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Derivation subalgebras of Lie algebras

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Abstract. Let L be a Lie algebra and I, J be two ideals of L. If $\mathrm{Der}^I_J(L)$ denotes the set of all derivations of L whose images are in I and send J to zero, then we give necessary and sufficient conditions under which $\mathrm{Der}^I_J(L)$ is equal to some special subalgebras of the derivation algebra of L. We also consider finite dimensional Lie algebra for which the center of the set of inner derivations, $Z(\mathrm{IDer}(L))$, is equal to the set of central derivations of L, $\mathrm{Der}_z(L)$, and give a characterisation of such Lie algebras.

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Introduction

Let L be a Lie algebra over an arbitrary field F. Let L^2 and $Z(L)$ denote the derived algebra and the center of L, respectively. A *derivation* of L is an F-linear transformation $\alpha : L \to L$ such that $\alpha([x, y]) = [\alpha(x), y] + [x, \alpha(y)]$ for all $x, y \in L$. We denote by $\mathrm{Der}(L)$ the vector space of all derivations of L, which itself forms a Lie algebra with respect to the commutator of linear transformations, called the *derivation algebra* of L. For all $x \in L$, the map $\mathrm{ad}_x : L \to L$ given by $y \to [x, y]$ is a derivation called the *inner derivation* corresponding to x. Clearly, the space $\mathrm{IDer}(L) = \{\mathrm{ad}_x \mid x \in L\}$ of inner derivations is an ideal of $\mathrm{Der}(L)$.

Let I and J be two ideals of L. By $\mathrm{Der}^I(L)$ we mean the subalgebra of $\mathrm{Der}(L)$ consisting of all derivations whose images are in I and by $\mathrm{Der}_J(L)$, we mean the subalgebra of $\mathrm{Der}(L)$ consisting of all derivations mapping J onto 0. The subalgebra $\mathrm{Der}^I(L) \cap \mathrm{Der}_J(L)$ is denoted by $\mathrm{Der}^I_J(L)$.

The relationships between the structure of L and $\mathrm{Der}(L)$ is studied by various authors in [1, 4, 6, 7, 8, 16, 19, 20]. Finding necessary and sufficient conditions under which subalgebras of $\mathrm{Der}(L)$ coincide seems to be an interesting problem. There are some results in this regard for a Lie algebra L. Leger [7] studied the equality $\mathrm{Der}(L) = \mathrm{IDer}(L)$. Tôgô [21], proved that if L is a Lie algebra over a field of characteristic zero such that $Z(L) \neq 0$, then

(i) $\mathrm{IDer}(L) = \mathrm{Der}_z(L)$ if and only if L is a Heisenberg algebra ($L^2 = Z(L)$ and $\dim Z(L) = 1$), and

(ii) $\mathrm{Der}(L) = \mathrm{Der}_z(L)$ if and only if L is abelian,

where $\mathrm{Der}_z(L)$ is the subalgebra of $\mathrm{Der}(L)$ consisting of all central derivations of L, that is, the set of all derivations of L mapping L into the center of L. It is easy to see that every element of $\mathrm{Der}_z(L)$ sends the derived algebra of L to 0. In other word, $\mathrm{Der}_z(L) = \mathrm{Der}_{L^2}^{Z(L)}(L)$.

Also, there are many papers about the study of complete Lie algebras. A Lie algebra is said to be *complete*, if its center is zero and all its derivations are inner. The definition of complete Lie algebras was introduced by Jacobson in [5] and later studied in [7, 12, 9, 13, 10, 11, 14, 15, 17, 18, 22].

The aim of this paper is to investigate the equalities $\mathrm{Der}_J^I(L) = Z(\mathrm{IDer}(L))$ and $\mathrm{Der}_J^I(L) = \mathrm{IDer}(L)$ in some special cases. As consequences of our main result, we obtain the result of Tôgô in [21] for finite dimensional Lie algebras and we give a characterisation of Lie algebras satisfying $Z(\mathrm{IDer}(L)) = \mathrm{Der}_z(L)$. Notice that for any Lie algebra L, the center of the set of inner derivations, $Z(\mathrm{IDer}(L))$, is always contained in $\mathrm{Der}_z(L)$.

In Section 3, we study the relationships between $\mathrm{Der}_J^I(L)$ and I, J up to isomorphism for some ideals of L and give a necessary and sufficient condition under which $\mathrm{Der}_{J_1}^{I_1}(L) = \mathrm{Der}_{J_2}^{I_2}(L)$. In Section 4, we discuss the equalities $Z(\mathrm{IDer}(L)) = \mathrm{Der}_J^I(L)$ and $\mathrm{IDer}(L) = \mathrm{Der}_J^I(L)$ in some special cases (Main Theorem and Corollaries). In Section 5, we will characterize all nilpotent Lie algebras of class 2 and naturally graded quasi-filiform Lie algebras satisfying $Z(\mathrm{IDer}(L)) = \mathrm{Der}_z(L)$. We also show that for filiform Lie algebras the above mentioned equality does not hold.

1 Preliminaries

Let A, B be two Lie algebras over a field F and $T(A, B)$ be the set of all linear transformations from A to B. Clearly, if B is an abelian Lie algebra, then $T(A, B)$ equipped with Lie bracket $[f, g](x) = [f(x), g(x)]$ for all $x \in A$ and $f, g \in T(A, B)$ is an abelian Lie algebra. Notice that in this equation the first Lie bracket is taken in $T(A, B)$ and the second Lie bracket is taken in B.

Given a Lie algebra L, the *lower central series* of L is defined as follows:

$$L = \gamma_1(L) \supseteq \gamma_2(L) \supseteq \cdots \supseteq \gamma_n(L) \supseteq \cdots ,$$

where $\gamma_2(L) = L^2$ is the derived algebra of L and $\gamma_n(L) = [\gamma_{n-1}(L), L]$.

Also, the *upper central series* of L is defined as

$$\{0\} = Z_0(L) \subseteq Z_1(L) \subseteq \cdots Z_n(L) \subseteq \cdots ,$$

where $Z_1(L) = Z(L)$ is the center of L and $Z_{n+1}(L)/Z_n(L) = Z(L/Z_n(L))$.

A Lie algebra L is *nilpotent* if there exists a non-negative integer k such that $\gamma_k(L) = 0$. The smallest integer k for which $\gamma_{k+1}(L) = 0$ is called the *nilpotency class* of L. A Lie algebra L of dimension n is *filiform* if $\dim \gamma_i(L) = n - i$ for all $2 \leq i \leq n$. These algebras have maximal nilpotency class $n - 1$. The nilpotent Lie algebras of class $n - 2$ are called *quasi-filiform* and those whose nilpotency class is 1 are abelian.

Let L be a nilpotent Lie algebra of class k over the field of complex numbers $\mathbb{C}$. Put $S_i = L$, if $i \leq 1$; $S_i = \gamma_i(L)$, if $2 \leq i \leq k$; and $S_i = \{0\}$, if $i > k$. Then, with L we can associate naturally a graded Lie algebra with the same nilpotency class, noted by gr$L$ and defined by $\mathrm{gr}L = \oplus_{i\in\mathbb{Z}} L_i$, where $L_i = \dfrac{S_i}{S_{i+1}}$. Because of the nilpotency of the algebra, the above gradation is finite, that is $\mathrm{gr}L = L_1 \oplus L_2 \oplus ... \oplus L_k$ with $[L_i, L_j] \subset L_{i+j}$, for $i + j \leq k$, $\dim L_1 \geq 2$ and $\dim L_i \geq 1$, for all $2 \leq i \leq k$. A Lie algebra L is said to be *naturally graded* if gr$L$ is isomorphic to L, denoted by $\mathrm{gr}L = L$.

The following theorem of Gómez and Jiménez-Merchán [2] gives a classification of naturally graded quasi-filiform Lie algebras of dimension n.

Theorem 1. *Every naturally graded quasi-filiform Lie algebra of dimension n over the field of complex numbers $\mathbb{C}$ is isomorphic to one of the following algebras:*

(i) If n is even to $L_{n-1} \oplus \mathbb{C}$, $\mathcal{T}_{(n,n-3)}$ or $\mathcal{L}_{(n,r)}$, with r odd and $3 \leq r \leq n-3$.

(ii) If n is odd to $L_{n-1} \oplus \mathbb{C}$, $Q_{n-1} \oplus \mathbb{C}$, $\mathcal{L}_{(n,n-2)}$, $\mathcal{T}_{(n,n-4)}$, $\mathcal{L}_{(n,r)}$, or $\mathcal{Q}_{(n,r)}$, with r odd and $3 \leq r \leq n-4$. In the cases of $n = 7$ and $n = 9$, we add algebras $\varepsilon_{(7,3)}$, $\varepsilon^1_{(9,5)}$ and $\varepsilon^2_{(9,5)}$.

Naturally graded quasi-filiform Lie algebras in Theorem 1 are defined in a basis $(X_0, X_1, ..., X_{n-1})$ as follows.

Split:
$L_{n-1} \oplus \mathbb{C}$ $(n \geq 4)$:
$[X_0, X_i] = X_{i+1},\ 1 \leq i \leq n-3$
$Q_{n-1} \oplus \mathbb{C}$ $(n \geq 7,\ n$ odd$)$:
$[X_0, X_i] = X_{i+1},\ 1 \leq i \leq n-3$
$[X_i, X_{n-2-i}] = (-1)^{i-1} X_{n-2},\ 1 \leq i \leq \frac{n-3}{2}$

Principal:
$\mathcal{L}_{(n,r)}$ $(n \geq 5, r$ odd, $3 \leq r \leq 2\lfloor \frac{n-1}{2} \rfloor - 1)$:
$[X_0, X_i] = X_{i+1},\ 1 \leq i \leq n-3$
$[X_i, X_{r-i}] = (-1)^{i-1} X_{n-1},\ 1 \leq i \leq \frac{r-1}{2}$
$\mathcal{Q}_{(n,r)}$ $(n \geq 7, n$ odd, r odd, $3 \leq r \leq n-4)$:
$[X_0, X_i] = X_{i+1},\ 1 \leq i \leq n-3$
$[X_i, X_{r-i}] = (-1)^{i-1} X_{n-1},\ 1 \leq i \leq \frac{r-1}{2}$
$[X_i, X_{n-2-i}] = (-1)^{i-1} X_{n-2},\ 1 \leq i \leq \frac{n-3}{2}$

Terminal:
$\mathcal{T}_{(n,n-3)}$ $(n$ even, $n \geq 6)$:
$[X_0, X_i] = X_{i+1},\ 1 \leq i \leq n-3$
$[X_{n-1}, X_1] = \frac{n-4}{2} X_{n-2}$
$[X_i, X_{n-3-i}] = (-1)^{i-1}(X_{n-3} + X_{n-1}),$
$1 \leq i \leq \frac{n-4}{2}$
$[X_i, X_{n-2-i}] = (-1)^{i-1} \frac{n-2-2i}{2} X_{n-2},$
$1 \leq i \leq \frac{n-4}{2}$

$\mathcal{T}_{(n,n-4)}$ $(n$ odd, $n \geq 7)$:
$[X_0, X_i] = X_{i+1},\ 1 \leq i \leq n-3$
$[X_{n-1}, X_i] = \frac{n-5}{2} X_{n-4+i}\ 1 \leq i \leq 2$
$[X_i, X_{n-4-i}] = (-1)^{i-1}(X_{n-4} + X_{n-1}),$
$1 \leq i \leq \frac{n-5}{2}$
$[X_i, X_{n-3-i}] = (-1)^{i-1} \frac{n-3-2i}{2} X_{n-3},$
$1 \leq i \leq \frac{n-5}{2}$
$[X_i, X_{n-2-i}] = (-1)^{i}(i-1) \frac{n-3-i}{2} X_{n-2},$
$2 \leq i \leq \frac{n-3}{2}$

$\varepsilon^1_{(9,5)}$:
$[X_0, X_i] = X_{i+1}, 1 \leq i \leq 6$
$[X_8, X_i] = 2X_{5+i}, 1 \leq i \leq 2$
$[X_1, X_4] = X_5 + X_8$
$[X_1, X_5] = 2X_6$
$[X_1, X_6] = 3X_7$
$[X_2, X_3] = -X_5 - X_8$
$[X_2, X_4] = -X_6$
$[X_2, X_5] = -X_7$

$\varepsilon^2_{(9,5)}$:
$[X_0, X_i] = X_{i+1}, 1 \leq i \leq 6$
$[X_8, X_i] = 2X_{5+i}, 1 \leq i \leq 2$
$[X_1, X_4] = X_5 + X_8$
$[X_1, X_5] = 2X_6$
$[X_1, X_6] = X_7$
$[X_2, X_3] = -X_5 - X_8$
$[X_2, X_4] = -X_6$
$[X_2, X_5] = X_7$
$[X_3, X_4] = -2X_7$

$\varepsilon_{(7,3)}$:
$[X_0, X_i] = X_{i+1}, 1 \leq i \leq 4$
$[X_6, X_i] = X_{3+i}, 1 \leq i \leq 2$
$[X_1, X_2] = X_3 + X_6$
$[X_1, X_i] = X_{i+1}, 3 \leq i \leq 4$

2 Special subalgebras of L and $\mathrm{Der}(L)$

Let L be a Lie algebra over a field F. Clearly, $Z(L) = \bigcap_{\alpha \in \mathrm{IDer}(L)} \mathrm{Ker}\alpha$ and $L^2 = \sum_{\alpha \in \mathrm{IDer}(L)} \mathrm{Im}\alpha$. The following lemma is useful for the proof of our main results.

Lemma 1. *Let I, J be two ideals of Lie algebra L such that $I \subseteq Z(L)$.*

Then

(i) $\mathrm{Der}_J^I(L) \cong T(L/(L^2+J), I)$ *as vector spaces, and*

(ii) if $I \subseteq J$*, then the above isomorphism turns into an isomorphism of Lie algebras, in particular* $\mathrm{Der}_J^I(L)$ *is abelian.*

Proof. For any $\alpha \in \mathrm{Der}_J^I(L)$, the map $\psi_\alpha : L/L^2+J \to I$ defined by $\psi_\alpha(x+L^2+J) = \alpha(x)$ for all $x \in L$ is a linear transformation. It is easy to see that the map $\psi : \mathrm{Der}_J^I(L) \to T(L/(L^2+J), I)$ defined by $\psi(\alpha) = \psi_\alpha$ is a one-to-one and onto linear transformation. Moreover, if $I \subseteq J$, then ψ is a Lie isomorphism, from which the result follows. QED

For each Lie algebra L and ideal J of L, let $C_L(J) = \{x \in L \mid [x,y] = 0 \ \forall y \in J\}$ denote the centralizer of J in L.

Let $\mathcal{D}$ be a subalgebra of $\mathrm{Der}(L)$ containing $\mathrm{IDer}(L)$. Then $E(L) = \bigcap_{\alpha\in\mathcal{D}} \mathrm{Ker}\alpha$ and $U(L) = \sum_{\alpha \in C} \mathrm{Im}\alpha$, where $C = C_{\mathcal{D}}\left(\mathrm{Der}^{E(L)}(L)\right)$, are ideals of L such that $E(L) \subseteq Z(L)$ and $L^2 \subseteq U(L)$. It is easy to see that

$$C = C_{\mathcal{D}}\left(\mathrm{Der}^{E(L)}(L)\right) = \left\{\alpha \in \mathcal{D} : \beta \circ \alpha = 0, \forall \beta \in \mathrm{Der}^{E(L)}(L)\right\}.$$

In particular, we can let $\mathcal{D} = \mathrm{Der}(L)$ or $\mathcal{D} = \mathrm{Der}^{L^2}(L)$.

Now, if we let $\mathrm{Der}_z(L) = \mathrm{Der}^{Z(L)}(L)$, $\mathrm{Der}_e(L) = \mathrm{Der}^{E(L)}(L)$, then by invoking the previous lemma we obtain the following result.

Corollary 1. *Let L be a Lie algebra. Then*

(i) $\mathrm{Der}_z(L) \cong T(L/L^2, Z(L))$ *as vector spaces and if* $Z(L) \subseteq L^2$*, then* $\mathrm{Der}_z(L)$ *is isomorphic to the abelian Lie algebra* $T(L/L^2, Z(L))$.

(ii) $\mathrm{Der}_e(L) \cong T(L/U(L), E(L))$ *as vector spaces and if* $E(L) \subseteq U(L)$*, then* $\mathrm{Der}_e(L)$ *is isomorphic to the abelian Lie algebra* $T(L/U(L), E(L))$.

(iii) If $\mathcal{D} = \mathrm{Der}(L)$*, then* $\mathrm{Der}_e(L)$ *is isomorphic to the abelian Lie algebras* $T(L/(L^2+E(L)), E(L))$ *and* $T(L/(U(L)+E(L)), E(L))$.

(iv) $\mathrm{Der}_{Z(L)}^{Z(L)}(L)$ *is isomorphic to the abelian Lie algebra* $T(L/(L^2+Z(L)), Z(L))$.

Proof. In view of the previous lemma it is enough to show that every element of L^2 and $U(L)$ is sent to zero by every element of $\mathrm{Der}_z(L)$ and $\mathrm{Der}_e(L)$, respectively. Also, if $\mathcal{D} = \mathrm{Der}(L)$, then every element of $E(L)$ is sent to zero by every element of $\mathrm{Der}_e(L)$. QED

Corollary 2. *Let L be a finite dimensional Lie algebra. Let I_1, I_2, J_1 and J_2 be ideals of L such that $I_1 \subseteq I_2 \subseteq Z(L)$, $J_2 \subseteq J_1$. Then* $\mathrm{Der}_{J_1}^{I_1}(L) \subseteq \mathrm{Der}_{J_2}^{I_2}(L)$. *Also* $\mathrm{Der}_{J_1}^{I_1}(L) = \mathrm{Der}_{J_2}^{I_2}(L)$ *if and only if* $I_1 = I_2$, $J_1 = J_2$.

Proof. It is obvious that if $I_1 \subseteq I_2, J_2 \subseteq J_1$, then $\mathrm{Der}_{J_1}^{I_1}(L) \subseteq \mathrm{Der}_{J_2}^{I_2}(L)$. Now suppose that $\mathrm{Der}_{J_1}^{I_1}(L) = \mathrm{Der}_{J_2}^{I_2}(L)$. Then by Lemma 1,

$$\dim T(L/(L^2 + J_1), I_1) = \dim T(L/(L^2 + J_2), I_2).$$

If $I_1 \subset I_2$ or $J_2 \subset J_1$, then $\dim T(L/(L^2 + J_1), I_1) < \dim T(L/(L^2 + J_2), I_2)$. By this contradiction we have $I_1 = I_2, J_1 = J_2$. The converse is clear. QED

Corollary 3. *Let L be a finite dimensional Lie algebra and I, J be two ideals of L such that $I \subseteq Z(L)$. Then* $\mathrm{Der}_J^I(L) = \mathrm{Der}_z(L)$ *if and only if* $I = Z(L)$ *and* $J \subseteq L^2$.

Proof. Since $I \subseteq Z(L), \mathrm{Der}_J^I(L) = \mathrm{Der}_{J+L^2}^I(L)$. Then by Corollary 2, $\mathrm{Der}_{J+L^2}^I(L)$ $= \mathrm{Der}_{L^2}^{Z(L)}(L)$ if and only if $I = Z(L), J + L^2 = L^2$ or, equivalently, $I = Z(L)$ and $J \subseteq L^2$. QED

3 Main theorem

The following lemma is useful for our study of $\mathrm{Der}_J^I(L)$.

Lemma 2. *Let I, J be two ideals of Lie algebra L and $K/I = Z(L/I)$. Then*

$$\mathrm{Der}_J^I(L) \cap \mathrm{IDer}(L) \cong \frac{K \cap C_L(J)}{Z(L)}.$$

In particular, if $\mathrm{Der}_J^I(L) \subseteq \mathrm{IDer}(L)$, *then*

$$\mathrm{Der}_J^I(L) \cong \frac{K \cap C_L(J)}{Z(L)}.$$

Proof. For each $x \in K \cap C_L(J)$, we have $\mathrm{ad}_x \in \mathrm{Der}_J^I(L) \cap \mathrm{IDer}(L)$. Now, define the map

$$\begin{array}{rcl} \psi : K \cap C_L(J) & \longrightarrow & \mathrm{Der}_J^I(L) \cap \mathrm{IDer}(L) \\ x & \longmapsto & \mathrm{ad}_x. \end{array}$$

Clearly, ψ is a Lie epimorphism such that $\mathrm{Ker}\psi = Z(L)$, as required. QED

Now, we are in the position to state and prove the main result of this section.

Main Theorem. *Let I, J be two ideals of a non-abelian Lie algebra L such that $I \subseteq Z(L)$. If $Z(L/I) = K/I$, then* *(i)* $Z(\mathrm{IDer}(L)) \subseteq \mathrm{Der}_J^I(L)$ *if and only if* $K = Z_2(L) \subseteq C_L(J)$,

(ii) if $I \subseteq J$ *and* $\dim Z_2(L)/Z(L) < \infty$*, then* $Z(\mathrm{IDer}(L)) = \mathrm{Der}_J^I(L)$ *if and only if* $K = Z_2(L) \subseteq C_L(J)$ *and* $T(L/(L^2 + J), I) \cong Z_2(L)/Z(L)$*. In particular,* $\mathrm{IDer}(L) = \mathrm{Der}_J^I(L)$ *if and only if* $J \subseteq Z(L)$*,* $L^2 \subseteq I$ *and* $T(L/J, I) \cong L/Z(L)$*.*

Proof. (i) First suppose that $K = Z_2(L) \subseteq C_L(J)$. It is obvious that for each $x \in L$, we have that $x \in Z_2(L)$ if and only if $\mathrm{ad}_x \in Z(\mathrm{IDer}(L))$. Now, assume that $\mathrm{ad}_x \in Z(\mathrm{IDer}(L))$. Then

$$\mathrm{ad}_x(y) = [x, y] \in [Z_2(L), L] = [K, L] \subseteq I$$

for all $y \in L$. Also, $\mathrm{ad}_x(y) = [x, y] \in [Z_2(L), J] = 0$ for all $y \in J$, which implies that $\mathrm{ad}_x \in \mathrm{Der}_J^I(L)$.

Conversely, suppose that $Z(\mathrm{IDer}(L)) \subseteq \mathrm{Der}_J^I(L)$. Since $[K, L] \subseteq I \subseteq Z(L)$, it follows that $K \subseteq Z_2(L)$. On the other hand, $\mathrm{ad}_x \in \mathrm{Der}_J^I(L)$ for all $x \in Z_2(L)$, which implies that $[Z_2(L), L] \subseteq I$ and hence $Z_2(L) \subseteq K$. Thus $K = Z_2(L)$. Moreover, $[x, y] = 0$ for all $x \in Z_2(L)$ and $y \in J$, from which it follows that $[Z_2(L), J] = 0$, that is, $K = Z_2(L) \subseteq C_L(J)$.

(ii) Suppose $Z(\mathrm{IDer}(L)) = \mathrm{Der}_J^I(L)$. By Lemma 1,

$$T(L/(L^2 + J), I) \cong Z_2(L)/Z(L).$$

Conversely, we have

$$\mathrm{Der}_J^I(L) \cong T\left(L/(L^2 + J), I\right) \cong \frac{Z_2(L)}{Z(L)} \cong Z(\mathrm{IDer}(L)).$$

On the other hand, $K = Z_2(L) \subseteq C_L(J)$, hence $Z(\mathrm{IDer}(L)) \subseteq \mathrm{Der}_J^I(L)$. But $\dim Z_2(L)/Z(L) < \infty$ so that the equality holds.

Now, if $\mathrm{IDer}(L) = \mathrm{Der}_J^I(L)$, then $\mathrm{IDer}(L) \subseteq \mathrm{Der}^{Z(L)}(L)$, which implies that L is nilpotent of class 2. Thus $K = Z_2(L) = C_L(J) = L$. Hence $J \subseteq Z(L)$, $L^2 \subseteq I$ and $T(L/J, I) \cong L/Z(L)$. Conversely, if $J \subseteq Z(L)$ and $L^2 \subseteq I$, then $\mathrm{IDer}(L) \subseteq \mathrm{Der}_J^I(L)$ and $T(L/J, I) \cong L/Z(L)$, from which the equality holds. The proof is complete. QED

Corollary 4. *Let* I*,* J *be two ideals of a finitely generated non-abelian Lie algebra* L *with* $Z(L) \neq 0$ *such that* $I \subseteq Z(L) \subseteq J$*. Then* $\mathrm{Der}_J^I(L) = \mathrm{IDer}(L)$ *if and only if* L *is a finite dimensional nilpotent Lie algebra of class 2,* $J = Z(L), L^2 \subseteq I$ *and* $\dim I = 1$

Proof. First we prove that if $L^2 \subseteq I$, then $T(L/Z(L), I) \cong L/Z(L)$ if and only if $\dim I = 1$. Since L is finitely generated and $L^2 \subseteq I \subseteq Z(L)$, by Lemma l(i) in [3], L is a finite dimensional Lie algebra. Then it is obvious that $T(L/Z(L), I) \cong L/Z(L)$ if and only if $\dim I = 1$. Now by main theorem(ii) $\mathrm{Der}_J^I(L) = \mathrm{IDer}(L)$ if and only if $J \subseteq Z(L), L^2 \subseteq I$ and $T(L/J, I) \cong L/Z(L)$ or equivalently $J = Z(L), L^2 \subseteq I$ and $\dim I = 1$ $\boxed{QED}$

If we put $I = J = Z(L)$ in the above corollary, then we obtain the following result immediately.

Corollary 5. *Let L be a finitely generated Lie algebra with* $Z(L) \neq 0$. *Then* $\mathrm{Der}_{Z(L)}^{Z(L)}(L) = \mathrm{IDer}(L)$ *if and only if L is finite dimensional abelian, or L is finite dimensional nilpotent Lie algebra of class 2 and* $\dim Z(L) = 1$.

The following corollary is the result of Tôgô [21, Theorem 3] for Lie algebras of finite dimension.

Corollary 6. *Let L be a finite dimensional Lie algebra with* $Z(L) \neq 0$. *Then* $\mathrm{Der}_z(L) = \mathrm{IDer}(L)$ *if and only if* $L^2 = Z(L)$ *and* $\dim Z(L) = 1$, *in which case L is a Heisenberg algebra.*

Proof. Let $L^2 = Z(L)$, $\dim Z(L) = 1$. By putting $I = Z(L)$ and $J = L^2$ in the Corollary 4, we have $\mathrm{Der}_z(L) = \mathrm{IDer}(L)$.

Conversely suppose that $\mathrm{Der}_z(L) = \mathrm{IDer}(L)$. First we show that $L^2 = Z(L)$. If $\mathrm{Der}_z(L) = \mathrm{IDer}(L)$, then $L^2 \subseteq Z(L)$ and $\psi : \mathrm{Der}_z(L) \longrightarrow T(L/Z(L), L^2)$ defined by $\psi(\alpha) = \psi_\alpha$, where $\psi_\alpha(x + Z(L)) = \alpha(x)$ is a Lie isomorphism. Suppose on the contrary that $L^2 \subsetneq Z(L)$. Then $\dim \frac{L}{Z(L)} < \dim \frac{L}{L^2}$ and $\dim L^2 < \dim Z(L)$. Thus $\dim T(L/Z(L), L^2) < \dim T(L/L^2, Z(L))$, which is a contradiction. Now by Corollary 4, $\dim Z(L) = 1$. $\boxed{QED}$

4 Characterization

It is easy to see that a central derivation of a Lie algebra L commutes with every inner derivation. Also, for any Lie algebra L, the center of the Lie algebra of inner derivations is always contained in the set of central derivations of L, because $\mathrm{Der}_z(L) = C_{\mathrm{Der}(L)}(\mathrm{IDer}(L))$. In this section we consider finite dimensional Lie algebras for which this lower bound is attained, that is $\mathrm{Der}_z(L) = Z(\mathrm{IDer}(L))$, and give a characterisation of such Lie algebras.

Theorem 2. *Let L be a finite dimensional Lie algebra with* $Z(L) \neq 0$. *If* $\mathrm{Der}_z(L) = Z(\mathrm{IDer}(L))$, *then* $Z(L) \subseteq L^2$. *Also* $\mathrm{Der}_z(L) = Z(\mathrm{IDer}(L))$ *if and only if* $T(L/L^2, Z(L)) \cong Z_2(L)/Z(L)$.

Proof. Let $\mathrm{Der}_z(L) = Z(\mathrm{IDer}(L))$. Then $\mathrm{Der}_z(L) = \mathrm{Der}^{Z(L)}_{Z(L)}(L)$ and $\mathrm{Der}^{Z(L)}_{Z(L)}(L) \cong T(L/(L^2 + Z(L)), Z(L))$, by Corollary 1(iv). Therefore we have

$$\mathrm{Der}^{Z(L)}_{Z(L)}(L) \cong T(L/(L^2 + Z(L)), L^2 \cap Z(L)),$$

because $\psi_\alpha(x + L^2 + Z(L)) \in L^2 \cap Z(L)$ for all $x \in L$. Thus $\dim(L^2 \cap Z(L)) = \dim Z(L)$. This implies that $Z(L) \subseteq L^2$.

Now, if $\mathrm{Der}_z(L) = Z(\mathrm{IDer}(L))$, by putting $I = Z(L)$ and $J = L^2$ in the main theorem, $T(L/L^2, Z(L)) \cong \frac{Z_2(L)}{Z(L)}$.

Conversely, if $T(L/L^2, Z(L)) \cong \frac{Z_2(L)}{Z(L)}$, then since for any Lie algebra L, $Z(\mathrm{IDer}(L)) \subseteq \mathrm{Der}_z(L)$ and $\dim \mathrm{Der}_z(L) = \dim T(L/L^2, Z(L))$, we conclude that $\mathrm{Der}_z(L) = Z(\mathrm{IDer}(L))$. QED

Clearly, for any nontrivial abelian Lie algebra L, $Z(\mathrm{IDer}(L)) \subsetneq \mathrm{Der}_z(L)$.

Corollary 7. *Let L be a finite dimensional nilpotent Lie algebra of class 2. Then $\mathrm{Der}_z(L) = Z(\mathrm{IDer}(L))$ if and only if $L^2 = Z(L)$ and $\dim Z(L) = 1$, in which case L is a Heisenberg algebra.*

Proof. Let L be a finite dimensional nilpotent Lie algebra of class 2. Then $\mathrm{IDer}(L)$ is an abelian Lie algebra. Now, the result follows from Corollary 6. QED

Corollary 8. *Let L be a nilpotent Lie algebra of finite dimension such that dim$(Z_2(L)/Z(L)) = 1$. Then $Z(\mathrm{IDer}(L) \subsetneq \mathrm{Der}_z(L)$. In particular, this holds if L is a filiform Lie algebra.*

Proof. It is well known that $\dim(L/L^2) \geq 2$ and $\dim Z(L) \geq 1$ for a nilpotent Lie algebra. Thus $\dim T(L/L^2, Z(L)) \geq 2$, so can not be isomorphic to $Z_2(L)/Z(L)$. Hence the result follows from Theorem 2. QED

Corollary 9. *Let L be a naturally graded quasi-filiform Lie algebra of dimension n over the field of complex numbers $\mathbb{C}$. Then $\mathrm{Der}_z(L) = Z(\mathrm{IDer}(L))$ if and only if $L = \mathcal{T}_{(n,n-3)}$ (n even, $n \geq 6$)*

Proof. Let L be an n-dimensional naturally graded quasi-filiform Lie algebra and $\mathrm{Der}_z(L) = Z(\mathrm{IDer}(L))$. It is well known that $1 \leq \dim(Z_2(L)/Z(L)) \leq 2$ for any quasi-filiform Lie algebra. Therefore by Theorem 2, $\dim Z(\mathrm{IDer}(L)) = 2$ and hence $\dim(L/L^2) = 2$ and $\dim Z(L) = 1$.

If n is even, then by Theorem 1 L is isomorphic to one of the algebras $L_{n-1} \oplus \mathbb{C}$, $\mathcal{T}_{(n,n-3)}$, or $\mathcal{L}_{(n,r)}$, with r odd and $3 \leq r \leq n-3$. If $L = L_{n-1} \oplus \mathbb{C}$, then $\dim(L/L^2) = 3$, $\dim Z(L) = 2$. If $L = \mathcal{L}_{(n,r)}$, then $\dim(L/L^2) = \dim Z(L) = 2$.

Therefore equality does not hold. If $L = \mathcal{T}_{(n,n-3)}$, then $\dim(L/L^2) = 2$ and $\dim Z(L) = 1$. Since $\gamma_{n-3}(L) \subseteq Z_2(L)$ and $\dim \gamma_{n-3}(L) = 3$, $\dim Z_2(L) = 3$. Therefore $\dim Z(\mathrm{IDer}(L)) = 2$. This implies that $\mathrm{Der}_z(L) = Z(\mathrm{IDer}(L))$.

If n is odd, then by Theorem 1 L is isomorphic to one of the algebras $L_{n-1} \oplus \mathbb{C}$, $Q_{n-1} \oplus \mathbb{C}$, $\mathcal{L}_{(n,n-2)}$, $\mathcal{T}_{(n,n-4)}$, $\mathcal{L}_{(n,r)}$, or $\mathcal{Q}_{(n,r)}$, with r odd and $3 \leq r \leq n-4$. If $L = L_{n-1} \oplus \mathbb{C}$ or $L = Q_{n-1} \oplus \mathbb{C}$, then $\dim(L/L^2) = 3$ and $\dim Z(L) = 2$. If $L = \mathcal{L}_{(n,n-2)}$, $L = \mathcal{L}_{(n,r)}$ or $L = \mathcal{Q}_{(n,r)}$, then $\dim(L/L^2) = \dim Z(L) = 2$. If $L = \mathcal{T}_{(n,n-4)}$, then it is easy to see that $Z(\mathrm{IDer}(L)) = \langle ad_{X_{n-3}} \rangle$ and hence $\dim Z(\mathrm{IDer}(L)) = 1$. Therefore for these Lie algebras equality does not hold.

In the case of $n = 7$ and $n = 9$, we have algebras $\varepsilon_{(7,3)}$, $\varepsilon^1_{(9,5)}$ and $\varepsilon^2_{(9,5)}$. For these algebras, $\dim Z(\mathrm{IDer}(L)) = 1$ and so $Z(\mathrm{IDer}(L)) \subsetneq \mathrm{Der}_z(L)$. QED

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