{L}_\Gamma(\tilde{J}^*(dG))$ for some superfunction G, then, $\Gamma(G)$ is a Lagrangian superfunction.

2 Preliminaries

In this section we give a brief introduction to involutive distributions, emphasizing aspects that apply to the study of Lagrangian supermechanics. This section is an abbreviated version of [1]. Let $\mathcal{M} = (M,\mathcal{A}_M)$ be an $(m,n)-$dimensional graded manifold, the sheaf of left $\mathcal{A}_M-$modules of derivations of a graded manifold $\mathcal{M}$ is the subsheaf of $End\mathcal{A}_M$ whose sections are linear graded derivations and denoted by $Der\mathcal{A}_M$ the sheaf of graded derivations of $\mathcal{A}_M$. Let $\mathcal{D}$ be a locally free sheaf of $\mathcal{A}_M-$modules. $\mathcal{D}$ is a direct subsheaf of $Der\mathcal{A}_M$ of rank (r,s), if for each point $p \in M$ there is an open subset U over which any set of generators $\{D_i, D\mu | 1 \le i \le r, 1 \le \mu \le s\}$ of the module $\mathcal{D}(U)$ can be enlarged to a set

$$\left\{ C_u, D_i, D_\mu, C_\alpha \;\middle|\; {}^{1\le i\le r}_{r+1\le u\le m} \; and \; {}^{1\le\mu\le s}_{s+1\le\alpha\le n} \; {}^{|C_u|=0}_{|D_i|=0} \; and \; {}^{|D_\mu|=1}_{|C_\alpha|=1} \right\}$$

of free generators of $Der\mathcal{A}_M$. A direct subsheaf $\mathcal{D}$ of $Der\mathcal{A}_M$ is involutive if $[\mathcal{D},\mathcal{D}] \subset \mathcal{D}$ (see [12]).

Theorem 2.1. (Frobenius [12]). Let $\mathcal{D} \subset Der\mathcal{A}_M$ be a direct subsheaf of rank (r,s). Then, $\mathcal{D}$ is involutive, if and only if for each $p \in M$ there exist a coordinate system $\{(y_i;\zeta_\mu) | 1 \le i \le m, 1 \le \mu \le n\}$, defined in a neighborhood $\mathcal{U} = (U,\mathcal{A}_U)$of p, such that,

$$\mathcal{D} = \left\langle \frac{\partial}{\partial y_i}; \frac{\partial}{\partial \zeta_\mu} \right\rangle, 1 \le i \le r, \quad 1 \le \mu \le s.$$

Let $\mathcal{D}$ be an involutive direct subsheaf of $Der\mathcal{A}_M$ of rank (r,s) such that $2r \le m$, $2s \le n$ and Γ a homogeneous graded vector field on $\mathcal{M}$ such that $\Gamma \notin \mathcal{D}$. Set

$$\mathcal{C} := \mathcal{D} + [\Gamma,\mathcal{D}] = \{D_1 + [\Gamma, D_2] : D_1, D_2 \in \mathcal{D}\}.$$

For each $p \in M$, since $\mathcal{D}$ is involutive, according to Theorem 2.1, there is a coordinate neighborhood $\mathcal{U}$ of p and coordinates $(q_u, y_i; \theta_\alpha, \zeta_\mu)$, for $u = 1, 2, \cdots, m-r$, $i = 1, 2, \cdots, r$, $\alpha = 1, 2, \cdots, n-s$ and $\mu = 1, 2, \cdots, s$ such that $\mathcal{D}|_{\mathcal{U}} = \langle \frac{\partial}{\partial y_i}; \frac{\partial}{\partial \zeta_\mu} \rangle$. In this coordinate system, the local representation of Γ is

$$\Gamma|_{\mathcal{U}} = \Gamma_u \frac{\partial}{\partial q_u} + \Gamma_i \frac{\partial}{\partial y_i} + \Gamma'_\alpha \frac{\partial}{\partial \theta_\alpha} + \Gamma'_\mu \frac{\partial}{\partial \zeta_\mu}, \tag{2.1}$$

where $\Gamma_u, \Gamma_i, \Gamma'_\mu$, and Γ'_α are smooth superfunctions on $\mathcal{U}$. We want to find the conditions under which two graded vector fields $[\Gamma|_{\mathcal{U}}, \frac{\partial}{\partial y_i}]$ and $[\Gamma|_{\mathcal{U}}, \frac{\partial}{\partial \zeta_\mu}]$ are linearly independent. The local representation of these graded vector fields are

$$\begin{aligned} [\Gamma|_{\mathcal{U}}, \frac{\partial}{\partial y_i}] &\equiv -\frac{\partial \Gamma_u}{\partial y_i}\frac{\partial}{\partial q_u} - \frac{\partial \Gamma'_\alpha}{\partial y_i}\frac{\partial}{\partial \theta_\alpha} \qquad (mod\, \mathcal{D}), \\ [\Gamma|_{\mathcal{U}}, \frac{\partial}{\partial \zeta_\mu}] &\equiv -(-1)^{|\Gamma_u|}\frac{\partial \Gamma_u}{\partial \zeta_\mu}\frac{\partial}{\partial q_u} + (-1)^{|\Gamma'_\alpha|}\frac{\partial \Gamma'_\alpha}{\partial \zeta_\mu}\frac{\partial}{\partial \theta_\alpha} \qquad (mod\, \mathcal{D}). \end{aligned}$$

If the local coefficients of these graded vector fields are zero, then we have $[\Gamma, \mathcal{D}] \subset \mathcal{D}$, and they are dependent vector fields. Therefore we suppose that $[\Gamma, \mathcal{D}] \cap \mathcal{D} = \{0\}$, i.e., if $D \in \mathcal{D}$ and $[\Gamma, D] \in \mathcal{D}$ then $D = 0$, and in the next theorem we show that they are linearly independent. Here a brief description of the geometry of Γ and $\mathcal{D}$ is given.

Theorem 2.2. Suppose that the graded vector field Γ on $\mathcal{M}$ is such that $[\Gamma, \mathcal{D}] \cap \mathcal{D} = \{0\}$.
(1) If $|\Gamma| = 0$, $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ have maximal ranks respectively r and s. Then $rank_p(\mathcal{C}) = (2r, 2s)$.
(2) If $|\Gamma| = 1$, then

- for $m = 2r, n = 2s$,
  - if $r \leq s$, both $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ have the same maximal rank r,
  - if $r > s$ both $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ have the same maximal rank s,
- for $m = 2r + 1, n = 2s$,
  - if $r < s$, $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ have maximal ranks respectively $r + 1$ and r,
  - if $r \geq s$ both $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ have the same maximal rank s,
- for $m = 2r, n = 2s + 1$,
  - if $r \leq s$, both $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ have the same maximal rank r,

  - if $r > s$, $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ have maximal ranks respectively s and $s+1$,

- for $m = 2r+1, n = 2s+1$,
  - if $r = s$, both $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ have the same maximal rank r,
  - if $r < s$, $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ have maximal ranks respectively $r+1$ and r,
  - if $r > s$, $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ have maximal ranks respectively s and $s+1$.

In each of these cases, $rank_p(\mathcal{C}) = (r + rank[\Gamma, \mathcal{D}]_0, s + rank[\Gamma, \mathcal{D}]_1)$.

Proof. Let $p \in M$ and $\mathcal{U}$ a coordinate neighborhood of p with local coordinates $(q_u, y_i; \theta_\alpha, \zeta_\mu)$, as above. Let $D^i(p)[\Gamma, \frac{\partial}{\partial y_i}](p) + D^\mu(p)[\Gamma, \frac{\partial}{\partial \zeta_\mu}](p) = 0$. A simple computation will shows that we have

$$\begin{cases} D^i \frac{\partial \Gamma_u}{\partial y_i} + (-1)^{|\Gamma_u|} D^\mu \frac{\partial \Gamma_u}{\partial \zeta_\mu} \equiv 0 \quad (mod\ \mathcal{D}), \\ D^i \frac{\partial \Gamma'_\alpha}{\partial y_i} - (-1)^{|\Gamma'_\alpha|} D^\mu \frac{\partial \Gamma'_\alpha}{\partial \zeta_\mu} \equiv 0 \quad (mod\ \mathcal{D}). \end{cases} \tag{2.2}$$

Let

$$J_\Gamma := \begin{pmatrix} \frac{\partial \Gamma_u}{\partial y_i} & \frac{\partial \Gamma'_\alpha}{\partial y_i} \\ (-1)^{|\Gamma_u|} \frac{\partial \Gamma_u}{\partial \zeta_\mu} & -(-1)^{|\Gamma'_\alpha|} \frac{\partial \Gamma'_\alpha}{\partial \zeta_\mu} \end{pmatrix}.$$

If $|\Gamma| = 0$ we see that the matrices

$$\left(\frac{\partial \Gamma_u}{\partial y_i} \right)_{1 \le i \le r, 1 \le u \le m-r}, \left(\frac{\partial \Gamma'_\alpha}{\partial \zeta_\mu} \right)_{1 \le \mu \le s, 1 \le \alpha \le n-s},$$

have maximal ranks respectively r and s. Let $rank_p J_\Gamma = (r, s)$. By permuting the $\Gamma_{u'}$ and $\Gamma'_{\alpha'}$, we may therefore assume that the matrices

$$\left(\frac{\partial \Gamma_{u'}}{\partial y_i} \right)_{1 \le i \le r, 1 \le u' \le r}, \left(\frac{\partial \Gamma'_{\alpha'}}{\partial \zeta_\mu} \right)_{1 \le \mu \le s, 1 \le \alpha' \le s},$$

are invertible at p. Then from (2.2) we conclude that $D^i(p) = 0 = D^\mu(p)$. This means that the graded vector fields $[\Gamma, \frac{\partial}{\partial y_i}]$ and $[\Gamma, \frac{\partial}{\partial \zeta_\mu}]$, $(1 \le i \le r$, and $1 \le \mu \le s)$ are linearly independent at p, thus $rank_p(\mathcal{C}) = (2r, 2s)$.

(2) Let $|\Gamma| = 1$. We may choose $m = 2r+1, n = 2s+1, \quad r < s$. Then the matrices

$$\left(\frac{\partial \Gamma'_\alpha}{\partial y_i} \right)_{1 \le i \le r, 1 \le \alpha \le n-s}, \left(\frac{\partial \Gamma_u}{\partial \zeta_\mu} \right)_{1 \le \mu \le s, 1 \le u \le m-r},$$

are even and have maximal ranks respectively r and $r+1$. A computation similar to part (1) shows that the odd graded vector fields $[\Gamma, \frac{\partial}{\partial y_i}]$ and the even graded vector fields $[\Gamma, \frac{\partial}{\partial \zeta_\mu}]$, $\quad (1 \leq i \leq r$, and $1 \leq \mu \leq r+1)$ are linearly independent at p, so $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ have maximal ranks respectively $r+1$ and r, and $rank_p(\mathcal{C}) = (2r+1, s+r)$. Similarly, one may choose $m = 2r+1, n = 2s+1, \quad r > s$, etc. We will therefore have twelve types of possibilities for m, n, r and s. In each of these cases, the matrices $\left(\frac{\partial \Gamma'_\alpha}{\partial y_i}\right)$, $\left(\frac{\partial \Gamma_u}{\partial \zeta_\mu}\right)$ are even and have maximal ranks and we have a number of odd graded vector fields $[\Gamma, \frac{\partial}{\partial y_i}]$ and a number of the even graded vector fields $[\Gamma, \frac{\partial}{\partial \zeta_\mu}]$ which are linearly independent at p. QED

Hereafter, unless otherwise stated, we will assume that the graded vector field Γ satisfies the conditions of Theorem 2.2, and $[\mathcal{C}, \mathcal{C}] \subset \mathcal{C}$. From the above theorem, we see that if $\{D_i, D_\mu\}$ is a local basis of $\mathcal{D}$ consisting of coordinate fileds $\partial/\partial y_i$ and $\partial/\partial \zeta_\mu$ of a local coordinate system $(q_u, y_i; \theta_\alpha, \zeta_\mu)$ and if we set $C_a = [\Gamma, D_a]$ (for $|D_a| = 0$) and $C_b = [\Gamma, D_b]$ (for $|D_b| = 1$), then $\{C_a, D_i, D_\mu, C_b\}$ is a local basis for $\mathcal{C}$, where $\{C_a, C_b\}$ are generators of $[\Gamma, \mathcal{D}]$. Thus $\mathcal{C}$ is a direct subsheaf of $Der\mathcal{A}_M$. Moreover, $C_a = [\Gamma, D_a]$ and $C_b = [\Gamma, D_b]$ are respectively odd and even graded vector fields whenever Γ is odd and $|D_a| = 0$, $|D_b| = 1$.

Now we consider a graded tensor field on $\mathcal{C}$ which can be extended to a tensor field on $\mathcal{M}$ as a nonlinear connection similar to the classical case, see [17, 19]. Consider the tensor field $J : \mathcal{C} \to \mathcal{C}$ of type (1,1) by

$$J(X_a) = 0 \quad and \quad J(Y_a) = -X_a, \quad X_a \in \{D_i, D_\mu\}, Y_a = [\Gamma, X_a]. \tag{2.3}$$

It is called pseudo almost tangent structure (see also [5, 9]). Clearly $J^2 = 0$ and $|J| = |\Gamma|$. If $|\Gamma| = 0$ then $Im\ J = Ker\ J = \mathcal{D}$.

Take a graded vector field Γ such that $[\Gamma, \mathcal{C}] \subset \mathcal{C}$ and consider the morphism $-\mathcal{L}_\Gamma J$. For any $C \in \mathcal{C}$ and $D \in \mathcal{D}$, we have

$$(\mathcal{L}_\Gamma J)(C) := [\Gamma, J(C)] - (-1)^{|J||\Gamma|} J[\Gamma, C],$$

and

$$(-\mathcal{L}_\Gamma J)(D) = -[\Gamma, J(D)] + (-1)^{|J||\Gamma|} J[\Gamma, D] = -(-1)^{|J||\Gamma|} D,$$

thus $(-\mathcal{L}_\Gamma J)^2(D) = D$. Also,

$$\begin{aligned} (-\mathcal{L}_\Gamma J)([\Gamma, D]) &= -[\Gamma, J([\Gamma, D])] + (-1)^{|J||\Gamma|} J[\Gamma, [\Gamma, D]] \\ &= [\Gamma, D] + (-1)^{|J||\Gamma|} J[\Gamma, [\Gamma, D]]. \end{aligned}$$

Since $J[\Gamma, [\Gamma, D]] \in \mathcal{D}$, we have $(-\mathcal{L}_\Gamma J)(J[\Gamma, [\Gamma, D]]) = -(-1)^{|J||\Gamma|} J[\Gamma, [\Gamma, D]]$, and therefore

$$(-\mathcal{L}_\Gamma J)^2 \Big([\Gamma, D]\Big) = \Big([\Gamma, D] + ((-1)^{|J||\Gamma|} - 1) J\Big[\Gamma, [\Gamma, D]\Big]\Big).$$

If $|\Gamma| = 0$, it is clear that $(\mathcal{L}_\Gamma J)^2 = Id$.

Remark 2.3. Let $\mathcal{D} \subset Der\mathcal{A}_M$ be an involutive direct subsheaf with even and odd generators $\{D_i, D_\mu\}$, $(1 \leq i \leq r$, and $1 \leq \mu \leq s)$ such that $[X_a, X_b] = 0, \forall X_a, X_b \in \{D_i, D_\mu\}$. We want to find the conditions under which for all a, b, $[X_a, Y_b] \in \mathcal{D}$, where $Y_a = [\Gamma, X_a]$. If we change basis to

$$\widehat{D}_i = A_{ij}D_j + B_{i\nu}D_\nu, \quad \widehat{D}_\mu = E_{\mu j}D_j + F_{\mu\nu}D_\nu,$$

where $A_{ij}, B_{i\nu}, E_{\mu j}$ and $F_{\mu\nu}$ are superfunctions and

$$G = (G_{ab}) = \begin{pmatrix} A_{ij} & B_{i\nu} \\ E_{\mu j} & F_{\mu\nu} \end{pmatrix},$$

and if we again assume that $\hat{X}_a, \hat{X}_b \in \{\hat{D}_i, \hat{D}_\mu\}$, then the necessary and sufficient conditions for $[\widehat{X}_a, \widehat{X}_b] = 0$, is that

$$\forall d, \quad G_{ac}X_c(G_{bd}) = (-1)^{|G_{bc}X_c||G_{ad}X_d|}G_{bc}X_c(G_{ad}),$$

(we use the Einstein convention, that is, repeated indices denotes summation over their range). For example, $[\widehat{D}_i, \widehat{D}_\mu] = 0$ if and only if

$$\begin{cases} A_{ij}D_j(E_{\mu k}) + B_{i\nu}D_\nu(E_{\mu k}) = E_{\mu j}D_j(A_{ik}) + F_{\mu\nu}D_\nu(A_{ik}), \\ A_{ij}D_j(F_{\mu\omega}) + B_{i\nu}D_\nu(F_{\mu\omega}) = E_{\mu j}D_j(B_{i\omega}) + F_{\mu\nu}D_\nu(B_{i\omega}). \end{cases}$$

Also C_i and C_μ change to

$$\begin{aligned} \widehat{C}_i &= \Gamma(A_{ij})D_j + \Gamma(B_{i\nu})D_\nu + A_{ij}C_j + (-1)^{|\Gamma|}B_{i\nu}C_\nu, \\ \widehat{C}_\mu &= \Gamma(E_{\mu j})D_j + \Gamma(F_{\mu\nu})D_\nu + (-1)^{|\Gamma|}E_{\mu j}C_j + F_{\mu\nu}C_\nu. \end{aligned}$$

On the other hand, $\mathcal{C}$ is involutive, then there are superfunctions $\alpha^c_{ab}, \beta^c_{ab} \in \mathcal{A}_M$ such that

$$[X_a, Y_b] = \alpha^c_{ab}X_c + \beta^c_{ab}Y_c, \quad X_a, X_c \in \{D_i, D_\mu\}, \quad Y_b, Y_c \in \{C_i, C_\mu\}.$$

Therefore $[\Gamma, [X_a, X_b]] = 0$. From the graded Jacobi identity, we have

$$0 = [\Gamma, [X_a, X_b]] = -(-1)^{(|\Gamma|+|X_a|)|X_b|}[X_b, Y_a] + (-1)^{|\Gamma||X_a|}[X_a, Y_b],$$

and the superfunctions $\alpha^c_{ab}, \beta^c_{ab}$ are:

- symmetric for lower case Latin indexes,

- symmetric up to $(-1)^{|\Gamma|}$, i.e., $\beta^c_{ab} = (-1)^{|\Gamma|}\beta^c_{ba}$, if one of the lower indices is Greek,
- antisymmetric for lower case Greek indexes.

If we change the basis of $\mathcal{D}$ to $\widehat{X}_a = G_{ab}X_b$, then we have

$$\begin{aligned}[\widehat{X}_a, \widehat{Y}_c] &= \Big[G_{ab}X_b, \Gamma(G_{cd})X_d + (-1)^{|\Gamma||G_{cd}|}G_{cd}Y_d\Big] \\ &\equiv \Big\{(-1)^{|\Gamma||G_{cd}|}G_{ab}X_b(G_{cd}) + (-1)^{|X_b||G_{ce}|+|\Gamma||G_{ce}|}G_{ab}G_{ce}\beta^d_{be}\Big\}Y_d \quad (mod\, \mathcal{D}).\end{aligned}$$

Thus $[\widehat{X}_a, \widehat{Y}_c] \in \mathcal{D}$ if and only if for each d,

$$(-1)^{|\Gamma||G_{cd}|}G_{ab}X_b(G_{cd}) + (-1)^{|G_{ce}|(|\Gamma|+|X_b|)}G_{ab}G_{ce}\beta^d_{be} = 0.$$

Theorem 2.4. Let Γ be an even graded vector field on $\mathcal{M}$ and let $\mathcal{D}$ and $\mathcal{C}$ be involutive, as above. We can find local supercoordinates $(t_l, x_i, y_i; \tau_\rho, \eta_\mu, \zeta_\mu)$ on $\mathcal{M}$, $l = 1, \cdots, m-2r$, $i = 1, \cdots, r, \rho = 1, \cdots, n-2s$, $\mu = 1, \cdots, s$, such that

$$D_i = \frac{\partial}{\partial y_i}, \qquad C_i \equiv -\frac{\partial}{\partial x_i} \quad (mod\, \mathcal{D}), \tag{2.4}$$

$$D_\mu = \frac{\partial}{\partial \zeta_\mu}, \qquad C_\mu \equiv -\frac{\partial}{\partial \eta_\mu} \quad (mod\, \mathcal{D}). \tag{2.5}$$

Proof. By the Frobenius theorem, we take a coordinate neighborhood $\mathcal{U}$ of $p \in \mathcal{M}$, with supercoordinates $(q_u, y_i; \theta_\alpha, \zeta_\mu), u = 1, \cdots, m-r,\ i = 1, \cdots, r, \alpha = 1, \cdots, n-s$, $\mu = 1, \cdots, s$, such that $D_i = \partial/\partial y_i$ and $D_\mu = \partial/\partial \zeta_\mu$. Let $\mathcal{U}$ be the image of a product of open subsets $\mathcal{U}_1 \subset \mathcal{R}^{m-r|n-s}$ and $\mathcal{U}_2 \subset \mathcal{R}^{r|s}$, where $0 \in \mathcal{U}_2$ (c.f. [20]). Then $y_i = 0, \zeta_\mu = 0$ define a graded submanifold (in the sense of 3.2.1 of [10]) $\mathcal{N} = (N, \mathcal{A}_N)$ of $\mathcal{U}$ of graded dimension $(m-r, n-s)$. Denote by $pr : \mathcal{U} \to \mathcal{N}$ the corresponding projection morphism, then $pr_*(D_i) = pr_*(D_\mu) = 0$. It is clear that the restrictions of C_i and C_μ to $\mathcal{U}$ are pr_*-projectable to $\mathcal{N}$. We denote pr_*-projections of C_j and C_ν by graded vector fields $\bar{C}_j$ and $\bar{C}_\nu$ on $\mathcal{N}$ respectively. Let us denote by $\bar{\mathcal{C}}$ the graded direct subsheaf on $\mathcal{N}$ spanned by $\bar{C}_i$ and $\bar{C}_\mu$, i.e., $\bar{\mathcal{C}} = \langle \bar{C}_i; \bar{C}_\mu\rangle$. Since $\mathcal{C}$ is involutive, $\bar{\mathcal{C}}$ is an involutive direct subsheaf of rank (r, s). Now again repeat the Theorem 2.2 for the graded manifold $\mathcal{N}$, the graded vector field Γ, and graded direct subsheaf $\bar{\mathcal{C}}$ of rank (r, s), then we may choose supercoordinates $(t_l, x_i; \tau_\rho, \eta_\mu)$ on $\mathcal{N}$, where $l = 1, \cdots, m-2r, \rho = 1, \cdots, n-2s$, such that the rank of supermatrix $\bar{J}_\Gamma$ in this case is (r, s).Then with respect to the supercoordinates $(t_l, x_i, y_i; \tau_\rho, \eta_\mu, \zeta_\mu)$ on $\mathcal{U}$ we have

$$D_i = \frac{\partial}{\partial y_i}, \qquad C_i \equiv P_{ij}(x_i; \eta_\mu)\frac{\partial}{\partial x_j} + Q_{i\nu}(x_i; \eta_\mu)\frac{\partial}{\partial \eta_\nu} \qquad (mod\, \mathcal{D}),$$

$$D_\mu = \frac{\partial}{\partial \zeta_\mu}, \qquad C_\mu \equiv R_{\mu j}(x_i; \eta_\mu)\frac{\partial}{\partial x_j} + S_{\mu\nu}(x_i; \eta_\mu)\frac{\partial}{\partial \eta_\nu} \qquad (mod\, \mathcal{D}),$$

where the coefficients $P_{ij}, Q_{i\nu}, R_{\mu j}$, and $S_{\mu\nu}$ are the components of a nonsingular supermatrix. Let

$$\begin{pmatrix} A_{ij} & B_{i\nu} \\ E_{\mu j} & F_{\mu\nu} \end{pmatrix}$$

be its inverse. If we set

$$\widehat{D}_i = A_{ij}(t,x;\tau,\eta)D_j + B_{i\nu}(t,x;\tau,\eta)D_\nu,$$
$$\widehat{D}_\mu = E_{\mu j}(t,x;\tau,\eta)D_j + F_{\mu\nu}(t,x;\tau,\eta)D_\nu,$$

then

$$\widehat{C}_i = [\Gamma, \widehat{D}_i] = \Gamma(A_{ij})D_j + (-1)^{|\Gamma|}\Gamma(B_{i\nu})D_\nu + A_{ij}C_j + B_{i\nu}C_\nu,$$
$$\widehat{C}_\mu = [\Gamma, \widehat{D}_\mu] = (-1)^{|\Gamma|}\Gamma(E_{\mu j})D_j + \Gamma(F_{\mu\nu})D_\nu + E_{\mu j}C_j + F_{\mu\nu}C_\nu.$$

A simple computation shows that

$$\widehat{D}_i = A_{ij}\frac{\partial}{\partial y_j} + B_{i\nu}\frac{\partial}{\partial \zeta_\nu}, \qquad \widehat{C}_i \equiv \frac{\partial}{\partial x_i} \qquad (mod\, \mathcal{D}),$$
$$\widehat{D}_\mu = E_{\mu j}\frac{\partial}{\partial y_j} + F_{\mu\nu}\frac{\partial}{\partial \zeta_\nu}, \qquad \widehat{C}_\mu \equiv \frac{\partial}{\partial \eta_\mu} \qquad (mod\, \mathcal{D}).$$

Therefore, a new change of the coordinates

$$\widehat{t}_l = t_l, \quad \widehat{y}_i = P_{ij}(x_i;\eta_\mu)y_j + Q_{i\nu}(x_i;\eta_\mu)\zeta_\nu, \quad \widehat{x}_i = -x_i,$$
$$\widehat{\tau}_\rho = \tau_\rho, \quad \widehat{\zeta}_\mu = R_{\mu j}(x_i;\eta_\mu)y_j + S_{\mu\nu}(x_i;\eta_\mu)\zeta_\nu, \quad \widehat{\eta}_\mu = -\eta_\mu,$$

may be performed to bring the local basis of $Der\mathcal{A}_M$ into the form

$$\frac{\partial}{\partial \widehat{t}_l} \equiv \frac{\partial}{\partial t_l} \quad (mod\, \mathcal{D}), \quad \frac{\partial}{\partial \widehat{x}_i} \equiv -\frac{\partial}{\partial x_i} \quad (mod\, \mathcal{D}), \quad \frac{\partial}{\partial \widehat{y}_i} = A_{ij}\frac{\partial}{\partial y_j} + E_{i\nu}\frac{\partial}{\partial \zeta_\nu},$$
$$\frac{\partial}{\partial \widehat{\tau}_\rho} \equiv \frac{\partial}{\partial \tau_\rho} \quad (mod\, \mathcal{D}), \quad \frac{\partial}{\partial \widehat{\eta}_\mu} \equiv -\frac{\partial}{\partial \eta_\mu} \quad (mod\, \mathcal{D}), \quad \frac{\partial}{\partial \widehat{\zeta}_\mu} = B_{\mu j}\frac{\partial}{\partial y_j} + F_{\mu\nu}\frac{\partial}{\partial \zeta_\nu},$$

and this completes the proof. QED

Theorem 2.5. Let $\mathcal{D}$ and $\mathcal{C}$ be involutive and assume that $|\Gamma| = 1$. We can find local supercoordinates $(t_l, x_a, y_i; \tau_\rho, \eta_b, \zeta_\mu)$ on $\mathcal{M}$, $l = 1, \cdots, l_1$, $a = 1, \cdots, a_1$, $i = 1, \cdots, r, \rho = 1, \cdots, \rho_1$, $b = 1, \cdots, b_1$, $\mu = 1, \cdots, s$, such that

$$D_i = \frac{\partial}{\partial y_i}, \qquad C_b = [\Gamma, \frac{\partial}{\partial y_b}] \equiv -\frac{\partial}{\partial \eta_b} \quad (mod\, \mathcal{D}), \tag{2.6}$$

$$D_\mu = \frac{\partial}{\partial \zeta_\mu}, \qquad C_a = [\Gamma, \frac{\partial}{\partial \zeta_a}] \equiv \frac{\partial}{\partial x_a} \quad (mod\, \mathcal{D}), \tag{2.7}$$

and l_1, a_1, b_1 and ρ_1 are given as in the following table (Table 1):

$\dim\mathcal{M}=(m,n)$	$r=s$	$r<s$	$r>s$
$(2r,2s)$	$l_1=0,a_1=r,$ $b_1=s,\rho_1=0$	$l_1=0,a_1=r,$ $b_1=r,\rho_1=s-r$	$l_1=r-s,a_1=s,$ $b_1=s,\rho_1=0$
$(2r+1,2s)$	$l_1=1,a_1=s,$ $b_1=s,\rho_1=0$	$l_1=0,a_1=r+1,$ $b_1=r,\rho_1=s-r$	$l_1=r+1-s,a_1=s,$ $b_1=s,\rho_1=0$
$(2r,2s+1)$	$l_1=0,a_1=s,$ $b_1=s,\rho_1=1$	$l_1=0,a_1=r,$ $b_1=r,\rho_1=s+1-r$	$l_1=r-s,a_1=s,$ $b_1=s+1,\rho_1=0$
$(2r+1,2s+1)$	$l_1=1,a_1=s,$ $b_1=s,\rho_1=1$	$l_1=0,a_1=r+1,$ $b_1=r,\rho_1=s+1-r$	$l_1=r+1-s,a_1=s,$ $b_1=s+1,\rho_1=0$

Table 1. The range of indices l_1, a_1, b_1 and ρ_1

Proof. We consider the result in Theorem 2.4 and apply it to the case that $|\Gamma|=1$. There is a coordinate neighborhood $\mathcal{U}$ of $p\in\mathcal{M}$, with supercoordinates $(q_u, y_i;\theta_\alpha,\zeta_\mu)$, $u=1,\cdots,m-r,\ i=1,\cdots,r,\alpha=1,\cdots,n-s,\ \mu=1,\cdots,s,$ a graded submanifold $\mathcal{N}=(N,\mathcal{A}_N)$ of $\mathcal{U}$ of graded dimension $(m-r,n-s)$ and the corresponding projection morphism $pr:\mathcal{U}\to\mathcal{N}$, such that $pr_*(D_i)=pr_*(D_\mu)=0$.

As we have seen in Theorem 2.2, we have twelve types of possibilities for m,n,r and s. We may choose $m=2r+1, n=2s+1,\quad r<s$ to prove the theorem and a similar proof can also be performed in other cases.

Since $rank_p(\mathcal{C})=(2r+1,s+r)$, we assume that

$$\mathcal{C}=\left\langle C_a, D_i, D_\mu, C_b \middle| \begin{smallmatrix} a=1,\cdots,r+1\\ b=1,\cdots,r\end{smallmatrix} \ and\ \begin{smallmatrix} C_a=[\Gamma,\frac{\partial}{\partial\zeta_a}]\\ C_b=[\Gamma,\frac{\partial}{\partial y_b}]\end{smallmatrix}\ and\ \begin{smallmatrix}|C_a|=0\\ |C_b|=1\end{smallmatrix}\right\rangle.$$

Then $\bar{\mathcal{C}}=\langle\bar{C}_a;\bar{C}_b\rangle$ is an involutive direct subsheaf of $Der\mathcal{A}_M$ and $rank_p\bar{\mathcal{C}}=(r+1,r)$, where $\bar{C}_a$ and $\bar{C}_b$ are pr_*-projections of C_a and C_b on $\mathcal{N}$ respectively. We can continue the method discussed in Theorem 2.2, but there is another way to find the local generators of $\mathcal{C}$. By the Frobenius theorem, we take a local coordinate $(x'_a;\tau'_\rho,\eta'_b), \rho=1,...,s+1-r$, on $\mathcal{N}$ of $\tilde{pr}(p)$, such that

$$\bar{C}_a=\frac{\partial}{\partial x'_a},\bar{C}_b=\frac{\partial}{\partial\eta'_b}.$$

Then there exists a coordinate neighborhood $\mathcal{U}'$ of $p\in\mathcal{M}$, with local coordinates

$$\left\{pr^*x'_a, y'_i; pr^*\tau'_\rho, pr^*\eta'_b,\zeta'_\alpha) \middle| \begin{smallmatrix} a=1,\cdots,r+1\\ b=1,\cdots,r\end{smallmatrix}\ and\ \ i=1,...,r\ and\ \begin{smallmatrix}\alpha=1,...,s\\ \rho=1,...,s+1-r\end{smallmatrix}\right\}$$

such that $D_i=\frac{\partial}{\partial y'_i},\quad D_\alpha=\frac{\partial}{\partial\zeta'_\alpha}, C_a\equiv\frac{\partial}{\partial pr^*x'_a}, C_b\equiv\frac{\partial}{\partial pr^*\eta'_b}(mod\,\mathcal{D})$. We shall write these coordinates as $\{x_a, y_i;\tau_\rho,\eta_b,\zeta_\alpha\}$. Thus, a new change of the coordinates $\{x_a, y_i;\, \tau_\rho, \eta_b, \zeta_\alpha\} \mapsto \{x_a, y_i; \tau_\rho, \eta_b, \zeta_\alpha\}$ may be performed to complete the proof. QED

Proposition 2.6. If both $\mathcal{D}$ and $\mathcal{C}$ are involutive, then there is a graded commuting basis $\{X_a\}$ of $\mathcal{D}$ such that for all a, b we have $[X_a, Y_b] \in \mathcal{D}$, where $X_a \in \{D_i, D_\mu\}$ and $Y_a = [\Gamma, X_a]$.

Proof. This is an immediate consequence of Theorems 2.4 and 2.5. QED

Now we want to write the local form of Γ in a local supercoordinate system. There are two cases to consider, $\Gamma \in \mathcal{C}$ and $\Gamma \notin \mathcal{C}$.

Theorem 2.7. Let $\mathcal{D}$ and $\mathcal{C}$ be involutive and $\Gamma \in \mathcal{C}$. Assume that the set $N = \{z \in M : \Gamma(z) \in \mathcal{D}(z)\} \subset M$, is nonempty.
(1) If $|\Gamma| = 0$, then we may choose supercoordinates $(t_l, x_i, y_i; \tau_\rho, \eta_\mu, \zeta_\mu), i = 1, ..., r,\ l = 1, ..., m-2r, \mu = 1, ...s, \rho = 1, ..., n-2s$, with respect to which

$$\Gamma = y_i \frac{\partial}{\partial x_i} + \Gamma_i(t, x, y; \tau, \eta, \zeta) \frac{\partial}{\partial y_i} + \zeta_\mu \frac{\partial}{\partial \eta_\mu} + \Gamma'_\mu(t, x, y; \tau, \eta, \zeta) \frac{\partial}{\partial \zeta_\mu}.$$

(2) If $|\Gamma| = 1$, then we may choose supercoordinates $(t_l, x_a, y_i; \tau_\rho, \eta_b, \zeta_\mu)$, as described in Theorem 2.5, with respect to which

$$\Gamma = \zeta_a \frac{\partial}{\partial x_a} + \Gamma_i(t, x, y; \tau, \eta, \zeta) \frac{\partial}{\partial y_i} + y_b \frac{\partial}{\partial \eta_b} + \Gamma'_\mu(t, x, y; \tau, \eta, \zeta) \frac{\partial}{\partial \zeta_\mu}.$$

Proof. Since $\Gamma \in \mathcal{C}$, with respect to the basis $\{C_i, D_i; D_\mu, C_\mu\}$ for $\mathcal{C}$ (see Proposition 2.6), we have

$$\Gamma = f_i D_i + \bar{f}_\mu D_\mu + g_i C_i + \bar{g}_\mu C_\mu, \tag{2.8}$$

where $f_i, \bar{f}_\mu, g_i$, and $\bar{g}_\mu$ are superfunctions on $\mathcal{M}$. Then from

$$\begin{aligned} C_i &= [\Gamma, D_i] \equiv -D_i(g_j)C_j - D_i(\bar{g}_\mu)C_\mu \quad (mod\ \mathcal{D}), \\ C_\mu &= [\Gamma, D_\mu] \equiv -(-1)^{|\Gamma|} D_\mu(g_i)C_i - (-1)^{|\Gamma|} D_\mu(\bar{g}_\nu)C_\nu \quad (mod\ \mathcal{D}), \end{aligned}$$

we conclude that $D_i(g_j) = -\delta_i^j, D_i(\bar{g}_\mu) = 0 = D_\mu(g_i)$, and $D_\mu(\bar{g}_\nu) = -(-1)^{|\Gamma|}\delta_\mu^\nu$, so

$$\begin{pmatrix} D_i(g_j) & D_i(\bar{g}_\nu) \\ D_\mu(g_j) & D_\mu(\bar{g}_\nu) \end{pmatrix}$$

is nonsingular. Note that in this computation we use $[X_a, Y_b] \in \mathcal{D}$, where $X_a \in \{D_i, D_\mu\}$ and $Y_a = [\Gamma, X_a]$. We sketch the proof of the theorem for the case (1). By the Frobenius theorem we take a coordinate neighborhood $\mathcal{U}$ in $\mathcal{M}$, with supercoordinates $(q_u, y_i; \theta_\alpha, \zeta_\mu)$ such that $D_i = \partial/\partial y_i$ and $D_\mu = \partial/\partial \zeta_\mu$, (see

Theorem 2.4). Now consider the Jacobian matrix of the superfunctions g_i and $\bar{g}_\mu$ with respect to D_i and D_μ, it is nonsingular of rank (r,s). Then for each $p \in N$, $\Gamma(p) \in \mathcal{D}$ if and only if $g_i(p) = 0$ and $\bar{g}_\mu(p) = 0$. Thus we have a closed embedded supermanifold $\mathcal{N}$ as a subsupermanifold of $\mathcal{M}$ with the base manifold given by $\{\tilde{g}_i^{-1}(0)|i=1,...,r\} \cap \{\tilde{g}_\mu^{-1}(0)|i=1,...,s\}$. From Theorem 2.4 we can take supercoordinates $(t_l, x_i, y_i; \tau_\rho, \eta_\mu, \zeta_\mu)$ such that

$$C_i \equiv -\frac{\partial}{\partial x_i} \quad (mod\, \mathcal{D}), \qquad C_\mu \equiv -\frac{\partial}{\partial \eta_\mu} \quad (mod\, \mathcal{D}).$$

Then $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial \eta_\mu}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial \zeta_\mu}\}$ evaluated at p, is another basis for $\mathcal{C}$, and in this supercoordinates

$$\Gamma = \gamma_i \frac{\partial}{\partial x_i} + \gamma'_\mu \frac{\partial}{\partial \eta_\mu} + \Gamma_i \frac{\partial}{\partial y_i} + \Gamma'_\mu \frac{\partial}{\partial \zeta_\mu},$$

where $\gamma_i, \gamma'_\mu, \Gamma_i$, and Γ'_μ are superfunctions of $(t,x,y;\tau,\eta,\zeta)$. Note that $\Gamma \in \mathcal{C}$, so in the local form of Γ, the coefficients of $\frac{\partial}{\partial t_l}$ and $\frac{\partial}{\partial \tau_\rho}$ are zero. We will find the coefficients γ_i and γ'_μ by using $C_i = [\Gamma, \frac{\partial}{\partial y_i}]$ and $C_\mu = [\Gamma, \frac{\partial}{\partial \zeta_\mu}]$. Thus we have

$$C_i \equiv -\frac{\partial \gamma_j}{\partial y_i}\frac{\partial}{\partial x_j} - \frac{\partial \gamma'_\mu}{\partial y_i}\frac{\partial}{\partial \eta_\mu} \ (mod\, \mathcal{D}), \quad C_\mu \equiv -\left(\frac{\partial \gamma_i}{\partial \zeta_\mu}\frac{\partial}{\partial x_i} + \frac{\partial \gamma'_\nu}{\partial \zeta_\mu}\frac{\partial}{\partial \eta_\nu}\right) \ (mod\, \mathcal{D}).$$

For each $p \in N$, $\gamma_i(p) = 0$ and $\gamma'_\mu(p) = 0$, then from

$$\frac{\partial \gamma_j}{\partial y_i} = \delta_i^j, \quad \frac{\partial \gamma'_\nu}{\partial y_i} = 0 = \frac{\partial \gamma_j}{\partial \zeta_\mu}, \quad \frac{\partial \gamma'_\nu}{\partial \zeta_\mu} = \delta_\mu^\nu,$$

we have $\gamma_i(t,x,y;\tau,\eta,\zeta) = y_i, \qquad \gamma'_\mu(t,x,y;\tau,\eta,\zeta) = \zeta_\mu$.

(2) Let $|\Gamma| = 1$, we consider again Γ as a linear combination of $\{C_i, D_i; D_\mu, C_\mu\}$ as (2.8). Then $|g_i| = 0$ and $|\bar{g}_\mu| = 1$ and the Jacobian matrix of the superfunctions g_i and $\bar{g}_\mu$ with respect to D_i and D_μ, is nonsingular of rank (r,s). Repeat the above analysis for this case and using Theorem 2.5, then there exist a coordinate system $(t_l, x_a, y_i; \tau_\rho, \eta_b, \zeta_\mu)$ on $\mathcal{M}$, $l = 1, \cdots, l_1$, $a = 1, \cdots, a_1$, $i = 1, \cdots, r$, $\rho = 1, \cdots, \rho_1$, $b = 1, \cdots, b_1$, $\mu = 1, \cdots, s$, such that

$$D_i = \frac{\partial}{\partial y_i}, \qquad C_a = [\Gamma, \frac{\partial}{\partial y_a}] \equiv -\frac{\partial}{\partial \eta_a} \quad (mod\, \mathcal{D}), \tag{2.9}$$

$$D_\mu = \frac{\partial}{\partial \zeta_\mu}, \qquad C_\mu = [\Gamma, \frac{\partial}{\partial \zeta_\mu}] \equiv \frac{\partial}{\partial x_\mu} \quad (mod\, \mathcal{D}), \tag{2.10}$$

and l_1, a_1, b_1 and ρ_1 are given in the Table 1. According to this table, we have twelve types of possibilities for m, n, r, s. In each of these cases we may write Γ

in the form

$$\Gamma = \gamma_a \frac{\partial}{\partial x_a} + \gamma'_b \frac{\partial}{\partial \eta_b} + \Gamma_i \frac{\partial}{\partial y_i} + \Gamma'_\mu \frac{\partial}{\partial \zeta_\mu},$$

and it is a straightforward matter to verify that

$$\frac{\partial \gamma_a}{\partial \zeta_\mu} = \delta^a_\mu, \quad \frac{\partial \gamma'_\alpha}{\partial \zeta_\mu} = 0 = \frac{\partial \gamma'_b}{\partial y_a}, \quad \frac{\partial \gamma'_\alpha}{\partial y_a} = \delta^\alpha_a.$$

For each $p \in N$, $\gamma_a(p) = 0$ and $\gamma'_\mu(p) = 0$, then $\gamma_a = \zeta_a$ and $\gamma'_b = y_b$ and this completes the proof. QED

Theorem 2.8. Let $\mathcal{D}$ and $\mathcal{C}$ be involutive. Let Γ be everywhere independent of $\mathcal{C}$ and $[\Gamma, \mathcal{C}] \subset \mathcal{C}$.
(1) If $|\Gamma| = 0$, then we may choose supercoordinates $(t_1, x_i, y_i; \tau_1, \eta_\mu, \zeta_\mu), i = 1, ..., r,\ \mu = 1, ...s$, such that

- (a) $\Gamma = \frac{\partial}{\partial t_1} + y_i \frac{\partial}{\partial x_i} + \Gamma_i \frac{\partial}{\partial y_i} + \zeta_\mu \frac{\partial}{\partial \eta_\mu} + \Gamma'_\mu \frac{\partial}{\partial \zeta_\mu}$, if the coefficient of $\frac{\partial}{\partial \tau_1}$ in Γ is zero,
- (b) $\Gamma = \tau_1 \frac{\partial}{\partial \tau_1} + y_i \frac{\partial}{\partial x_i} + \Gamma_i \frac{\partial}{\partial y_i} + \zeta_\mu \frac{\partial}{\partial \eta_\mu} + \Gamma'_\mu \frac{\partial}{\partial \zeta_\mu}$, if the coefficient of $\frac{\partial}{\partial t_1}$ in Γ is zero,
- (c) $\Gamma = \frac{\partial}{\partial t_1} + \tau_1 \frac{\partial}{\partial \tau_1} + y_i \frac{\partial}{\partial x_i} + \Gamma_i \frac{\partial}{\partial y_i} + \zeta_\mu \frac{\partial}{\partial \eta_\mu} + \Gamma'_\mu \frac{\partial}{\partial \zeta_\mu}$, if the coefficients of $\frac{\partial}{\partial t_1}$ and $\frac{\partial}{\partial \tau_1}$ in Γ are nonzero, where Γ_i and Γ'_μ are superfunctions on $\mathcal{M}$.

(2) If $|\Gamma| = 1$, then we may choose supercoordinates $(t_l, x_a, y_i; \tau_\rho, \eta_b, \zeta_\mu)$, as described in Theorem 2.5, with respect to which

$$\Gamma = \phi_l \frac{\partial}{\partial t_l} + \zeta_a \frac{\partial}{\partial x_a} + \Gamma_i(t, x, y; \tau, \eta, \zeta) \frac{\partial}{\partial y_i} + \varphi_\rho \frac{\partial}{\partial \tau_\rho} + y_b \frac{\partial}{\partial \eta_b} + \Gamma'_\mu(t, x, y; \tau, \eta, \zeta) \frac{\partial}{\partial \zeta_\mu}.$$

where ϕ_l and φ_ρ are independent of x, y, η and ζ and Γ_i and Γ'_μ are superfunctions on $\mathcal{M}$.

Proof. Using the same arguments we used in the proof of Theorem 2.4 we can take supercoordinates $(t_l, x_i, y_i; \tau_\rho, \eta_\mu, \zeta_\mu)$ such that

$$\begin{aligned} \Gamma &\equiv \phi_l(t, x, y; \tau, \eta, \zeta) \frac{\partial}{\partial t_l} + \varphi_\rho(t, x, y; \tau, \eta, \zeta) \frac{\partial}{\partial \tau_\rho} \quad (mod\ \mathcal{C}). \\ &= \phi_l \frac{\partial}{\partial t_l} + \varphi_\rho \frac{\partial}{\partial \tau_\rho} + \gamma_i \frac{\partial}{\partial x_i} + \gamma'_\mu \frac{\partial}{\partial \eta_\mu} + \Gamma_i \frac{\partial}{\partial y_i} + \Gamma'_\mu \frac{\partial}{\partial \zeta_\mu}. \end{aligned}$$

From $[\Gamma, \mathcal{C}] \subset \mathcal{C}$ we conclude that the derivative of ϕ and φ with respect to x, y, η, and ζ are zero, then ϕ and φ depend only on the t and τ. According to the procedure given in the previous theorem, we have

$$\frac{\partial \gamma_j}{\partial y_i} = \delta_i^j, \quad \frac{\partial \gamma'_\nu}{\partial y_i} = 0 = \frac{\partial \gamma_j}{\partial \zeta_\mu}, \quad \frac{\partial \gamma'_\nu}{\partial \zeta_\mu} = \delta_\mu^\nu.$$

Then

$$\gamma_i(t, x, y; \tau, \eta, \zeta) = y_i + h_i(t, x; \tau, \eta), \qquad \gamma'_\mu(t, x, y; \tau, \eta, \zeta) = \zeta_\mu + k_\mu(t, x; \tau, \eta).$$

Now if $\Gamma \neq 0$, since ϕ and φ are arbitrary, we may choose $\phi \neq 0$ and $\varphi = 0$ to finding the local expression for Γ. Similarly, one may choose $\phi = 0$ and $\varphi \neq 0$, etc. So by a transformation of the coordinates t_l we may take $\Gamma \equiv \frac{\partial}{\partial t_1} \quad (mod\, \mathcal{C})$.

Consider the new change of supercoordinates

$$\begin{array}{lll} \widehat{t}_1 = t_1, & \widehat{y}_i = y_i + h_i(t, x; \tau, \eta), & \widehat{x}_i = x_i, \\ \widehat{\tau}_1 = \tau_1, & \widehat{\zeta}_\mu = \zeta_\mu + k_\mu(t, x; \tau, \eta), & \widehat{\eta}_\mu = \eta_\mu. \end{array}$$

Therefore

$$\begin{aligned} \Gamma \;=\; & \frac{\partial}{\partial \widehat{t}_1} + \widehat{y}_i \frac{\partial}{\partial \widehat{x}_i} + \left(\Gamma_i + \frac{\partial h_i}{\partial t_1} + \widehat{y}_j \frac{\partial h_i}{\partial x_j} + \widehat{\zeta}_\mu \frac{\partial h_i}{\partial \eta_\mu} \right) \frac{\partial}{\partial \widehat{y}_i} \\ & + \widehat{\zeta}_\mu \frac{\partial}{\partial \widehat{\eta}_\mu} + \left(\Gamma'_\mu + \frac{\partial k_\mu}{\partial t_1} + \widehat{y}_i \frac{\partial k_\mu}{\partial x_i} + \widehat{\zeta}_\nu \frac{\partial k_\mu}{\partial \eta_\nu} \right) \frac{\partial}{\partial \widehat{\zeta}_\mu}. \end{aligned}$$

Now let $\widehat{\Gamma}_i = \Gamma_i + \frac{\partial h_i}{\partial t_1} + \widehat{y}_j \frac{\partial h_i}{\partial x_j} + \widehat{\zeta}_\mu \frac{\partial h_i}{\partial \eta_\mu}$ and $\widehat{\Gamma}_\mu = \Gamma'_\mu + \frac{\partial k_\mu}{\partial t_1} + \widehat{y}_i \frac{\partial k_\mu}{\partial x_i} + \widehat{\zeta}_\nu \frac{\partial k_\mu}{\partial \eta_\nu}$, this completes the proof.

(2) The proof of this part follows simply from the above discussion and Theorems 2.5 and 2.7. QED

3 dynamical symmetry of super SODE

As mentioned in the previous section, for a given graded vector field Γ on $\mathcal{M} = (M, \mathcal{A}_M)$ and a direct subsheaf $\mathcal{D}$ of $Der\mathcal{A}_M$ of rank (r, s), we have another direct subsheaf $\mathcal{C} := \mathcal{D} + [\Gamma, \mathcal{D}]$ of $Der\mathcal{A}_M$ such that $rank_p(\mathcal{C}) = (r + rank[\Gamma, \mathcal{D}]_0, s + rank[\Gamma, \mathcal{D}]_1)$. If $|\Gamma| = 0$, $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ have maximal ranks respectively r and s, and if $|\Gamma| = 1$, depending on the dimension of the graded manifold $\mathcal{M}$, there are several cases for introducing the maximal rank of $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ (see Theorem 2.2).

Here, we will only consider the situation $|\Gamma| = 0$, $m = 2r + 1$ and $n = 2s$. In this case we showed that for each $p \in \mathcal{M}$, there is a coordinate neighborhood $\mathcal{U}$

of p and coordinates $(t, x_i, y_i; \eta_\mu, \zeta_\mu)$, for $i = 1, 2, \cdots, r$ and $\mu = 1, 2, \cdots, s$ such that $\mathcal{D}|_\mathcal{U} = \langle \frac{\partial}{\partial y_i}; \frac{\partial}{\partial \zeta_\mu} \rangle$, the local expression of the graded vector field $\Gamma \in Der\mathcal{A}_M$ is

$$\Gamma = \frac{\partial}{\partial t} + y_i \frac{\partial}{\partial x_i} + \Gamma_i(t, x_i, y_i; \eta_\mu, \zeta_\mu) \frac{\partial}{\partial y_i} + \zeta_\mu \frac{\partial}{\partial \eta_\mu} + \Gamma'_\mu(x_i, y_i; \eta_\mu, \zeta_\mu) \frac{\partial}{\partial \zeta_\mu},$$

and we have

$$dt(\Gamma) = 1, \theta_i(\Gamma) = 0 \text{ and } \overline{\theta}_\mu(\Gamma) = 0 \text{ for } i \in \{1, \cdots, r\},\ \mu \in \{1, \cdots, s\},$$

where $\{dt, \theta_i = dx_i - y_i dt, \phi_i = dy_i - \Gamma_i dt; \overline{\theta}_\mu = d\eta_\mu - \zeta_\mu dt, \overline{\phi}_\mu = d\zeta_\mu - \Gamma'_\mu dt\}$ is a local basis of the set of contact 1-forms.
First, we recall some basic relations from the graded tensor calculus necessary to this paper.

Lemma 3.1. [11] Suppose $X, Y \in Der\mathcal{A}_M$. For each section ψ of $(Der\mathcal{A}_M)^*$ and $(1, 1)$ tensor filed $T = Z \otimes \omega$, we have

$$\begin{aligned} \psi(T(X)) &= (-1)^{|T||\psi|}(T(\psi))(X), \\ \mathcal{L}_X(\psi(Y)) &= (\mathcal{L}_X\psi)(Y) + (-1)^{|\psi||X|}\psi(\mathcal{L}_X Y), \\ \mathcal{L}_X(T(Y)) &= (\mathcal{L}_X T)(Y) + (-1)^{|X||T|}T(\mathcal{L}_X Y), \\ \mathcal{L}_{[X,Y]} &= [\mathcal{L}_X, \mathcal{L}_Y]. \end{aligned} \tag{3.1}$$

In local coordinate system $(t, x_i, y_i; \eta_\mu, \zeta_\mu)$, we may define a new graded tensor field $\tilde{J}$ of type (1,1) on $\mathcal{M}$,

$$\tilde{J} = J - \Delta \otimes dt$$

where $\Delta = y_i \dfrac{\partial}{\partial y_i} + \zeta_\mu \dfrac{\partial}{\partial \zeta_\mu}$ and J is defined by (2.3). It is clear that

(1) $\tilde{J}(\Gamma) = \tilde{J}(\dfrac{\partial}{\partial y_i}) = \tilde{J}(\dfrac{\partial}{\partial \zeta_\mu}) = 0,$

(2) $\tilde{J}(\dfrac{\partial}{\partial t}) = -\Delta.$

With respect to the above contact 1-forms, it reads

$$\tilde{J} = \frac{\partial}{\partial y_i} \otimes \theta_i - \frac{\partial}{\partial \zeta_\mu} \otimes \overline{\theta}_\mu.$$

The dual operator of $\tilde{J}$, which is used for the action on 1-forms, will be denoted by $\tilde{J}^* = \theta_i \otimes \dfrac{\partial}{\partial y_i} + \overline{\theta}_\mu \otimes \dfrac{\partial}{\partial \zeta_\mu}$, i.e., $(\tilde{J}^*(\alpha))(X) = \alpha(\tilde{J}(X))$. We have

$$\tilde{J}^2 = 0, \tag{3.2}$$

$$(\mathcal{L}_\Gamma \tilde{J})(\Gamma) = 0, \tag{3.3}$$

$$\mathcal{L}_\Gamma \tilde{J} \circ \tilde{J} = -\tilde{J} \circ \mathcal{L}_\Gamma \tilde{J} = \tilde{J}, \tag{3.4}$$

$$(\mathcal{L}_\Gamma \tilde{J})^2 = I - \Gamma \otimes dt, \tag{3.5}$$

$$\mathcal{L}_\Gamma \tilde{J}^* \circ \tilde{J}^* = -\tilde{J}^* \circ \mathcal{L}_\Gamma \tilde{J}^* = -\tilde{J}^*, \tag{3.6}$$

$$(\mathcal{L}_\Gamma \tilde{J}^*)^2 = I - \Gamma \otimes dt. \tag{3.7}$$

In local coordinate system $(t, x_i, y_i; \eta_\mu, \zeta_\mu)$ on $\mathcal{U}$, $y_i = 0$ and $\zeta_\mu = 0$ define a subsupermanifold $\mathcal{N} = (N, \mathcal{A}_N)$ of $\mathcal{U}$ of graded dimension $(r+1, s)$(in the sense of 3.2.1 of [10]). Denote by $pr : \mathcal{U} \to \mathcal{N}$ the corresponding projection morphism, then $pr_*(D_i) = pr_*(D_\mu) = 0$.

For every graded vector field $Z = f\dfrac{\partial}{\partial t} + g^i \dfrac{\partial}{\partial x_i} + h^\mu \dfrac{\partial}{\partial \eta_\mu}$ on $\mathcal{N}$, there is a unique graded vector field $Z^{(1)}$ on $\mathcal{U}$, such that $Z^{(1)}$ projects onto Z, also $\mathcal{L}_{Z^{(1)}}\theta_i$ and $\mathcal{L}_{Z^{(1)}}\overline{\theta}_\mu$ are linear combinations of the basic contact 1-forms θ_i and $\overline{\theta}_\mu$. In local coordinates $(t, x_i, y_i; \eta_\mu, \zeta_\mu)$, the local expression of such a graded vector field is:

$$Z^{(1)} = f\frac{\partial}{\partial t} + g^i \frac{\partial}{\partial x_i} + h^\mu \frac{\partial}{\partial \eta_\mu} + (\dot{g}^i - y_i \dot{f})\frac{\partial}{\partial y_i} + (\dot{h}^\mu - (-1)^{|Z|}\zeta_\mu \dot{f})\frac{\partial}{\partial \zeta_\mu},$$

where the notation $\dot{u}$, with u a superfunction of (t, x_i, η_μ), means $\dot{u} = \dfrac{\partial u}{\partial t} + y_j \dfrac{\partial u}{\partial x_j} + \zeta_\mu \dfrac{\partial u}{\partial \eta_\mu}$. A simple calculation shows that $\mathcal{L}_{Z^{(1)}}\Gamma = -\dot{f}\Gamma + \bar{Z}$, where $\bar{Z} \in \mathcal{D}$.

We associate to Γ, a set of graded vector fields $(Der\mathcal{A}_M)_\Gamma$ defined by,

$$(Der\mathcal{A}_M)_\Gamma = \{X \in Der\mathcal{A}_M \mid \tilde{J}(\mathcal{L}_\Gamma X) = 0 \text{ and } dt(X) = 0\}. \tag{3.8}$$

The local expression for $X \in (Der\mathcal{A}_M)_\Gamma$ is

$$X = f^i \frac{\partial}{\partial x_i} + \Gamma(f^i)\frac{\partial}{\partial y_i} + g^\mu \frac{\partial}{\partial \eta_\mu} + \Gamma(g^\mu)\frac{\partial}{\partial \zeta_\mu}, \tag{3.9}$$

where f^i and g^μ are smooth superfunctions on $\mathcal{M}$. We have

$$\tilde{J}(\mathcal{L}_\Gamma(fX)) = \tilde{J}(\Gamma(f)X + f\mathcal{L}_\Gamma X) = \Gamma(f)\tilde{J}(X) + f\tilde{J}(\mathcal{L}_\Gamma X) = \Gamma(f)\tilde{J}(X).$$

This shows that if $X \in (Der\mathcal{A}_M)_\Gamma$ and if $f \in \mathcal{A}_M$ is a superfunction satisfying $\Gamma(f) = 0$, then $fX \in (Der\mathcal{A}_M)_\Gamma$.

Now let $X \in (Der\mathcal{A}_M)_\Gamma$ and $f \in \mathcal{A}_M$, since $\tilde{J}^2 = 0$, $\tilde{J} \circ \mathcal{L}_\Gamma \tilde{J} = -\tilde{J}$ and $\tilde{J}(\mathcal{L}_\Gamma X) = 0$, we have

$$\begin{aligned}\tilde{J}(\mathcal{L}_\Gamma(fX + \Gamma(f)\tilde{J}(X))) &= \tilde{J}(\Gamma(f)X + f\mathcal{L}_\Gamma X + \Gamma^2(f)\tilde{J}(X) + \Gamma(f)\mathcal{L}_\Gamma(\tilde{J}(X))) \\ &= \tilde{J}(\Gamma(f)X + \Gamma(f)\mathcal{L}_\Gamma(\tilde{J}(X)) + f\mathcal{L}_\Gamma X) = 0.\end{aligned}$$

This shows that $fX + \Gamma(f)\tilde{J}(X) \in (Der\mathcal{A}_M)_\Gamma$. Thus $(Der\mathcal{A}_M)_\Gamma$ can be endowed with an $\mathcal{A}_M$-module structure by means of the product

$$f \star X = fX + \Gamma(f)\tilde{J}(X), \quad f \in \mathcal{A}_M,\ X \in (Der\mathcal{A}_M)_\Gamma.$$

Definition 3.2. A pseudo-dynamical symmetry of Γ is a graded vector field $X \in Der\mathcal{A}_M$ such that $[\Gamma, X] = 0$.

Proposition 3.3. If $X \in (Der\mathcal{A}_M)_\Gamma$ and $\mathcal{L}_\Gamma X \in (Der\mathcal{A}_M)_\Gamma$, then X is a pseudo-dynamical symmetry of Γ.

Proof. Let $X \in (Der\mathcal{A}_M)_\Gamma$, then $\tilde{J}(\mathcal{L}_\Gamma X) = 0$ and we have $\theta_i(\mathcal{L}_X\Gamma) = 0$, $\overline{\theta}_\mu(\mathcal{L}_X\Gamma) = 0$. Since $\mathcal{L}_\Gamma X \in (Der\mathcal{A}_M)_\Gamma$, so $dt([\Gamma, X]) = 0$ and

$$\theta_i(\mathcal{L}_{[\Gamma,X]}\Gamma) = \overline{\theta}_\mu(\mathcal{L}_{[\Gamma,X]}\Gamma) = 0.$$

Applying the Jacobi identity repeatedly gives

$$\begin{aligned}0 =&\theta_i(\mathcal{L}_{[\Gamma,X]}\Gamma) = \theta_i(\mathcal{L}_\Gamma([X,\Gamma]) \\ =&\mathcal{L}_\Gamma(\theta_i(\mathcal{L}_X\Gamma)) - (\mathcal{L}_\Gamma\theta_i)(\mathcal{L}_X\Gamma) = -(\mathcal{L}_\Gamma\theta_i)(\mathcal{L}_X\Gamma) \\ =& -\phi_i([X,\Gamma]),\end{aligned}$$

and

$$\begin{aligned}0 =&\overline{\theta}_\mu(\mathcal{L}_{[\Gamma,X]}\Gamma) = \overline{\theta}_\mu(\mathcal{L}_\Gamma([X,\Gamma]) \\ =&\mathcal{L}_\Gamma(\overline{\theta}_\mu(\mathcal{L}_X\Gamma)) - (\mathcal{L}_\Gamma\overline{\theta}_\mu)(\mathcal{L}_X\Gamma) = -(\mathcal{L}_\Gamma\overline{\theta}_\mu)(\mathcal{L}_X\Gamma) \\ =& -\overline{\phi}_\mu([X,\Gamma]).\end{aligned}$$

All of the 1-forms $dt, \theta_i, \overline{\theta}_\mu, \phi_i, \overline{\phi}_\mu$ on $[X, \Gamma]$ are zero, so this vector field is zero. QED

Now we restrict ourselves to a subset of $(Der\mathcal{A}_M)^*$ which consists of those 1-forms α for which $\tilde{J}^*(\mathcal{L}_\Gamma\alpha) = 0$ and $\alpha(\Gamma) = 0$. We denote this subset by $\mathcal{M}^*_\Gamma$. Thus

$$\mathcal{M}^*_\Gamma = \{\alpha \in (Der\mathcal{A}_M)^* \mid \tilde{J}^*(\mathcal{L}_\Gamma\alpha) = 0 \text{ and } \alpha(\Gamma) = 0\}. \tag{3.10}$$

The local expression for $\alpha \in \mathcal{M}_\Gamma^*$ is

$$\alpha = \alpha^i\phi_i + \overline{\alpha}^\mu\overline{\phi}_\mu - (\Gamma(\alpha^i) + \alpha^j\frac{\partial\Gamma_j}{\partial y_i} + \overline{\alpha}^\nu\frac{\partial\Gamma'_\nu}{\partial y_i})\theta_i - (\Gamma(\overline{\alpha}^\mu) - \alpha^j\frac{\partial\Gamma_j}{\partial\zeta_\mu} + \overline{\alpha}^\nu\frac{\partial\Gamma'_\nu}{\partial\zeta_\mu})\overline{\theta}_\mu, \tag{3.11}$$

where α^i and $\overline{\alpha}^\mu$ are smooth superfunctions on $\mathcal{M}$. For $\alpha \in \mathcal{M}_{\Gamma^*}$ and $f \in \mathcal{A}_M$, since $(\tilde{J}^*)^2 = 0$, $\tilde{J}^* \circ \mathcal{L}_\Gamma\tilde{J}^* = \tilde{J}^*$ and $\tilde{J}^*(\mathcal{L}_\Gamma\alpha) = 0$, (see (3.6)) we have

$$\tilde{J}^*(\mathcal{L}_\Gamma(f\alpha - \Gamma(f)\tilde{J}^*(\alpha))) = \tilde{J}^*(\Gamma(f)\alpha + f\mathcal{L}_\Gamma\alpha - \Gamma^2(f)\tilde{J}^*(\alpha) - \Gamma(f)\mathcal{L}_\Gamma(\tilde{J}^*(\alpha))) = 0.$$

Thus $\mathcal{M}_\Gamma^*$ can be endowed with an $\mathcal{A}_M$-module structure by means of the product

$$f \star \alpha = f\alpha - \Gamma(f)\tilde{J}^*(\alpha), \quad f \in \mathcal{A}_M,\ \alpha \in \mathcal{M}_\Gamma^*.$$

The map $p_\Gamma^* : (Der\mathcal{A}_M)^* \to (Der\mathcal{A}_M)^*$, given by

$$p_\Gamma^*(\alpha) = \alpha - \tilde{J}^*(\mathcal{L}_\Gamma\alpha) - \alpha(\Gamma)dt, \tag{3.12}$$

is a morphisme of $\mathcal{A}_M$-modules that is a projection map onto $\mathcal{M}_\Gamma^*$:

$$\begin{aligned} p_{\Gamma^*}(f\alpha) =& f\alpha - \tilde{J}^*(\mathcal{L}_\Gamma(f\alpha)) - f\alpha(\Gamma)dt \\ =&(f\alpha - \Gamma(f)\tilde{J}^*(\alpha)) - (f\tilde{J}^*(\mathcal{L}_\Gamma\alpha) - \Gamma(f)(\tilde{J}^*)^2(\mathcal{L}_\Gamma\alpha)) \\ &- (f\alpha(\Gamma)dt - \Gamma(f)\tilde{J}^*(\alpha(\Gamma)dt)) \\ =& f \star p_{\Gamma^*}(\alpha). \end{aligned}$$

Also

$$\begin{aligned} \tilde{J}^*(\mathcal{L}_\Gamma(p_\Gamma^*(\alpha)) =& \tilde{J}^*(\mathcal{L}_\Gamma\alpha - \mathcal{L}_\Gamma(\tilde{J}^*(\mathcal{L}_\Gamma\alpha)) - \mathcal{L}_\Gamma(\alpha(\Gamma)dt)) \\ =& \tilde{J}^*(\mathcal{L}_\Gamma\alpha) - (\tilde{J}^* \circ \mathcal{L}_\Gamma\tilde{J}^*)(\mathcal{L}_\Gamma\alpha) = 0, \end{aligned}$$

and

$$p_\Gamma^*(\alpha)(\Gamma) = 0.$$

We associate to Γ a subsheaf of graded 1-forms $(Der\mathcal{A}_M)_\Gamma^*$, such that each section ψ of $(Der\mathcal{A}_M)_\Gamma^*$ has the property

$$\mathcal{L}_\Gamma(\tilde{J}^*(\psi)) = \psi. \tag{3.13}$$

The local expression for ψ is

$$\psi = a^i\phi_i + \Gamma(a^i)\theta_i + b^\mu\overline{\phi}_\mu + \Gamma(b^\mu)\overline{\theta}_\mu, \tag{3.14}$$

where, a^i and b^μ are smooth superfunctions on $\mathcal{M}$. Now let ψ be a section of $(Der\mathcal{A}_M)_{\Gamma^*}$ and $f \in \mathcal{A}_M$, since $(\tilde{J}^*)^2 = 0$ and $\mathcal{L}_\Gamma(\tilde{J}^*(\psi)) = \psi$, we have

$$\begin{aligned}\mathcal{L}_\Gamma(\tilde{J}^*(f\psi + \Gamma(f)\tilde{J}^*(\psi)) &= \mathcal{L}_\Gamma(f\tilde{J}^*(\psi) + \Gamma(f)(\tilde{J}^*)^2(\psi)) \\ &= f\psi + \Gamma(f)\tilde{J}^*(\psi).\end{aligned}$$

Thus $(Der\mathcal{A}_M)^*_\Gamma$ can be endowed with an $\mathcal{A}_M$-module structure by means of the product

$$f \star \psi = f\psi + \Gamma(f)\tilde{J}^*(\psi), \quad f \in \mathcal{A}_M,\ \psi \in (Der\mathcal{A}_M)^*_\Gamma.$$

Lemma 3.4. If $f \in \mathcal{A}_M$ and df is a section of $(Der\mathcal{A}_M)^*_\Gamma$, then $\Gamma(f) = 0, \Gamma(\frac{\partial f}{\partial y_i}) = \frac{\partial f}{\partial x_i}$, and $\Gamma(\frac{\partial f}{\partial \zeta_\mu}) = \frac{\partial f}{\partial \eta_\mu}$.

Proof. It suffices to take into account the local expressions for Γ. QED

Definition 3.5. The graded vector field Γ is called a pseudo-Lagrangian vector field if there exists $L \in \mathcal{A}_M$ such that $i_\Gamma\omega_L = 0$, where $\omega_L = -d\theta_L$ and θ_L is the Poincaré-cartan 1-form $\theta_L = Ldt + \tilde{J}^*(dL)$. L is called a pseudo-Lagrangian superfunction for Γ.

Equivalently, the graded vector field Γ is called a pseudo-Lagrangian graded vector field if there exists $L \in \mathcal{A}_M$ such that

$$\Gamma(\frac{\partial L}{\partial y_i}) = \frac{\partial^2 L}{\partial t\partial y_i} + y_j\frac{\partial^2 L}{\partial x_j\partial y_i} + \Gamma_j\frac{\partial^2 L}{\partial y_j\partial y_i} + \zeta_\nu\frac{\partial^2 L}{\partial \eta_\nu\partial y_i} + \Gamma'_\nu\frac{\partial^2 L}{\partial \zeta_\nu\partial y_i} = \frac{\partial L}{\partial x_i}, \tag{3.15}$$

$$\Gamma(\frac{\partial L}{\partial \zeta_\mu}) = \frac{\partial^2 L}{\partial t\partial \zeta_\mu} + y_j\frac{\partial^2 L}{\partial x_j\partial \zeta_\mu} + \Gamma_j\frac{\partial^2 L}{\partial y_j\partial \zeta_\mu} + \zeta_\nu\frac{\partial^2 L}{\partial \eta_\nu\partial \zeta_\mu} + \Gamma'_\nu\frac{\partial^2 L}{\partial \zeta_\nu\partial \zeta_\mu} = \frac{\partial L}{\partial \eta_\mu}. \tag{3.16}$$

Remark 3.6. If L is a pseudo-Lagrangian function for the pseudo-Lagrangian graded vector field Γ, then $i_\Gamma\theta_L = L$, thus $\mathcal{L}_\Gamma\theta_L = dL$ and vice versa.

Let Γ be a pseudo-Lagrangian graded vector field. From $\mathcal{L}_\Gamma dt = 0$ and $\tilde{J}^*(dt) = 0$, we conclude that

$$\begin{aligned}\mathcal{L}_\Gamma(\tilde{J}^*(dL - (\mathcal{L}_\Gamma L)dt)) - dL + \mathcal{L}_\Gamma(L)dt =& \mathcal{L}_\Gamma(\tilde{J}^*(dL)) - dL + \mathcal{L}_\Gamma(Ldt) \\ =& \mathcal{L}_\Gamma(\tilde{J}^*(dL) + Ldt) - dL \\ =& \mathcal{L}_\Gamma\theta_L - dL = 0.\end{aligned}$$

Thus $dL - (\mathcal{L}_\Gamma L)dt$ is a section of $(Der\mathcal{A}_M)^*_\Gamma$. We summarize as follows:

Lemma 3.7. If Γ is a pseudo-Lagrangian graded vector field then $dL - (\mathcal{L}_\Gamma L)dt$ is a section of $(Der\mathcal{A}_M)^*_\Gamma$.

Given ψ as a section of $(Der\mathcal{A}_M)^*_\Gamma$, let f be an arbitrary element of $\mathcal{A}_M$ such that $\psi + fdt$ is an exact 1-form. Then there exist a superfunction $L \in \mathcal{A}_M$ such that $\psi + fdt = dL$. From (3.14), we have $i_\Gamma\psi = 0$ and then $i_\Gamma dL = f$. Since $\mathcal{L}_\Gamma(\tilde{J}^*(\psi)) = \psi$ we have

$$\begin{aligned}0 =&\mathcal{L}_\Gamma(\tilde{J}^*(dL - fdt)) - dL + fdt\\ =&\mathcal{L}_\Gamma(\tilde{J}^*(dL)) - dL + (i_\Gamma dL)dt\\ =&\mathcal{L}_\Gamma(\tilde{J}^*(dL)) - dL + (\mathcal{L}_\Gamma L)dt\\ =&\mathcal{L}_\Gamma(\tilde{J}^*(dL) + Ldt) - dL\end{aligned}$$

therefore $\mathcal{L}_\Gamma\theta_L = dL$ and L is a pseudo-Lagrangian for Γ. We summarize these results in the following theorem.

Theorem 3.8. Let ψ be a section of $(Der\mathcal{A}_M)^*_\Gamma$. Let f be an arbitrary element of $\mathcal{A}_M$ such that $\psi + fdt$ is an exact 1-form, then Γ is a pseudo-Lagrangian vector field.

Theorem 3.9. The graded vector field Γ is a pseudo-Lagrangian vector field if and only if there exists a closed 1-form α on $\mathcal{M}$ such that $\mathcal{L}_\Gamma(p^*_\Gamma(\alpha)) = 0$.

Proof. Let L be a pseudo-Lagrangian for Γ and α a solution of the equation $i_\Gamma\alpha = L$. Then

$$\begin{aligned}0 =&dL - \mathcal{L}_\Gamma\theta_L\\ =&d(i_\Gamma\alpha) - \mathcal{L}_\Gamma(\alpha(\Gamma)dt + \tilde{J}^*(\mathcal{L}_\Gamma\alpha))\\ =&\mathcal{L}_\Gamma\{\alpha - \alpha(\Gamma)dt - \tilde{J}^*(\mathcal{L}_\Gamma\alpha)\}\\ =&\mathcal{L}_\Gamma(p^*_\Gamma(\alpha)),\end{aligned}$$

so that $p^*_\Gamma(\alpha)$ is Γ-invariant. Finally, let α be a closed 1-form such that $\mathcal{L}_\Gamma(p^*_\Gamma(\alpha)) = 0$. If we take $L = i_\Gamma\alpha$ then

$$p^*_\Gamma(\alpha) = \alpha - \alpha(\Gamma)dt - \tilde{J}^*(\mathcal{L}_\Gamma\alpha) = \alpha - (Ldt + \tilde{J}^*(dL)) = \alpha - \theta_L,$$

and

$$\mathcal{L}_\Gamma(p^*_\Gamma(\alpha)) = \mathcal{L}_\Gamma(\alpha - \theta_L) = \mathcal{L}_\Gamma\alpha - \mathcal{L}_\Gamma\theta_L = dL - \mathcal{L}_\Gamma\theta_L.$$

then, the superfunction L is a Lagrangian for Γ. $\boxed{QED}$

Let Γ be a Lagrangian sode vector field, that is, there exists $L \in \mathcal{A}_M$ such that $i_\Gamma \omega_L = 0$, or in equivalent way $\mathcal{L}_\Gamma \theta_L = dL$. Let $X = f^i \frac{\partial}{\partial x_i} + \Gamma(f^i)\frac{\partial}{\partial y_i} + g^\mu \frac{\partial}{\partial \eta_\mu} + \Gamma(g^\mu)\frac{\partial}{\partial \zeta_\mu} \in (Der\mathcal{A}_M)_\Gamma$. If $i_X\omega_L$ is a member of the set $\mathcal{M}^*_\Gamma$, then its local representation in coordinates $(t, x_i, y_i; \eta_\mu, \zeta_\mu)$, is similar to (3.11) such that the coefficients of ϕ_i and $\bar{\phi}_\mu$ are given by

$$\alpha^i = (-1)^{|X||L|}\frac{\partial^2 L}{\partial y_j \partial y_i} f^j - (-1)^{|X|(|L|+1)+|L|}\frac{\partial^2 L}{\partial \zeta_\nu \partial y_i} g^\nu, \tag{3.17}$$

and

$$\overline{\alpha}^\mu = -(-1)^{|X|(|L|+1)+|L|}\frac{\partial^2 L}{\partial y_j \partial \zeta_\mu} f^j - (-1)^{|X||L|}\frac{\partial^2 L}{\partial \zeta_\nu \partial \zeta_\mu} g^\nu \tag{3.18}$$

respectively. Also, if $[X, \Gamma] = 0$, then $i_X \circ i_\Gamma = i_\Gamma \circ i_X$, and from $i_\Gamma\omega_L = 0$, we have

$$\mathcal{L}_\Gamma(i_X\omega_L) = \mathcal{L}_X(i_\Gamma\omega_L) = 0.$$

This means that the map $\varrho_L : (Der\mathcal{A}_M) \to (Der\mathcal{A}_M)^*$, given by $\varrho_L(X) = i_X\omega_L$ maps $(Der\mathcal{A}_M)_\Gamma$ in $\mathcal{M}^*_\Gamma$. Also ϱ_L maps symmetries of Γ on Γ-invariant 1-forms in $\mathcal{M}^*_\Gamma$. If $L \in \mathcal{A}_M$ is regular, i.e. the matrix $\begin{pmatrix} \frac{\partial^2 L}{\partial y_j \partial y_i} & \frac{\partial^2 L}{\partial \zeta_\nu \partial y_i} \\ \frac{\partial^2 L}{\partial y_j \partial \zeta_\mu} & \frac{\partial^2 L}{\partial \zeta_\nu \partial \zeta_\mu} \end{pmatrix}$ is nonsingular, then ϱ_L is a surjective map.

Proposition 3.10. Let X be a pseudo-dynamical symmetry of Γ. If $\varrho_L(X)$ is an exact 1-form then $\mathcal{L}_X\omega_L = 0$.

Proof. Let F be a superfunction on $\mathcal{M}$, such that $\varrho_L(X) = dF$, then

$$\mathcal{L}_X\omega_L = -(i_X \circ d + d \circ i_X)(d\theta_L) = -i_\Gamma \circ d(d\theta_L) - d^2F = 0.$$

QED

Proposition 3.11. Let X be a pseudo-dynamical symmetry of Γ. If there exist a closed 1-form α such that $\varrho_L(X) = p^*_\Gamma(\alpha)$, then $i_\Gamma\alpha$ is a pseudo-Lagrangian superfunction and $\omega_L = d(\varrho_L(X)) = \mathcal{L}_X\omega_L$.

Proof. Since X is a pseudo-dynamical symmetry of Γ, $\varrho_L(X)$ is Γ-invariant 1-forms, then there exist a closed 1-form α such that

$$\mathcal{L}_\Gamma(p^*_\Gamma(\alpha)) = \mathcal{L}_\Gamma(\varrho_L(X)) = 0.$$

So, from Proposition 3.9, the superfunction $i_\Gamma\alpha$ is a Lagrangian for Γ, also

$$d(\varrho_L(X)) = d(\alpha - \theta_L) = \omega_L,$$

and

$$d(\varrho_L(X)) = d(i_X\omega_L) = (\mathcal{L}_X - i_X \circ d)(\omega_L) = \mathcal{L}_X\omega_L.$$

QED

4 Pseudo-adjoint symmetry

Given the graded vector field Γ.

Definition 4.1. A 1-form ψ as a section of $(Der\mathcal{A}_M)^*_\Gamma$ is a pseudo-adjoint symmetry of Γ if $\mathcal{L}_\Gamma\psi$ is a section of $(Der\mathcal{A}_M)^*_\Gamma$.

It is instructive to look at the coordinate expression of the pseudo-adjoint symmetry of Γ. If $\psi = a^i\phi_i + \Gamma(a^i)\theta_i + b^\mu\overline{\phi}_\mu + \Gamma(b^\mu)\overline{\theta}_\mu$ is a section of $(Der\mathcal{A}_M)^*$, then we have

$$\begin{aligned}\mathcal{L}_\Gamma\psi =&(2\Gamma(a^i) + a^j\frac{\partial\Gamma_j}{\partial y_i} + b^\nu\frac{\partial\Gamma'_\nu}{\partial y_i})\phi_i + (2\Gamma(b^\mu) - a^j\frac{\partial\Gamma_j}{\partial\zeta_\mu} + b^\nu\frac{\partial\Gamma'_\nu}{\partial\zeta_\mu})\overline{\phi}_\mu\\ &+ (\Gamma\Gamma(a^i) + a^j\frac{\partial\Gamma_j}{\partial x_i} + b^\nu\frac{\partial\Gamma'_\nu}{\partial x_i})\theta_i + (\Gamma\Gamma(b^\mu) - a^j\frac{\partial\Gamma_j}{\partial\eta_\mu} + b^\nu\frac{\partial\Gamma'_\nu}{\partial\eta_\mu})\overline{\theta}_\mu.\end{aligned}$$

Therefore, $\mathcal{L}_\Gamma\psi$ is a section of $(Der\mathcal{A}_M)^*_\Gamma$ if and only if we have for $i \in \{1, \dots, r\}$ and $\mu \in \{1, \dots, s\}$

$$\Gamma\Gamma(a^i) + \Gamma(a^j\frac{\partial\Gamma_j}{\partial y_i}) + \Gamma(b^\nu\frac{\partial\Gamma'_\nu}{\partial y_i}) - a^j\frac{\partial\Gamma_j}{\partial x_i} - b^\nu\frac{\partial\Gamma'_\nu}{\partial x_i} = 0, \tag{4.1}$$

$$\Gamma\Gamma(b^\mu) - \Gamma(a^j\frac{\partial\Gamma_j}{\partial\zeta_\mu}) + \Gamma(b^\nu\frac{\partial\Gamma'_\nu}{\partial\zeta_\mu}) + a^j\frac{\partial\Gamma_j}{\partial\eta_\mu} - b^\nu\frac{\partial\Gamma'_\nu}{\partial\eta_\mu} = 0. \tag{4.2}$$

We see that this is a system of second-order differential equations for the superfunctions a^i and b^μ.

Definition 4.2. A 1-form β on $\mathcal{M}$ is a Γ-basic if $i_\Gamma\beta = 0$ and $i_\Gamma d\beta = 0$.

Equivalently, β is Γ-basic if $i_\Gamma\beta = 0$ and $\mathcal{L}_\Gamma\beta = 0$.

Let ψ be a pseudo-adjoint symmetry of Γ. Using the definition, straightforward computations show that

$$\mathcal{L}_\Gamma(\mathcal{L}_\Gamma(\tilde{J}^*(\psi)) - \mathcal{L}_\Gamma\tilde{J}^*(\psi)) = \mathcal{L}_\Gamma\psi,$$

and

$$\mathcal{L}_\Gamma(\mathcal{L}_\Gamma \tilde{J}^*(\psi)) = \mathcal{L}_\Gamma(\mathcal{L}_\Gamma(\tilde{J}^*(\psi)) - \psi) = 0,$$

since $\mathcal{L}_\Gamma(\tilde{J}^*(\mathcal{L}_\Gamma\psi)) = \mathcal{L}_\Gamma\psi$ and $\psi \in (Der\mathcal{A}_M)^*_\Gamma$. Now, let $\beta = \mathcal{L}_\Gamma \tilde{J}^*(\psi)$, then

$$i_\Gamma(\beta) = (\mathcal{L}_\Gamma \tilde{J}^*(\psi))(\Gamma) = \psi(\mathcal{L}_\Gamma \tilde{J}^*(\Gamma)) = 0,$$

so β is Γ-basic. If ψ_1 and ψ_2 are adjoint symmetries giving rise to the same β, we have $\mathcal{L}_\Gamma \tilde{J}^*(\psi_1 - \psi_2) = 0$, which means that $\psi_1 = \psi_2$. On the other hand, if β is a Γ-basic form and $\psi := \mathcal{L}_\Gamma \tilde{J}^*(\beta)$, then we have

$$\begin{aligned}\mathcal{L}_\Gamma(\tilde{J}^*(\psi)) &= \mathcal{L}_\Gamma(\tilde{J}^* \circ (\mathcal{L}_\Gamma \tilde{J}^*)(\beta)) \\ &= \mathcal{L}_\Gamma(\tilde{J}^*(\beta)) = \mathcal{L}_\Gamma \tilde{J}^*(\beta) + \tilde{J}^*(\mathcal{L}_\Gamma\beta) = \mathcal{L}_\Gamma \tilde{J}^*(\beta) = \psi,\end{aligned}$$

where in the above formula, we used (3.1) and (3.6). This shows that ψ is a section of $(Der\mathcal{A}_M)^*_\Gamma$. Also, from (3.7), we have

$$\beta = (\mathcal{L}_\Gamma \tilde{J}^*)^2(\beta) = \mathcal{L}_\Gamma \tilde{J}^*(\psi) = \mathcal{L}_\Gamma(\tilde{J}^*(\psi)) - \tilde{J}^*(\mathcal{L}_\Gamma\psi) = \psi - \tilde{J}^*(\mathcal{L}_\Gamma\psi).$$

Then

$$\mathcal{L}_\Gamma\beta = \mathcal{L}_\Gamma\psi - \mathcal{L}_\Gamma(\tilde{J}^*(\mathcal{L}_\Gamma\psi)).$$

β is a Γ-basic form, thus $\mathcal{L}_\Gamma\psi$ is a section of $(Der\mathcal{A}_M)^*_\Gamma$ and hence ψ is a pseudo-adjoint symmetry of Γ.

We summarize the conclusion drawn from this calculation as follows.

Proposition 4.3. The tensor field $\mathcal{L}_\Gamma \tilde{J}^*$ determines a bijection between the set of pseudo-adjoint symmetries and the set of Γ-basic forms.

Remark 4.4. If ψ is a pseudo-adjoint symmetry of Γ and if there exists a superfunction G on $\mathcal{M}$ such that $\mathcal{L}_\Gamma \tilde{J}^*(\psi) = dG$ then dG is a Γ-basic form and $\Gamma(G) = 0$. On the other hand, if we assume that for each $G \in \mathcal{A}_M$, $\Gamma(G) = 0$, then dG is a Γ-basic form and from proposition 4.3, there is a pseudo-adjoint symmetry ψ of Γ such that $\mathcal{L}_\Gamma \tilde{J}^*(\psi) = dG$.

Proposition 4.5. Let ψ be a pseudo-adjoint symmetry of Γ such that $\psi = \mathcal{L}_\Gamma(\tilde{J}^*(dG))$ for some superfunction G. Then, $\Gamma(G)$ is a Lagrangian superfunction. Conversely, if $\Gamma(G)$ is a Lagrangian superfunction, $\mathcal{L}_\Gamma(\tilde{J}^*(dG))$ is a pseudo-adjoint symmetry of Γ.

Proof. From applying $\mathcal{L}_\Gamma \tilde{J}^*$ to both sides $\psi = \mathcal{L}_\Gamma(\tilde{J}^*(dG))$, we have

$$\mathcal{L}_\Gamma \tilde{J}^*(\psi) = \mathcal{L}_\Gamma \tilde{J}^*(\mathcal{L}_\Gamma \tilde{J}^*(dG) + \tilde{J}^*(\mathcal{L}_\Gamma(dG))) = dG - \tilde{J}^*(\mathcal{L}_\Gamma(dG)) = dG - \tilde{J}^*(d\Gamma(G)).$$

So

$$\mathcal{L}_\Gamma(\mathcal{L}_\Gamma(\tilde{J}^*(\psi)) - \tilde{J}^*(\mathcal{L}_\Gamma\psi)) = \mathcal{L}_\Gamma(dG - \tilde{J}^*(d\Gamma(G))).$$

If ψ be an adjoint symmetry, then

$$\mathcal{L}_\Gamma(\tilde{J}^*(d\Gamma(G))) = d\Gamma(G).$$

This means that $d\Gamma(G)$ is a section of $(Der\mathcal{A}_M)^*_\Gamma$, and then $\Gamma(G)$ is a Lagrangian superfunction. Conversely, let $\Gamma(G)$ be a Lagrangian superfunction, i.e. $\mathcal{L}_\Gamma(\tilde{J}^*(d\Gamma(G))) = d\Gamma(G)$. we have

$$\begin{aligned}\mathcal{L}_\Gamma(\tilde{J}^*(\mathcal{L}_\Gamma(\tilde{J}^*(dG)))) =& \mathcal{L}_\Gamma(\tilde{J}^*(\mathcal{L}_\Gamma\tilde{J}^*(dG) + \tilde{J}^*(\mathcal{L}_\Gamma dG)))\\ =& \mathcal{L}_\Gamma((\tilde{J}^* \circ \mathcal{L}_\Gamma\tilde{J}^*)(dG))\\ =& \mathcal{L}_\Gamma(\tilde{J}^*(dG)),\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_\Gamma(\tilde{J}^*(\mathcal{L}_\Gamma(\mathcal{L}_\Gamma(\tilde{J}^*(dG))))) =& \mathcal{L}_\Gamma(\mathcal{L}_\Gamma(\tilde{J}^*(\mathcal{L}_\Gamma(\tilde{J}^*(dG)))) - \mathcal{L}_\Gamma\tilde{J}^*(\mathcal{L}_\Gamma(\tilde{J}^*(dG))))\\ =& \mathcal{L}_\Gamma(\mathcal{L}_\Gamma(\tilde{J}^*(dG)) - \mathcal{L}_\Gamma\tilde{J}^*(\mathcal{L}_\Gamma\tilde{J}^*(dG) + \tilde{J}^*(\mathcal{L}_\Gamma dG)))\\ =& \mathcal{L}_\Gamma(\mathcal{L}_\Gamma(\tilde{J}^*(dG))) - \mathcal{L}_\Gamma(dG) + \mathcal{L}_\Gamma(\tilde{J}^*(d\Gamma(G)))\\ =& \mathcal{L}_\Gamma(\mathcal{L}_\Gamma(\tilde{J}^*(dG))),\end{aligned}$$

and therefore $\mathcal{L}_\Gamma(\tilde{J}^*(dG))$ is a pseudo-adjoint symmetry of Γ. QED

5 The tangent supermanifold and an inverse problem

An example of a graded manifold $\mathcal{M}$ that covers many of the concepts described in the previous sections is the supermanifold $(R^{1|0} \times TM', \mathcal{A}_{R^{1|0}\times TM'})$, where $(M', \mathcal{A}_{M'})$ is a graded manifold of dimension (r, s). By choosing this, we are able to find a generalization of the adjoint symmetry method for time-dependent second-order equations to the graded case. In this geometrical setting, the inverse problem is considered.

Let $(t, x_i, y_i; \eta_\mu, \zeta_\mu)$, for $i = 1, 2, \cdots, r$ and $\mu = 1, 2, \cdots, s$, be local coordinates on $(R^{1|0} \times TM', \mathcal{A}_{R^{1|0}\times TM'})$, where (x, η) are local coordinates on $(M', \mathcal{A}_{M'})$ and t is referred to as the even coordinate of $R^{1|0}$. Consider a graded vector field

$$\Gamma = \frac{\partial}{\partial t} + y_i\frac{\partial}{\partial x_i} + \Gamma_i(t, x_i, y_i; \eta_\mu, \zeta_\mu)\frac{\partial}{\partial y_i} + \zeta_\mu\frac{\partial}{\partial \eta_\mu} + \Gamma'_\mu(x_i, y_i; \eta_\mu, \zeta_\mu)\frac{\partial}{\partial \zeta_\mu},$$

which corresponds to a system of super second order ordinary differential equations on $(R^{1|0} \times TM', \mathcal{A}_{R^{1|0}\times TM'})$. Necessary and sufficient conditions for Γ to derive from a Lagrangian superfunction are investigated in the previous sections. As we have indicated, if α is a closed 1-form on $(R^{1|0} \times TM', \mathcal{A}_{R^{1|0}\times TM'})$, such that $\mathcal{L}_\Gamma(p_\Gamma^*(\alpha)) = 0$, then $L = i_\Gamma\alpha$ is a Lagrangian superfunction for Lagrangian graded vector field Γ. On the other hand if Γ is a Lagrangian graded vector field, then there exists a closed 1-form α such that $\mathcal{L}_\Gamma(p_\Gamma^*(\alpha)) = 0$, see Theorem 3.9.

Also, if ψ be a pseudo-adjoint symmetry of Γ such that $\psi = \mathcal{L}_\Gamma(\tilde{J}^*(dG))$ for some superfunction G, then, $\Gamma(G)$ is a Lagrangian superfunction. Conversely, if $\Gamma(G)$ is a Lagrangian superfunction, $\mathcal{L}_\Gamma(\tilde{J}^*(dG))$ is a adjoint symmetry of Γ, see Theorem 4.5.

References

[1] E. Azizpour and M. H. Zarifi: *Graded vector fields and involutive distributions on graded manifolds,* J. Dyn. Syst. Geom. Theory, **16** (2018), no. 2, 101-127.

[2] I. Bucataru and O. Constantinescu: *Helmholtz conditions and symmetries for the time dependent case of the inverse problem of the calculus of variations,* J. Geom. Phys. **60** (2010), no. 11, 1710-1725.

[3] J. F. Carinena and H. Figueroa *Recursion operators and constants of motion in supermechanics*, Differential Geometry and its Applications. **10** (1999), 191-202.

[4] J. F. Carinena and H. Figueroa *A geometrical version of Noethers theorem in supermechanics*, Rep. Math. Phys., **34** (1994), 277-303.

[5] J. F. Carinena and H. Figueroa *Hamiltonian versus Lagrangian formulations of supermechanics*, J. Phys. A, **30** (1997), 2705-2724.

[6] J. F. Carinena and H. Figueroa: *Singular Lagrangians in supermechanics*, Differential Geom. Appl., **18** (2003), 33-46.

[7] J. F. Carinena and E. Martinez: *Symmetry theory and Lagrangian inverse problem for time-dependent second-order differential equations,* J. Phys. A **22** (1989), no. 14, 2659-2665.

[8] J. F. Carinena, C. Lopez and E. Martinez: *A geometric characterisation of Lagrangian second-order differential equations,* Inverse Problems, **5** (1989) 691-705.

[9] L. A. Ibort and J. M. Solano: *Geometrical foundations of Lagrangian supermechanics and supersymmetry*, Rep. Math. Phys., **32** (1993), 385-409.

[10] D. Leites: *Intoduction to the theory of supermanifolds,* Russian. Math. Surveys, **35** (1980), 1-64.

[11] D. Leites: Seminar on supersymmetry (v. 1. Algebra and calculus: Main chapters) (J. Bernstein, D. Leites, V. Molotkov, V. Shander). MCCME, Moscow, 2011, 410 pp.

[12] J. Monterde and J. Munoz-Masque and O. A. Sanchez-Valenzuela: *Geometric properties of involutive distributions on graded manifolds,* Indag. Math., N.S., **8** (1997), 217-246.

[13] P. Morando and S. Pasquero: *The symmetry in the structure of dynamical and adjoint symmetries of second-order differential equations,* J. Phys. A **28** (1995), no. 7, 1943-1955.

[14] W. Sarlet: *Construction of adjoint symmetries for systems of second-order and mixed first- and second-order ordinary differential equations. Algorithms and software for symbolic analysis of nonlinear systems,* Math. Comput. Modelling **25** (1997), no. 8-9, 39-49.

[15] W. Sarlet, F. Cantrijn and M. Crampin: *A new look at second-order equations and Lagrangian mechanics,* J. Phys. A: Math. Gen. **17** (1984) 1999-2009.

[16] W. Sarlet, G. E. Prince and M. Crampin: *Adjoint symmetries for time-dependent second-order equations,* J. Phys. A **23** (1990), no. 8, 1335-1347.

[17] S. Vacaru and H. Dehnen: *Locally anisotropic structures and nonlinear connections in Einstein and gauge gravity*, *Gen. Rel. Grav.* **35** (2003) 209-250.

[18] S. I. Vacaru: *Superstrings in higher order extensions of Finsler superspaces, Nucl. Phys.* **B494** (1997) no. 3, 590-656.

[19] S. I. Vacaru: Interactions, strings and isotopies in higher order anisotropic superspaces, Hadronic Press, Palm Harbor, FL, USA, 1998.

[20] V. S. Varadarajan: Supersymmetry for mathematicians: an introduction, Courant Lecture Notes Series, New York, 2004.