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A Conditional Expectation on the Tensor Product of Exel-Laca algebras

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Abstract. We show that the ultragraph C^*-algebra $C^*(\mathcal{G}_1 \times \mathcal{G}_2)$ can be embedded in $C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)$ as a $*$-subalgebra. We then use this fact to investigate the existence of a conditional expectation on the tensor product of Exel-Laca algebras onto a certain subalgebra.

Keywords: ultragraph C^*-algebra, Exel-Laca algebras, conditional expectation

MSC 2000 classification: primary 46L05, secondary 46L55

Introduction

The Cuntz-Krieger algebras were introduced and studied in [3] for binary-valued matrices with finite index. A direct extension of these algebras is the Exel-Laca algebras of infinite matrices with $\{0, 1\}$-entries [4]. Another generalization of the Cuntz-Krieger algebras is the C^*-algebras of directed graphs [6, 1, 5]. It is shown in [5] that for directed graph G with no sinks and sources, the C^*-algebra $C^*(G)$ is canonically isomorphic to the Exel-Laca algebra $\mathcal{O}_{A_G}$, where A_G is the edge matrix of G.

The motivation of the definition of ultragraphs C^*-algebras is to unify the theory of graph C^*-algebras and Exel-Laca algebras [9]. In ultragraphs, the range of each edge is allowed to be a nonempty set of vertices. Any C^*-algebra of a directed graph can be considered as an ultragraph C^*-algebra and the C^*-algebras of ultragraphs with no singular vertices are precisely the Exel-Laca algebras. Furthermore, the class of ultragraph C^*-algebras are strictly larger than this class of directed graphs as well as the class of Exel-Laca algebras.

This paper is motivated by a natural question, which is the existence of a conditional expectation from $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ onto a subalgebra of $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ isomorphic to $\mathcal{O}_{d_1 d_2}$ [2]. We extend this question to Exel-Laca algebras. For ultragraphs $\mathcal{G}_1$ and $\mathcal{G}_2$, we show that $C^*(\mathcal{G}_1 \times \mathcal{G}_2)$ is isomorphic to a certain subalgebra of $C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)$. By setting some conditions on $\mathcal{G}_1$ and $\mathcal{G}_2$, we see that $C^*(\mathcal{G}_1 \times \mathcal{G}_2)$ is isomorphic to the fixed point algebra $\big(C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)\big)^\beta$, where β is an action from the unit circle. Finally, we show that there is a conditional expectation from the tensor product $\mathcal{O}_A \otimes \mathcal{O}_B$ onto $\mathcal{O}_{AB}$.

1 Preliminaries

In this section, we briefly review the basic definitions and properties of ultragraph C^*-algebras which will be used in the next section. For more details about the ultragraph C^*-algebras, we refer the reader to [9, 7].

An *ultragraph* $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ consists of countable sets G^0 of vertices and $\mathcal{G}^1$ of edges, the source map $s : \mathcal{G}^1 \to G^0$ and the range map $r : \mathcal{G}^1 \to \mathcal{P}(G^0)\backslash\{\emptyset\}$, where $\mathcal{P}(G^0)$ is the collection of all subsets of G^0. A vertex $v \in G^0$ is called a *sink* if $|s^{-1}(v)| = 0$ and an *infinite emitter* if $|s^{-1}(v)| = \infty$. A *singular* vertex is a vertex that is either a sink or an infinite emitter. The ultragraph is *row-finite* if each vertex emits at most finitely many edges.

For a set X, a subcollection of $\mathcal{P}(X)$ is called an *algebra* if it is closed under the set operations $\cup$ and $\cap$. If $\mathcal{G}$ is an ultragraph, we write $\mathcal{G}^0$ for the smallest algebra in $\mathcal{P}(G^0)$ containing $\{v, r(e) : v \in G^0 \text{ and } e \in \mathcal{G}^1\}$.

Definition 1. Let $\mathcal{G}$ be an ultragraph. A *Cuntz-Krieger $\mathcal{G}$-family* consists of projections $\{p_A : A \in \mathcal{G}^0\}$ and partial isometries $\{s_e : e \in \mathcal{G}^1\}$ with mutually orthogonal ranges such that

(1) $p_\emptyset = 0$, $p_A p_B = p_{A\cap B}$ and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ for all $A, B \in \mathcal{G}^0$;

(2) $s_e^* s_e = p_{r(e)}$ for $e \in \mathcal{G}^1$;

(3) $s_e s_e^* \leq p_{s(e)}$ for $e \in \mathcal{G}^1$;

(4) $p_v = \sum_{s(e)=v} s_e s_e^*$ whenever $0 < |s^{-1}(v)| < \infty$.

The C^*-algebra $C^*(\mathcal{G})$ is the universal C^*-algebra generated by a Cuntz-Krieger $\mathcal{G}$-family.

A *path* in ultragraph $\mathcal{G}$ is a sequence $\alpha = e_1 e_2 \cdots e_n$ of edges with $s(e_{i+1}) \in r(e_i)$ for $1 \leq i \leq n-1$. We say that the path α has *length* $|\alpha| := n$ and we write $\mathcal{G}^*$ for the set of finite paths. The maps r, s extend to $\mathcal{G}^*$ in an obvious way.

By [9, Remark 2.13], we have

$$C^*(\mathcal{G}) = \overline{\operatorname{span}}\left\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{G}^*, A \in \mathcal{G}^0, \text{ and } r(\alpha) \cap r(\beta) \cap A \neq \emptyset\right\},$$

where $s_\alpha := s_{e_1}s_{e_2}\cdots s_{e_n}$ if $\alpha = e_1e_2\cdots e_n$ and $s_\alpha := p_A$ if $\alpha = A$.

The universal property of $C^*(\mathcal{G})$ gives an action $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(\mathcal{G})$, which is characterized on generators by $\gamma_z(p_A) = p_A$ and $\gamma_z(s_e) = zs_e$ for $A \in \mathcal{G}^0$, $e \in \mathcal{G}^1$ and $z \in \mathbb{T}$. It is called the *gauge action* for $C^*(\mathcal{G})$. The $*$-subalgebra $\{a \in C^*(\mathcal{G}) : \int_{\mathbb{T}} \gamma_z(a)dz = a\}$, denoted $C^*(\mathcal{G})^\gamma$, is called the *fixed-point algebra* of $C^*(\mathcal{G})$.

2 Tensor Product

A C^*-algebra $\mathcal{A}$ is called *nuclear* if both the injective and projective C^*-cross norms on $\mathcal{A}\otimes\mathcal{B}$ are equal for every C^*-algebra $\mathcal{B}$. In [8, Theorem 30] it is shown that all ultragraph C^*-algebras are nuclear. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two ultragraphs. If $C^*(\mathcal{G}_1) = C^*(s,p)$ and $C^*(\mathcal{G}_2) = C^*(t,q)$, then one can show that

$$C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2) = \overline{\operatorname{span}}\big\{s_\alpha p_A s_\beta^* \otimes t_\mu q_B t_\nu^*\big\},$$

where $\alpha, \beta \in \mathcal{G}_1^*$, $A \in \mathcal{G}_1^0$, $\mu, \nu \in \mathcal{G}_2^*$ and $B \in \mathcal{G}_2^0$.

Definition 2. The *Cartesian product* of ultragraphs $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G} := \mathcal{G}_1 \times \mathcal{G}_2$, is the ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$, where $G^0 := G_1^0 \times G_2^0$, $\mathcal{G}^1 := \mathcal{G}_1^1 \times \mathcal{G}_2^1$ and $s : \mathcal{G}^1 \to G^0$ and $r : \mathcal{G}^1 \to \mathcal{P}(G_1^0 \times G_2^0)$ are the maps defined by $s(e,f) := \big(s_1(e), s_2(f)\big)$ and $r(e,f) := r_1(e) \times r_2(f)$, respectively.

Remark 1. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two ultragraphs. If $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$, then by [9, Lemma 2.12], we have

$$\mathcal{G}^0 = \left\{ \bigcup_{j=1}^{k} \bigcap_{i_j=1}^{n_j} A_{i_j} : A_{i_j} \in \big\{(v,w), r(e,f) : (v,w) \in G^0, (e,f) \in \mathcal{G}^1\big\} \right\}.$$

We see in the next theorem that there exist a Cuntz-Krieger $\mathcal{G}$-family $\{S,P\}$ in the tensor product $C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)$ such that $C^*(S,P) = C^*(\mathcal{G})$.

Theorem 1. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be ultragraphs and let $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$. Then $C^*(\mathcal{G})$ can be embedded in $C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)$ as a $*$-subalgebra.*

Proof. Let $C^*(\mathcal{G}_1) = C^*(s,p)$ and $C^*(\mathcal{G}_2) = C^*(\tilde{s},\tilde{p})$. We begin by construct a Cuntz-Krieger $\mathcal{G}$-family $\{S,P\}$ in $C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)$. Define $P_{(v,w)} = p_v \otimes \tilde{p}_w$ and $P_{r(e,f)} = p_{r_1(e)} \otimes \tilde{p}_{r_2(f)}$ for every $(v,w) \in G^0$ and $(e,f) \in \mathcal{G}^1$. By Remark 1, we generate the projections $\{P_A : A \in \mathcal{G}^0\}$ by defining

$$P_{A\cap B} := P_A P_B \quad \text{and} \quad P_{A\cup B} := P_A + P_B - P_A P_B,$$

for every $A, B \in \big\{(v,w), r(e,f) : (v,w) \in G^0, (e,f) \in \mathcal{G}^1\big\}$. Also, we naturally define $S_{(e,f)} := s_e \otimes \tilde{s}_f$ for every $(e,f) \in \mathcal{G}^1$.

We show that $\{S, P\}$ is a Cuntz-Krieger $\mathcal{G}$-family in $C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)$. By Remark 1, $\{P_A : A \in \mathcal{G}^0\}$ is a set of projections satisfies Condition (1) of Definition 1.

To verify Condition (2) suppose that $(e, f) \in \mathcal{G}^1$. Then

$$\begin{aligned} S^*_{(e,f)} S_{(e,f)} &= (s_e \otimes \tilde{s}_f)^*(s_e \otimes \tilde{s}_f) = s^*_e s_e \otimes \tilde{s}^*_f \tilde{s}_f \\ &= p_{r_1(e)} \otimes \tilde{p}_{r_2(f)} = P_{r_1(e) \times r_2(f)} = P_{r(e,f)}. \end{aligned}$$

The Condition (3) of Definition 1 may be verified as Condition (2). Suppose that $0 < \left|s^{-1}(v, w)\right| < \infty$. Since $s^{-1}(v, w) = s_1^{-1}(v) \times s_2^{-1}(w)$, we have $0 < |s_1^{-1}(v)|, |s_2^{-1}(w)| < \infty$. Hence

$$\begin{aligned} \sum_{(e,f)\in s^{-1}(v,w)} S_{(e,f)} S^*_{(e,f)} &= \sum_{(e,f)\in s^{-1}(v,w)} (s_e \otimes \tilde{s}_f)(s_e \otimes \tilde{s}_f)^* \\ &= \sum_{(e,f)\in s_1^{-1}(v)\times s_2^{-1}(w)} s_e s^*_e \otimes \tilde{s}_f \tilde{s}^*_f \\ &= \sum_{f\in s^{-1}(w)} \sum_{e\in s^{-1}(v)} s_e s^*_e \otimes \tilde{s}_f \tilde{s}^*_f = p_v \otimes \tilde{p}_w = P_{(v,w)}. \end{aligned}$$

Thus $\{S, P\}$ is a Cuntz-Krieger $\mathcal{G}$-family in $C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)$. Now we show that $C^*(\mathcal{G}) \cong C^*(S, P)$. Let γ_1 be the gauge action on $C^*(\mathcal{G}_1)$. Define the action $\gamma_2 : \mathbb{T} \to C^*(\mathcal{G}_2)$ by $\gamma_2(z) = Id$ for every $z \in \mathbb{T}$. Also, let $\beta := \gamma_1 \otimes \gamma_2$ be the action of compact group $\mathbb{T}$ on $C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)$ defined by $\beta(z) = \gamma_1(z) \otimes \gamma_2(z)$. We have $\beta_z(P_A) = P_A$ and $\beta_z(S_{(e,f)}) = (\gamma_1)_z(s_e) \otimes (\gamma_2)_z \tilde{s}_f = z s_e \otimes \tilde{s}_f = z S_{(e,f)}$ for every $A \in \mathcal{G}^0$ and $(e, f) \in \mathcal{G}^1$. If $\{T, Q\}$ is the universal Cuntz-Krieger $\mathcal{G}$-family, then there is a homomorphism $\phi : C^*(\mathcal{G}) \to C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)$ such that $\phi(Q_A) = P_A$ and $\phi(T_{(e,f)}) = S_{(e,f)}$ for every $A \in \mathcal{G}^0$ and $(e, f) \in \mathcal{G}^1$. Since $\beta_z \circ \phi = \phi \circ \gamma_z$, it follows from the gauge-invariant Uniqueness Theorem [9, Theorem 6.8] that ϕ is injective and $C^*(S, P) \cong C^*(\mathcal{G})$. QED

2.1 Conditional Expectation

By setting some conditions on ultragraphs $\mathcal{G}_1$ and $\mathcal{G}_2$, we show that there exists a conditional expectation from $C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)$ onto $C^*(\mathcal{G}_1 \times \mathcal{G}_2)$. In the following we recall the definition of conditional expectation.

Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$. A linear map $E : \mathcal{A} \to \mathcal{B}$ is called a *projection* if $E(b) = b$ for every $b \in \mathcal{B}$. A *conditional expectation* from $\mathcal{A}$ onto $\mathcal{B}$ is a projection $E : \mathcal{A} \to \mathcal{B}$ such that $\|E(a)\| \leq \|a\|$ for all $a \in \mathcal{A}$.

Lemma 1. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be range-finite ultragraphs with no singular vertex and let $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$. If $C^*(\mathcal{G}_1) = C^*(s,p)$ and $C^*(\mathcal{G}_2) = C^*(\tilde{s},\tilde{p})$, then $C^*(\mathcal{G})$ is isomorphic to the subalgebra*

$$\mathcal{A} := \overline{\operatorname{span}}\{s_\alpha p_A s_\beta^* \otimes \tilde{s}_\mu \tilde{p}_B \tilde{s}_\nu^* : |\alpha| - |\beta| = |\mu| - |\nu|\}.$$

Proof. Let $\{S,P\}$ be the Cuntz-Krieger $\mathcal{G}$-family as defined in Theorem 1 and let $C^*(\mathcal{G}) = C^*(S,P)$. We note that $\alpha := (e_1,f_1)(e_2,f_2)\cdots(e_n,f_n)$ is a path in $\mathcal{G}$ if and only if $\alpha := e_1e_2\cdots e_n$ and $\mu := f_1f_2\cdots f_n$ are paths in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Thus $C^*(\mathcal{G}) \subseteq \mathcal{A}$.

Conversely, suppose that $s_\alpha p_A s_\beta^* \otimes \tilde{s}_\mu \tilde{p}_B \tilde{s}_\nu^* \in \mathcal{A}$. Since $\mathcal{G}_1$ and $\mathcal{G}_2$ are range-finite $p_{A_1} \otimes \tilde{p}_{A_2} \in C^*(\mathcal{G})$ for every $A_1 \in \mathcal{G}_1^0$ and $A_2 \in \mathcal{G}_2^0$. Suppose that $|\mu| > |\alpha|$. Decompose $\mu = \mu'\mu''$ and $\nu = \nu'\nu''$, where $|\mu'| = |\alpha|$ and $|\nu'| = |\beta|$. Due to the fact that $|\alpha| - |\beta| = |\mu| - |\nu|$ we have $|\mu''| = |\nu''|$. Since $\mathcal{G}_1$ is a row-finite ultragraph without sinks, for every $v \in G_1^0$ we have

$$p_v \otimes \tilde{s}_{\mu''}\tilde{p}_B\tilde{s}_{\nu''}^* = \sum_{\{\eta\in\mathcal{G}_1^*:|\eta|=|\mu''|,\ s_1(\eta)=v\}} s_\eta p_{r(\eta)} s_\eta^* \otimes \tilde{s}_{\mu''}\tilde{p}_B\tilde{s}_{\nu''}^* \in C^*(\mathcal{G}).$$

Due to the fact that $\mathcal{G}_1$ is range-finite $p_A \otimes \tilde{s}_{\mu''}\tilde{p}_B\tilde{s}_{\nu''}^* \in C^*(\mathcal{G})$. Hence

$$(s_\alpha \otimes \tilde{s}_{\mu'})(p_A \otimes \tilde{s}_{\mu''}\tilde{p}_B\tilde{s}_{\nu''}^*)(s_\beta^* \otimes \tilde{s}_{\nu'}^*) = s_\alpha p_A s_\beta^* \otimes \tilde{s}_\mu \tilde{p}_B \tilde{s}_\nu^* \in C^*(\mathcal{G}).$$

Suppose that $|\mu| < |\alpha|$. Then a similar argument as before and the assumption that $\mathcal{G}_2$ is range-finite ultragraph with no singular vertices imply that $s_\alpha p_A s_\beta^* \otimes \tilde{s}_\mu \tilde{p}_B \tilde{s}_\nu^* \in C^*(\mathcal{G})$. Thus $\mathcal{A} \subseteq C^*(\mathcal{G})$, as desired. *QED*

By using the above lemma, we construct a conditional expectation from $C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)$ onto $C^*(\mathcal{G}_1 \times \mathcal{G}_2)$.

Proposition 1. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be range-finite ultragraphs with no singular vertex and let $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$. Then there exists a conditional expectation from $C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)$ onto $C^*(\mathcal{G})$. In particular,*

$$C^*(\mathcal{G}) = \big(C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)\big)^\beta,$$

where β is an action from the unit circle $\mathbb{T}$ on $C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)$.

Proof. Let $C^*(\mathcal{G}_1) = C^*(s,p)$ and $C^*(\mathcal{G}_2) = C^*(\tilde{s},\tilde{p})$. Also, let γ_1 and γ_2 be the gauge actions of $C^*(\mathcal{G}_1)$ and $C^*(\mathcal{G}_2)$, respectively. Define the action β of $\mathbb{T}$ on $C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)$ by $\beta(z) = \gamma_1(z) \otimes \gamma_2(\overline{z})$ for all $z \in \mathbb{T}$. For every $s_\alpha p_A s_\beta^* \otimes$

$\tilde{s}_\mu \tilde{p}_B \tilde{s}_\nu^* \in C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)$ we have

$$\begin{aligned}\int_{\mathbb{T}} \beta_z(s_\alpha p_A s_\beta^* \otimes \tilde{s}_\mu \tilde{p}_B \tilde{s}_\nu^*)\, dz &= \int_{\mathbb{T}} (\gamma_1)_z(s_\alpha p_A s_\beta^*) \otimes (\gamma_2)_z(\tilde{s}_\mu \tilde{p}_B \tilde{s}_\nu^*)\, dz \\ &= \int_{\mathbb{T}} z^{|\alpha|-|\beta|} s_\alpha p_A s_\beta^* \otimes z^{-(|\mu|-|\nu|)} \tilde{s}_\mu \tilde{p}_B \tilde{s}_\nu^*\, dz \\ &= \int_{\mathbb{T}} z^{|\alpha|-|\beta|-(|\mu|-|\nu|)} \big(s_\alpha p_A s_\beta^* \otimes \tilde{s}_\mu \tilde{p}_B \tilde{s}_\nu^*\big)\, dz \\ &= \int_0^1 e^{2\pi i t\big(|\alpha|-|\beta|-(|\mu|-|\nu|)\big)} \big(s_\alpha p_A s_\beta^* \otimes \tilde{s}_\mu \tilde{p}_B \tilde{s}_\nu^*\big)\, dz.\end{aligned}$$

So if we define $E : C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2) \to C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2)$ by $E(x) = \int_{\mathbb{T}} \beta_z(x)\, dz$, then

$$E(s_\alpha p_A s_\beta^* \otimes \tilde{s}_\mu \tilde{p}_B \tilde{s}_\nu^*) = \begin{cases} s_\alpha p_A s_\beta^* \otimes \tilde{s}_\mu \tilde{p}_B \tilde{s}_\nu^* & \text{if } |\alpha| - |\beta| = |\mu| - |\nu|, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by Lemma 1, E is onto $C^*(\mathcal{G})$ and consequently $E : C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2) \to C^*(\mathcal{G})$ is a conditional expectation. *QED*

2.2 Exel-Laca algebras

The Exel-Laca algebras, denoted by $\mathcal{O}_A$, are generated by a set of partial isometries whose relations are determined by a countable binary-valued matrix A with no identically zero rows [4, Definition 8.1]. The C^*-algebras of ultragraphs with no singular vertices are precisely the Exel-Laca algebras. More precisely, for matrix A the ultragraph $\mathcal{G}_A := (G_A^0, \mathcal{G}_A^1, r, s)$ is defined by $G_A^0 := \{v_i : i \in I\}$, $\mathcal{G}_A^1 := I$, $s(i) := v_i$ and $r(i) := \{v_j : A(i,j) = 1\}$ for every $i \in I$. By [9, Theorem 4.5], the Exel-Laca algebra $\mathcal{O}_A$ is canonically isomorphic to ultragraph C^*-algebra $C^*(\mathcal{G}_A)$.

For some square matrices A and B, we show that there is a conditional expectation from $\mathcal{O}_A \otimes \mathcal{O}_B$ onto $\mathcal{O}_{AB}$. To do this we recall the following definition.

Definition 3 ([9]). Let $\mathcal{G}$ be an ultragraph. The edge matrix of an ultragraph $\mathcal{G}$ is the $\mathcal{G}^1 \times \mathcal{G}^1$ matrix $A_\mathcal{G}$ given by

$$A_\mathcal{G}(e,f) := \begin{cases} 1 & \text{if } s(f) \in r(e), \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4. Let A_1 and A_2 be infinite matrices with entries in $\{0,1\}$ and having no identically zero rows. We define $A_1 A_2 := A_{\mathcal{G}_{A_1} \times \mathcal{G}_{A_2}}$. If $A = [1]_{n\times n}$, then we have the Cuntz algebra $\mathcal{O}_n$. For $B = [1]_{m\times m}$ we have $AB = [1]_{nm\times nm}$.

Now we give the main result of this paper.

Theorem 2. *Let A and B be infinite matrices with entries in $\{0,1\}$ and having no identically zero rows. Then $\mathcal{O}_{AB}$ can be embedded in $\mathcal{O}_A \otimes \mathcal{O}_B$ as a $*$-subalgebra. If any row in A and B has at most finitely many non-zero elements, then there is a conditional expectation from $\mathcal{O}_A \otimes \mathcal{O}_B$ onto $\mathcal{O}_{AB}$. In particular, there is a conditional expectation from $\mathcal{O}_m \otimes \mathcal{O}_n$ onto $\mathcal{O}_{mn}$.*

Proof. By [9, Theorem 4.5] $C^*(\mathcal{G}_A) \cong \mathcal{O}_A$ and $C^*(\mathcal{G}_B) \cong \mathcal{O}_B$. Also, we have $\mathcal{O}_{AB} = \mathcal{O}_{A_{\mathcal{G}_{A_1} \times \mathcal{G}_{A_2}}} \cong C^*(\mathcal{G}_A \otimes \mathcal{G}_B)$. So the first assertion follows from Theorem 1. Now suppose that any row in A and B has at most finitely many non-zero elements. Then one can see that $\mathcal{G}_A$ and $\mathcal{G}_B$ are range-finite ultragraphs with no singular vertex. By Proposition 1 there is a conditional expectation from $\mathcal{O}_A \otimes \mathcal{O}_B$ onto $\mathcal{O}_{AB}$. QED

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