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On a perturbation of a class of Schrödinger systems in L^2-spaces

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Abstract. The aim of this short note is to prove a generation result of C_0-semigroups in $L^2(\mathbb{R}^d, \mathbb{C}^m)$, with the characterization of the domain of their generators, for a perturbation of a class of matrix Schrödinger operators by symmetric potential matrices whose entries can grow exponentially at infinity. A further perturbation by drift matrices with entries that can grow at most linearly at infinity, is considered. Finally, suitable assumptions which guarantee that the generated semigroups are analytic, are provided too.

Keywords: Matrix Schrödinger operators, characterization of the domain, perturbation theory, analytic semigroups

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Systems of parabolic equations with unbounded coefficients appear quite naturally in several settings, as in the study of backward-forward stochastic differential systems, of Nash equilibria to stochastic differential games, of the time-dependent Born-Openheimer theory and also in the study of the Navier-Stokes equations.

In contrast to the scalar theory of second-order elliptic operators with unbounded coefficients, that has been widely studied by several authors (see e.g., the recent monograph [11] and the references quoted therein), the study of systems with unbounded coefficients is at the beginning and it still presents many open fields of investigation.

The papers [1, 2, 3, 4, 5, 9, 10, 12, 13] study this type of systems in the framework of the semigroup theory with different strategies and in different settings.

In [2, 5] systems of parabolic equations associated to Kolmogorov elliptic operators coupled up to the first-order, are studied in the space of continuous and bounded functions. Under suitable assumptions, a vector-valued semigroup $(T(t))_{t\geq 0}$ which governs the problem is constructed.

Another important setting where to study these kind of operators are the usual L^p-spaces related to the Lebesgue measure despite the fact that, as the scalar case shows, these are not the L^p-spaces which fit best the properties of elliptic operators with unbounded coefficients. In particular, even if their realizations in these spaces generate strongly continuous or even analytic semigroups, the characterization of the domain of the infinitesimal generator, which is crucial to study the associated parabolic equation, is known only in a few cases. We refer the reader e.g., to [7, 8, 14, 15, 20] for the scalar case.

In the vector-valued case, systems of parabolic equations with unbounded coefficients are studied in the classical L^p-setting in [9, 4, 10, 11, 12, 13]. In [4], taking advantage of the results in [2], conditions that ensure that the semigroup $(T(t))_{t\geq 0}$ can be extrapolated to the L^p-scale are provided. Clearly, this approach does not give any information about the domain of the generator of $T(t)$ in L^p. Recently, the L^p-realization of matrix Schrödinger operators of the type $\mathcal{A} = \Delta + V$ was studied in [10], where the generation of a semigroup in $L^p(\mathbb{R}^d, \mathbb{C}^m)$ and the characterization of the domain $D(A_p)$ have been established throughout a noncommutative version of the Dore-Venni theorem due to S. Monniaux and J. Prüss (see [16]). The assumptions which allow to apply this result (see (3) and (4)) force the entries of the matrix-valued function V to grow polynomially and, more precisely, as $|x|^r$, $r \in [1, 2[$.

In [13] the authors obtain the same generation and regularity results of [10] for a more general class of potentials whose diagonal entries are functions of type $|x|^\alpha$, $\alpha \geq 1$, or even $e^{|x|}$. The idea consists in perturbing the operator A_p by a scalar potential, i.e., considering the operator $\mathcal{A} - vI$ with $v \in W^{1,\infty}_{\rm loc}(\mathbb{R}^d)$ such that $|\nabla v| \leq C \cdot v$, for some positive constant C. In this case by applying a theorem due to Okazawa (see [17]), they prove that the realization of $\mathcal{A} - vI$ in $L^p(\mathbb{R}^d, \mathbb{C}^m)$ generates a C_0-semigroup in $L^p(\mathbb{R}^d, \mathbb{C}^m)$ and describe the domain of the generator.

This note is a first preliminary step in the study of much more general systems of elliptic equations in L^p-spaces aimed at providing classes of systems for which the realization in $L^p(\mathbb{R}^d, \mathbb{C}^m)$ generates a "good" semigroup and a complete characterization of the domain of the infinitesimal generator is available. In particular, here we extend the results in [13] when $p = 2$ to the case of a matrix Schrödinger operators of the type $\mathcal{A}$ perturbed by a first-order term, whose coefficients can grow at most linearly at infinity, and a zero-order term consisting of a symmetric matrix W satisfying suitable assumptions.

We point out that, under our assumptions, diagonal matrices W which are not of the type vI can be considered, allowing for different growth rates for the diagonal entries of the potential matrix. We provide a concrete example of this type which is covered neither by the results in [13] nor by results in [10], where hypotheses (4) below is assumed. Indeed, our choice of the matrix-valued function W (see Example 1) implies that, for each $\gamma \in (0, 1/2)$, the function $\|D_j(V - sW)(-V + sW)^{-\gamma}\|$ is unbounded on $\mathbb{R}^d$ for any $s > 0$.

We observe that the L^2-setting is the starting point also for the approach in [12], based on form methods. The main difference is that the operators that we consider here are not necessarily symmetric and that Okazawa's theorem permits a description of the domain of the generator.

Finally, it is worth noting that, if $p \neq 2$, even in the case of a diagonal perturbation of the potential V, technical difficulties arise.

1 Notation and preliminaries

For every $m, d \in \mathbb{N}$, we denote by $|\cdot|$ the Euclidean norm on $\mathbb{C}^m$ and by $\langle \cdot, \cdot \rangle$ the Euclidean inner product on $\mathbb{C}^m$. By $C_c^\infty(\mathbb{R}^d, \mathbb{C}^m)$ we denote the space of infinitely differentiable functions with compact support and by $L^2(\mathbb{R}^d, \mathbb{C}^m)$ the usual Hilbert space of Lebesgue square integrable functions endowed with the inner product

$$(f, g) = \int_{\mathbb{R}^d} \sum_{j=1}^m \langle f_j(x), g_j(x) \rangle dx, \qquad f, g \in L^2(\mathbb{R}^d, \mathbb{C}^m)$$

and the induced norm $\|\cdot\|_2$. Finally $W^{k,p}(\mathbb{R}^d, \mathbb{C}^m)$ stands for the classical Sobolev space of order k.

We recall that an operator $L : D(L) \subseteq L^2(\mathbb{R}^d, \mathbb{C}^m)$ is said to be m-accretive if $\mathrm{Re}(Lu, u) \geq 0$ for every $u \in D(L)$ and $(\lambda + L)(D(L)) = L^2(\mathbb{R}^d, \mathbb{C}^m)$ for some $\lambda > 0$. In this case, $-L$ generates a contraction semigroup.

For sake of completeness, we recall the Okazawa's perturbation theorem [17, Theorem 1.6] in the Hilbert space setting (see also [19] for the result about analyticity).

Theorem 1. *Let A and B be linear m-accretive operators on a Hilbert space H. Let D be a core for A and assume that there exist constants $a, b, c \geq 0$ such that for every $u \in D$ and every $\varepsilon > 0$:*

$$\mathrm{Re}(Au, B_\varepsilon u) \geq -c\|u\|^2 - a\|B_\varepsilon u\| \, \|u\| - b\|B_\varepsilon u\|^2,$$

where B_ε denotes the Yosida approximation of B. If $s > b$, then $A + sB$ with domain $D(A) \cap D(B)$ is m-accretive. If, in addition, there exists $M \geq 0$ such that

$$\mathrm{Re}((A + sB)u, u) \geq M|\mathrm{Im}((A + sB)u, u)| \tag{1}$$

for any $u \in D(A) \cap D(B)$, then $A + sB$ is sectorial of angle less than $\pi/2$ and thus $-(A + sB)$ generates an analytic semigroup.

Consider the operator

$$\mathcal{A} = \Delta + V \tag{2}$$

on $L^2(\mathbb{R}^d, \mathbb{C}^m)$ where $V = [v_{ij}] : \mathbb{R}^d \to \mathbb{R}^{m\times m}$ is a measurable matrix-valued function such that $v_{ij} \in W^{1,\infty}_{loc}(\mathbb{R}^d)$,

$$\mathrm{Re}\langle V(x)\xi, \xi\rangle \leq -2|\xi|^2, \qquad x \in \mathbb{R}^d,\ \xi \in \mathbb{C}^m, \tag{3}$$

and there exists $\gamma \in [0, \frac{1}{2})$ such that

$$\sup_{x\in\mathbb{R}^d} \left|\partial_j V(x)(-V(x) - I)^{-\gamma}\right| < \infty, \qquad j = 1, \dots m. \tag{4}$$

Under these assumptions, the realization A of $\mathcal{A}$ with domain

$$D(A) = \{u \in W^{2,2}(\mathbb{R}^d, \mathbb{C}^m) : Vu \in L^2(\mathbb{R}^d, \mathbb{C}^m)\}$$

generates a contraction semigroup. This follows from applying [10, Corollary 3.3] to the sum $A_2 + V_2$, where $A_2 = \Delta - I$ and $V_2 = V + I$. Moreover, by [13, Proposition 2.3], $C^\infty_c(\mathbb{R}^d, \mathbb{C}^m)$ is a core for A.

It should be noted that the assumption (3) is not restrictive. We could consider potentials $\widetilde{V}$ satisfying

$$\mathrm{Re}\langle \widetilde{V}(x)\xi, \xi\rangle \leq \beta|\xi|^2, \qquad x \in \mathbb{R}^d,\ \xi \in \mathbb{C}^m,$$

for some $\beta > 0$. Then (3) would be clearly satisfied by the shifted potential $V = \widetilde{V} - (\beta + 2)I$. In this more general situation, we should require that

$$\sup_{x\in\mathbb{R}^d} \left|\partial_j \widetilde{V}(x)(-\widetilde{V}(x) + (\beta + 1)I)^{-\gamma}\right| < \infty, \qquad j = 1, \dots m.$$

2 Main result

Along this section let V be a matrix-valued function satisfying (3) and (4) and let $W = [w_{ij}] : \mathbb{R}^d \to \mathbb{R}^{m\times m}$ be a measurable matrix-valued function such that $w_{ij} \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$, $w_{ij} = w_{ji}$ and for all $x \in \mathbb{R}^d$ there exists $c(x) > 0$ such that

$$\mathrm{Re}\langle W(x)\xi, \xi\rangle \geq c(x)|\xi|^2, \qquad \xi \in \mathbb{C}^m. \tag{5}$$

Consider on $L^2(\mathbb{R}^d, \mathbb{C}^m)$ the multiplication operator

$$Bu = Wu, \qquad u \in D(B) = \{u \in L^2(\mathbb{R}^d, \mathbb{C}^m) : \ Wu \in L^2(\mathbb{R}^d, \mathbb{C}^m)\}.$$

Then $(B, D(B))$ is m-accretive. Indeed, by (5), it is immediate to check that $\mathrm{Re}(Bu, u) \geq 0$ for every $u \in D(B)$. Moreover, for every $x \in \mathbb{R}^d$, the matrix $I + W(x)$ is positive definite, hence it is invertible and

$$\|(I + W(x))^{-1}\| = \sup_{\xi \in \mathbb{C}^m \setminus \{0\}} \frac{|(I + W(x))^{-1}\xi|}{|\xi|} \leq \frac{1}{1 + c(x)} \leq 1.$$

Therefore, for any $f \in L^2(\mathbb{R}^d, \mathbb{C}^m)$, the function $u = (I + W)^{-1} f$ clearly belongs to $L^2(\mathbb{R}^d, \mathbb{C}^m)$ and $(I + B)u = f$.

For every $\varepsilon > 0$, set $W_\varepsilon(x) := W(x)(I + \varepsilon W(x))^{-1}$. Then the Yosida approximations of B are $B_\varepsilon u := B(I + \varepsilon B)^{-1} u = W_\varepsilon u$, for every $u \in L^2(\mathbb{R}^d, \mathbb{C}^m)$.

Observe that, since $W(x)$ is symmetric and positive definite, also $W_\varepsilon(x)$ is symmetric and positive definite.

Lemma 1. *Assume that*

$$\mathrm{Re}\langle V(x)\xi, W(x)\xi\rangle \leq 0, \qquad \xi \in \mathbb{C}^m, \ x \in \mathbb{R}^d. \tag{6}$$

Then, for every $u \in C_c^\infty(\mathbb{R}^d, \mathbb{C}^m)$*:*

$$\mathrm{Re} \int_{\mathbb{R}^d} \langle -Au(x), B_\varepsilon u(x)\rangle dx \geq -\frac{1}{4} \sum_{k=1}^{d} \int_{\mathbb{R}^d} |W_\varepsilon^{-\frac{1}{2}}(x)(D_k W_\varepsilon(x))u(x)|^2 dx.$$

Proof. Set $R = W_\varepsilon$, $R = [r_{ij}]$. Observe that for every $\eta \in \mathbb{C}^m$ and every $x \in \mathbb{R}^d$

$$\mathrm{Re}\langle V(x)(I + \varepsilon W(x))\eta, W(x)\eta\rangle \leq 0.$$

Hence, for every $\xi \in \mathbb{C}^m$ and $x \in \mathbb{R}^d$, setting $\eta = (I + \varepsilon W(x))^{-1}\xi$, we get that

$$\mathrm{Re}\langle V(x)\xi, W_\varepsilon(x)\xi\rangle \leq 0.$$

Then

$$\begin{aligned}
&\mathrm{Re} \int_{\mathbb{R}^d} \langle -Au(x), B_\varepsilon u(x)\rangle dx \\
&= -\mathrm{Re} \int_{\mathbb{R}^d} \langle \Delta u(x), Ru(x)\rangle dx - \mathrm{Re} \int_{\mathbb{R}^d} \langle V(x)u(x), R(x)u(x)\rangle dx \\
&\geq -\int_{\mathbb{R}^d} \mathrm{Re}\langle \Delta u(x), Ru(x)\rangle dx = \sum_{i=1}^{m} \int_{\mathbb{R}^d} \mathrm{Re}\langle \nabla u_i(x), \nabla (R(x)u(x))_i\rangle dx.
\end{aligned}$$

The assertion follows by observing that

$$\begin{aligned}
&\sum_{i=1}^{m}\mathrm{Re}\langle\nabla u_i,\nabla(Ru)_i\rangle=\sum_{k=1}^{d}\mathrm{Re}\bigg(\sum_{i,j=1}^{m}D_ku_i\overline{D_k(r_{ij}u_j)}\bigg)\\
=&\sum_{k=1}^{d}\mathrm{Re}\bigg(\sum_{i,j=1}^{d}r_{ij}D_ku_i\overline{D_ku_j}+\sum_{i,j=1}^{d}D_kr_{ij}D_ku_i\overline{u_j}\bigg)\\
=&\sum_{k=1}^{d}\langle RD_ku,D_ku\rangle+\mathrm{Re}\langle(D_kR)u,D_ku\rangle\\
=&\sum_{k=1}^{d}\left|R^{\frac12}D_ku+\frac12R^{-\frac12}(D_kR)u\right|^2-\frac14|R^{-\frac12}D_kRu|^2. \qquad \square
\end{aligned}$$

Theorem 2. *Assume that conditions* (3)*,* (4)*,* (5) *and* (6) *hold and that there exist* $a,b\geq 0$ *such that*

$$\sum_{k=1}^{d}|W_\varepsilon^{-\frac12}(D_kW_\varepsilon(x))\xi|^2\leq a\langle W_\varepsilon(x)\xi,\xi\rangle+b|W_\varepsilon(x)\xi|^2,\qquad \xi\in\mathbb{C}^m,\qquad (7)$$

for all $\varepsilon>0$ *and* $x\in\mathbb{R}^d$*. Then the operators* $L_s:=A-sW=\Delta+V-sW$*, endowed with the domain*

$$D(L)=\{u\in W^{2,2}(\mathbb{R}^d,\mathbb{C}^m):Vu,Wu\in L^2(\mathbb{R}^d,\mathbb{C}^m)\}\qquad (8)$$

generate contractive C_0*-semigroups in* $L^2(\mathbb{R}^d,\mathbb{C}^m)$ *for every* $s>\frac{b}{4}$*. Moreover, the graph norm of* $D(L)$ *is equivalent to the norm* $u\mapsto\|u\|_2+\|\Delta u\|_2+\|Vu\|_2+\|Wu\|_2$*.*

Proof. Fix $u\in C_c^\infty(\mathbb{R}^d,\mathbb{C}^m)$. Then,

$$\begin{aligned}
&\mathrm{Re}\int_{\mathbb{R}^d}\langle -Au(x),B_\varepsilon u(x)\rangle dx\geq-\frac14\sum_{k=1}^{d}\int_{\mathbb{R}^d}|W_\varepsilon(x)^{-\frac12}(D_kW_\varepsilon)(x)u(x)|^2dx\\
\geq&-\frac{a}{4}\int_{\mathbb{R}^d}\langle W_\varepsilon(x)u(x),u(x)\rangle dx-\frac{b}{4}\int_{\mathbb{R}^d}|W_\varepsilon(x)u(x)|^2dx\\
\geq&-\frac{a}{4}\|W_\varepsilon u\|_2\cdot\|u\|_2-\frac{b}{4}\|W_\varepsilon u\|_2.
\end{aligned}$$

The assertion now follows applying Theorem 1. $\square$

Corollary 1. *Assume that* W *is a diagonal matrix with positive entries* $w_i\in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$*,* $i=1,\dots,m$*, such that* $\mathrm{Re}\langle V(x)\xi,W(x)\xi\rangle\leq 0$ *for every* $\xi\in\mathbb{C}^m$ *and* $x\in\mathbb{R}^d$*. Further, assume that there exists* $C>0$ *such that*

$$|\nabla w_i(x)|\leq Cw_i(x)$$

for all $i = 1, \dots, m$ and $x \in \mathbb{R}^d$. Then (7) is satisfied with $b = 0$. In particular $\Delta + V - W$, endowed with the domain $D(L)$ generates a contractive C_0-semigroup in $L^2(\mathbb{R}^d, \mathbb{C}^m)$.

Proof. Note that $W_\varepsilon^{-\frac{1}{2}}(D_k W_\varepsilon)$ is the diagonal matrix with entries $\frac{D_k w_i}{w_i^{1/2}(1+\varepsilon w_i)^{3/2}}$, $(i = 1, \dots, m)$. Then, for every $\xi \in \mathbb{C}^m$ and $x \in \mathbb{R}^d$:

$$\sum_{k=1}^{d} |W_\varepsilon^{-\frac{1}{2}}(x)(D_k W_\varepsilon(x))\xi|^2 = \sum_{k=1}^{d}\sum_{i=1}^{m} \frac{(D_k w_i(x))^2}{w_i(x)(1+\varepsilon w_i(x))^3}|\xi_i|^2$$
$$\leq C^2 \sum_{i=1}^{d} \frac{w_i(x)}{(1+\varepsilon w_i(x))^3}|\xi_i|^2 \leq C^2 \sum_{i=1}^{d} \frac{w_i(x)}{(1+\varepsilon w_i(x))}|\xi_i|^2 = C^2 \langle W_\varepsilon(x)\xi, \xi\rangle$$

and we conclude as in the proof of Theorem 2. □

3 Further properties and examples

In this section we collect some properties of the realization of the operators $L_s := \Delta + V - sW$, $(s > \frac{b}{4})$ in $L^2(\mathbb{R}^d, \mathbb{C}^m)$ and of the associated semigroup. Moreover, we provide an example of perturbed Schrödinger systems to which all our results can be applied. Here, besides hypotheses (3) and (4) we assume that assumptions (5), (6) and (7) are satisfied. We start by proving that the natural domain $D(L)$ actually coincides with the maximal domain.

Proposition 1. *The following equality holds true:*

$$D(L) = \{u \in L^2(\mathbb{R}^d, \mathbb{C}^m) \cap W^{2,2}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^m) : L_s u \in L^2(\mathbb{R}^d, \mathbb{C}^m)\} =: D_{2,\max}(L_s).$$

for any $s > \frac{b}{4}$.

Proof. From (8), it is clear that $D(L) \subset D_{2,\max}(L_s)$ for any $s > \frac{b}{4}$. On the other hand, to show the other inclusion it suffices to prove that $\lambda - L_s$ is injective on $D_{2,\max}(L_s)$ for some $\lambda > 0$. To this aim, let $u \in D_{2,\max}(L_s)$ be such that $\lambda u - L_s u = 0$ and prove that $u \equiv 0$. Since the entries of the matrix-valued functions V and W are real we can assume that u is a real valued-function. Indeed, if u is a complex-valued solution to the previous equation, then its real and imaginary parts solve the same equation.

Let (ϑ_n) be a sequence of cut-off functions such that $\chi_{B_n} \leq \vartheta_n \leq \chi_{B_{2n}}$ and $\|\nabla\vartheta_n\|_\infty \leq Cn^{-1}$ for some positive constant C and any $n \in \mathbb{N}$. Multiplying the equation $\lambda u - L_s u = 0$ by $\vartheta_n^2 u$, using (3), (5) and integrating by parts over $\mathbb{R}^d$

we get

$$
\begin{aligned}
0 =& \int_{\mathbb{R}^d} \langle \lambda u - L_s u, \vartheta_n^2 u\rangle dx \\
=& \lambda \int_{\mathbb{R}^d} \vartheta_n^2 |u|^2 dx + \sum_{k=1}^m \int_{\mathbb{R}^d} |\nabla u_k|^2 \vartheta_n^2 dx + 2\sum_{k=1}^m \int_{\mathbb{R}^d} \vartheta_n \langle \nabla u_k, \nabla \vartheta_n\rangle u_k dx \\
&- \int_{\mathbb{R}^d} \langle Vu, u\rangle dx + s\int_{\mathbb{R}^d} \langle Wu, u\rangle dx \\
\geq& \lambda \int_{\mathbb{R}^d} \vartheta_n^2 |u|^2 dx - \int_{\mathbb{R}^d} |\nabla \vartheta_n|^2 |u|^2 dx \\
\geq& \lambda \int_{\mathbb{R}^d} \vartheta_n^2 |u|^2 dx - \|\nabla \vartheta_n\|_\infty^2 \int_{\mathbb{R}^d} |u|^2 dx \\
\geq& \lambda \int_{\mathbb{R}^d} \vartheta_n^2 |u|^2 dx - C^2 n^{-2} \int_{\mathbb{R}^d} |u|^2 dx,
\end{aligned}
$$

where we have estimated

$$2\sum_{k=1}^m \vartheta_n \langle \nabla u_k, \nabla \vartheta_n\rangle u_k \geq -\vartheta_n^2 |\nabla u|^2 - |\nabla \vartheta_n|^2 |u|^2.$$

Hence, letting n tend to ∞ we deduce that $\lambda\|u\|_2 \leq 0$, whence $u \equiv 0$. ☐

Now, under suitable assumptions on the matrix-valued functions V and W, we are able to prove that the resolvent operator of L_s is compact in $L^2(\mathbb{R}^d, \mathbb{C}^m)$.

Proposition 2. *Assume that there exist $s > b/4$ and a measurable function $\varrho : \mathbb{R}^d \to \mathbb{R}^+$ blowing up as $|x| \to \infty$ such that $|(V(x) - sW(x))\xi| \geq \varrho(x)|\xi|$ for any $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^m$. Then, the operator L_s has compact resolvent in $L^2(\mathbb{R}^d, \mathbb{C}^m)$. Consequently, its spectrum is discrete and consists of eigenvalues only.*

Proof. Even if the proof is rather classical, for the reader convenience we provide the details. To prove the statement, we fix $s > b/4$ and show that the unit ball $\mathcal{B}$ of $D(L)$ is relatively compact in $L^2(\mathbb{R}^d, \mathbb{C}^m)$. Since $D(L) \subset W^{2,2}(\mathbb{R}^d)$, $\|u(\cdot + h) - u\|_2$ tends to zero as $h \to 0$, uniformly with respect to $u \in \mathcal{B}$. So, we just need to show that the L^2-norm of functions in $\mathcal{B}$ is almost all concentrated in a large ball centered at zero. Recalling that the graph norm of $D(L)$ is equivalent to the norm $u \mapsto \|u\|_2 + \|\Delta u\|_2 + \|Vu\|_2 + \|Wu\|_2$ (see Theorem 2), it follows that

$$\int_{\mathbb{R}^d} |(V(x) - sW(x))u(x)|^2 dx \leq C\|u\|_{D(L)}^2 \leq C$$

for some positive constant C and every $u \in \mathcal{B}$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^d\setminus B(0,r)} |u|^2 dx \leq & \frac{1}{\inf_{x\in\mathbb{R}^d\setminus B(0,r)} \varrho^2(x)} \int_{\mathbb{R}^d} \varrho^2 |u|^2 dx \\ \leq & \frac{1}{\inf_{x\in\mathbb{R}^d\setminus B(0,r)} \varrho^2(x)} \int_{\mathbb{R}^d\setminus B(0,r)} |(V(x) - sW(x))u(x)|^2 dx \\ \leq & \frac{1}{\inf_{x\in\mathbb{R}^d\setminus B(0,r)} \varrho^2(x)} \int_{\mathbb{R}^d} |(V(x) - sW(x))u(x)|^2 dx \\ \leq & \frac{C}{\inf_{x\in\mathbb{R}^d\setminus B(0,r)} \varrho^2(x)}\end{aligned}$$

for any $u \in \mathcal{B}$. Since ϱ blows up at infinity, letting r tend to ∞ in the previous chain of inequalities we conclude that $\|u\|_{L^2(\mathbb{R}^d\setminus B(0,r))}$ vanishes, uniformly with respect to $u \in \mathcal{B}$ and we are done. □

As it was observed in [10, Example 3.5], hypotheses (3) and (4) are not sufficient to guarantee that the semigroup generated by the operator $\mathcal{A}$ in (2) is analytic in $L^2(\mathbb{R}^d, \mathbb{C}^m)$. The following theorem provides a sufficient condition in order that the perturbed operator L_s generates an analytic semigroup in $L^2(\mathbb{R}^d, \mathbb{C}^m)$.

Theorem 3. *Assume that there exists $M \geq 0$ such that*

$$\mathrm{Re}\langle(-V(x) + sW(x))\xi, \xi\rangle \geq M|\mathrm{Im}\langle(V(x) - sW(x))\xi, \xi\rangle| \tag{9}$$

for any $x \in \mathbb{R}^d$, $\xi \in \mathbb{C}^m$ and some $s > \frac{b}{4}$. Then, the strongly continuous semigroup generated by L_s in $L^2(\mathbb{R}^d, \mathbb{C}^m)$ is analytic too.

Proof. First of all we recall that $\mathrm{Re}(-\Delta u, u) \geq c_0|\mathrm{Im}(\Delta u, u)|$ for any $u \in D(L) \subset W^{2,2}(\mathbb{R}^d, \mathbb{C}^m)$ and any positive constant c_0 (see [18, Theorem 3.9]). Hence, for any $u \in D(L)$ we get

$$\begin{aligned}\mathrm{Re}(-L_s u, u) &= \mathrm{Re}(-\Delta u, u) + \mathrm{Re}(-Vu + sWu, u) \\ &\geq c_0|\mathrm{Im}(\Delta u, u)| + M\int_{\mathbb{R}^d} |\mathrm{Im}\langle(V(x) - sW(x))u(x), u(x)\rangle| dx \\ &\geq \min\{c_0, M\}|\mathrm{Im}(L_s u, u)|\end{aligned}$$

and, by Theorem 1, we deduce that $-L_s$ is sectorial of angle less than $\pi/2$ and then L_s generates an analytic semigroup in $L^2(\mathbb{R}^d, \mathbb{C}^m)$. □

The result in Theorem 2 can be slightly generalized, considering a class of elliptic systems with a drift term, i.e., systems of the type

$$\mathcal{L}_s = \Delta + \sum_{j=1}^{d} F_j D_j + V - sW.$$

Corollary 2. *Under the assumptions of Theorem 3, assume that the entries of the matrix-valued functions F_j $(j = 1, \dots, d)$ are measurable functions, which grow at most linearly at infinity. Further, assume that there exist $s > b/4$ and a positive function ϕ blowing up at infinity, faster than quadratically, such that*

$$|(V(x) - sW(x))\xi| \geq \phi(x)|\xi|, \qquad x \in \mathbb{R}^d,\ \xi \in \mathbb{C}^m. \tag{10}$$

Then, the realization $\widetilde{L}_s$ of the operator $\mathcal{L}_s$ in $L^2(\mathbb{R}^d, \mathbb{C}^m)$ with $D(L)$ as domain, generates an analytic semigroup.

Proof. To prove the assertion it is enough to show that, for every Lipschitz continuous function $\zeta : \mathbb{R}^d \to \mathbb{R}$ with positive infimum over $\mathbb{R}^d$, there exist $\varepsilon_0 > 0$ and, for every $\varepsilon \in (0, \varepsilon_0)$, a positive constant C_ε such that

$$\|\zeta \nabla v\|_2 \leq \varepsilon \|\Delta v\|_2 + C_\varepsilon \|\zeta^2 v\|_2 \tag{11}$$

for every $v \in W^{2,2}(\mathbb{R}^d)$ such that $\zeta^2 v \in L^2(\mathbb{R}^d)$. Once estimate (11) is proved, we can apply a classical perturbation argument for generators of analytic semigroups (see [6, Theorem III 2.10]) and complete the proof. Indeed, from our assumptions on ϕ, we can infer that

$$|x|^2 \leq \delta |x|^4 + C'_\delta \leq \delta \phi(x)^2 + C''_\delta, \qquad x \in \mathbb{R}^d,$$

for any $\delta > 0$ and some positive constants C'_δ and C''_δ, blowing up as δ tends to 0^+. Therefore, taking (10) into account and choosing $\zeta(x) = |x| + 1$ in (11), we get that

$$\begin{aligned} \|F_j D_j u\|_2 \leq& \varepsilon \|\Delta u\|_2 + C_\varepsilon (\delta \|\phi u\|_2 + C''_\delta \|u\|_2) \\ \leq& \varepsilon \|\Delta u\|_2 + C_\varepsilon (\delta |(V - sW)u\|_2 + C''_\delta \|u\|_2) \end{aligned}$$

for every $j = 1, \dots, d$, $\varepsilon \in (0, \varepsilon_0)$, $\delta > 0$ and $u \in D(L)$. Choosing δ such that $C_\varepsilon \delta = \varepsilon$, we immediately conclude that

$$\|F_j D_j u\|_2 \leq \varepsilon \left(\|\Delta u\|_2 + \|(V - sW)u\|_2\right) + C'''_\varepsilon \|u\|_2 \leq K(\varepsilon \|u\|_{D(L)} + C'''_\varepsilon \|u\|_2)$$

for every $j = 1, \dots, d$, $\varepsilon \in (0, \varepsilon_0)$ and $u \in D(L)$, the constant K, being independent of v and ε.

So, let us prove (11). First of all, we note that $C_c^\infty(\mathbb{R}^d)$ is dense in the set $D = \{v \in W^{2,2}(\mathbb{R}^d) : \zeta^2 v \in L^2(\mathbb{R}^d)\}$. This property can be easily checked first observing that the function in $W^{2,2}(\mathbb{R}^d)$ with compact support are dense in D and, then, by approximating any such function by a sequence of $C_c^\infty(\mathbb{R}^d)$-function through a convolution argument.

Based on this remark, we can limit ourselves to proving (11) for functions in $C_c^\infty(\mathbb{R}^d)$. We fix such a function v, $x_0 \in \mathbb{R}^d$ and a cut-off function $\vartheta : \mathbb{R}^d \to \mathbb{R}$

such that $\chi_{B(0,1)} \le \vartheta \le \chi_{B(0,2)}$. By integrating by parts and applying the Cauchy-Schwarz inequality we get

$$\begin{aligned}\|\zeta(x_0)\nabla(\vartheta_{x_0}v)\|_2 \le&\|\Delta(\vartheta_{x_0}v)\|_2^{\frac{1}{2}}\|\zeta^2(x_0)\vartheta_{x_0}v\|_2^{\frac{1}{2}}\\ \le&\varepsilon\|\Delta(\vartheta_{x_0}v)\|_2+\frac{1}{4\varepsilon}\|\zeta^2(x_0)\vartheta_{x_0}v\|_2,\end{aligned} \tag{12}$$

where, $\vartheta_{x_0}(x)=\vartheta((x-x_0)/\varrho(x_0))$ for every $x\in\mathbb{R}^d$, and $\varrho=(2\|\nabla\zeta\|_\infty)^{-1}\zeta$.

A direct computation shows that $2^{-1}\zeta(x_0)\le\zeta(x)\le 3\cdot 2^{-1}\zeta(x_0)$ for every $x\in B(x_0,2\rho(x_0))$. Hence, we can estimate

$$\begin{aligned}\|\zeta\nabla v\|_{L^2(B(x_0,\rho(x_0)))} \le&\|\zeta\nabla(\vartheta_{x_0}v)\|_{L^2(B(x_0,2\rho(x_0)))}\\ \le&\frac{3}{2}\|\zeta(x_0)\nabla(\vartheta_{x_0}v)\|_{L^2(B(x_0,2\rho(x_0)))}\end{aligned} \tag{13}$$

and

$$\|\zeta^2(x_0)\vartheta_{x_0}v\|_2\le 4\|\zeta^2 v\|_{L^2(B(x_0,2\rho(x_0)))}. \tag{14}$$

Replacing (13) and (14) into (12), and using that

$$||\vartheta_{x_0}||_{C^2_b(\mathbb{R}^d)}\le C$$

for some positive constant C, we get

$$\begin{aligned}\|\zeta\nabla v\|_{L^2(B(x_0,\rho(x_0)))}\le C'\bigg(&\varepsilon\|\Delta v\|_{L^2(B(x_0,2\rho(x_0)))}+\varepsilon\|\nabla v\|_{L^2(B(x_0,2\rho(x_0)))}\\ &+\varepsilon\|v\|_{L^2(B(x_0,2\rho(x_0)))}+\frac{1}{4\varepsilon}\|\zeta^2 v\|_{L^2(B(x_0,2\rho(x_0)))}\bigg).\end{aligned}$$

Since ζ has positive infimum over $\mathbb{R}^d$, we can estimate

$$\begin{aligned}&\|v\|_{L^2(B(x_0,2\rho(x_0)))}+\|\nabla v\|_{L^2(B(x_0,2\rho(x_0)))}\\ \le&C_1\Big(\|\zeta^2 v\|_{L^2(B(x_0,2\rho(x_0)))}+\|\zeta\nabla v\|_{L^2(B(x_0,2\rho(x_0)))}\Big)\end{aligned}$$

for some positive constant C_1, independent of v. Moreover, since ζ is Lipschitz continuous, there exists a countable covering of $\mathbb{R}^d$ consisting of the balls $B(x_n,\rho(x_n))$ such that only a finite number of the doubled balls $B(x_n,2\rho(x_n))$ overlap. As a byproduct, we can infer that

$$\|\zeta\nabla v\|_{L^2(\mathbb{R}^d)}\le C''\bigg(\varepsilon\|\Delta v\|_2+\varepsilon\|\zeta\nabla v\|_2+\varepsilon\|\zeta^2 v\|_2+\frac{1}{4\varepsilon}\|\zeta^2 v\|_2\bigg)$$

Choosing properly $\varepsilon>0$, we can complete the proof. □

Finally we provide an example of perturbed Schrödinger system to which all our results can be applied.

Example 1. Let $r \in [1,2)$ and $\alpha, \beta \in W^{1,\infty}_{\rm loc}(\mathbb{R}^d)$ be two positive functions such that $|\nabla\alpha(x)| \le C\alpha(x)$, $|\nabla\beta(x)| \le C\beta(x)$ for every $x \in \mathbb{R}^d$ and some positive constant C. For example, we could consider α and β of type $|x|^\delta + 1$, with $\delta > 1$, or $e^{a|x|}$ with $a > 0$. Consider the matrix-valued functions $\widetilde{V}$ and W defined by

$$\widetilde{V}(x) = \begin{pmatrix} 0 & 0 & |x|^r \\ 0 & -|x|^r & 0 \\ -|x|^r & 0 & 0 \end{pmatrix}, \qquad W(x) = \begin{pmatrix} \alpha(x) & 0 & 0 \\ 0 & \beta(x) & 0 \\ 0 & 0 & \alpha(x) \end{pmatrix}, \qquad x \in \mathbb{R}^d.$$

Since $\mathrm{Re}\langle \widetilde{V}(x)\xi, \xi\rangle = -|x|^r|\xi_2|^2 \le 0$ for every $x \in \mathbb{R}^d$ and every $\xi \in \mathbb{C}^3$, the matrix valued function $V = \widetilde{V} - 2I$ satisfies the assumption (3).

The matrix $\widetilde{V}(x)$ has eigenvalues $\pm|x|^r i$ and $-|x|^r$; hence there exists a matrix P (independent of x) such that

$$-V(x) - I = P^{-1}\begin{pmatrix} i|x|^r + 1 & 0 & 0 \\ 0 & -i|x|^r + 1 & 0 \\ 0 & 0 & |x|^r + 1 \end{pmatrix} P$$

Then, for every $\gamma > 0$

$$\begin{aligned} \|(-V(x) - I)^{-\gamma}\| &= \left\| P^{-1}\begin{pmatrix} i|x|^r + 1 & 0 & 0 \\ 0 & -i|x|^r + 1 & 0 \\ 0 & 0 & |x|^r + 1 \end{pmatrix}^{-\gamma} P \right\| \\ &\le \widetilde{C}(|x|^r + 1)^{-\gamma} \end{aligned}$$

for every $x \in \mathbb{R}^d$ and some positive constant $\widetilde{C}$. Since

$$D_jV(x) = r|x|^{r-2}x_j \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

choosing $\gamma \in]\frac{r-1}{r}, \frac{1}{2}[$ we get that $D_jV(x)(-V(x) - I)^{-\gamma}$ is uniformly bounded. Hence V satisfies also the assumption (4). Moreover,

$$\begin{aligned} \mathrm{Re}\langle V(x)\xi, W(x)\xi\rangle &= \mathrm{Re}\left\langle \begin{pmatrix} 0 & 0 & \alpha(x)|x|^r \\ 0 & -\beta(x)|x|^r & 0 \\ -\alpha(x)|x|^r & 0 & 0 \end{pmatrix}\xi, \xi \right\rangle - 2\langle W\xi, \xi\rangle \\ &\le -\beta(x)|x|^r|\xi_2|^2 \le 0 \end{aligned}$$

for every $x \in \mathbb{R}^d$. Hence, by Theorem 2 and Corollary 1, for any $s > 0$, the operator L_s, endowed with the domain $D(L)$, generates a strongly continuous semigroup of contractions in $L^2(\mathbb{R}^d, \mathbb{C}^3)$.

It should be observed that, if $|x|^r = O(\alpha(x))$ and $|x|^r = O(\beta(x))$, then $Wu \in L^2(\mathbb{R}^d, \mathbb{C}^3)$ implies that also $Vu \in L^2(\mathbb{R}^d, \mathbb{C}^3)$. Hence, in this case, the domain of L_s is $D(L) = \{W^{2,2}(\mathbb{R}^d, \mathbb{C}^3) : Wu \in L^2(\mathbb{R}^d, \mathbb{C}^3)\}$.

Moreover, since $|V(x)\xi| \geq \sqrt{|x|^{2r} + 4}|\xi|$ for any $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^3$, Proposition 2 can be applied to deduce that, for any $s > 0$, the operator $\Delta + V - sW$ has compact resolvent in $L^2(\mathbb{R}^d, \mathbb{C}^3)$.

Further, if $\alpha(x) \geq C_0|x|^r$ for any $x \in \mathbb{R}^d$ and some positive constant C_0, then condition (9) is satisfied for any $s > 0$ and $M = 2C_0^{-1}$. Indeed,

$$\mathrm{Re}\langle W(x)\xi, \xi\rangle = \alpha(x)(|\xi_1|^2 + |\xi_3|^2) + \beta(x)|\xi_2|^2$$

whereas

$$\begin{aligned}|\mathrm{Im}\langle (V(x) - sW(x))\xi, \xi\rangle| =& s\,|\mathrm{Im}\langle V(x)\xi, \xi\rangle| \\ =& 2s|x|^r|\mathrm{Im}(\xi_3)\mathrm{Re}(\xi_1) - \mathrm{Im}(\xi_1)\mathrm{Re}(\xi_3)|\end{aligned}$$

for any $x \in \mathbb{R}^d$ and any $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{C}^3$. Recalling that $\mathrm{Re}\langle -V(x)\xi, \xi\rangle = |x|^r|\xi_2|^2 + 2|\xi|^2$, we infer that, for any $x \in \mathbb{R}^d$, $\xi \in \mathbb{C}^3$ and $s > 0$,

$$\begin{aligned}|\mathrm{Im}\langle (V(x) - sW(x))\xi, \xi\rangle| &\leq 4s|x|^r|\xi_1||\xi_3| \leq 2s|x|^r(|\xi_1|^2 + |\xi_3|^2) \\ &\leq 2sC_0^{-1}\alpha(x)(|\xi_1|^2 + |\xi_3|^2) \\ &\leq 2C_0^{-1}\mathrm{Re}\langle (-V(x) + sW(x))\xi, \xi\rangle,\end{aligned}$$

as claimed. Hence, for any $s > 0$, the operator $-(\Delta + V - sW)$ is sectorial in $L^2(\mathbb{R}^d, \mathbb{C}^3)$ of angle less than $\pi/2$, then $\Delta + V - sW$ generates an analytic semigroup in $L^2(\mathbb{R}^d, \mathbb{C}^3)$. Finally, if the function $\min\{\alpha, \beta\}$ blows up at infinity faster than quadratically, then we can apply Corollary 2 and perturb the operator $A + v - sW$ by a drift term of the type $\sum_{j=1}^d F_j D_j$, where the entries of the matrix-valued functions F_j grow at most linearly at infinity. Indeed, in this case $\alpha(x) \geq C_0|x|^r$ for any $x \in \mathbb{R}^d$ and some positive constant C_0 and also assumption (10) is satisfied with $\phi(x) \simeq \min\{\alpha(x), \beta(x)\}$ as $|x| \to \infty$ since, it is easy to show that

$$|(V(x) - sW(x))\xi|^2 \geq \min\{|x|^{2r} + (2 + s\alpha(x))^2, (|x|^r + 2 + s\beta(x))^2\}|\xi|^2,$$

for any $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^3$.

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