Inequalities concerning the \((p,k)\)-gamma and \((p,k)\)-polygamma functions

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Abstract. In this paper, we establish several inequalities involving the \((p,k)\)-gamma and \((p,k)\)-polygamma functions. Among other tools, we employ the mean value theorem, the Hermite-Hadamard’s inequality, Petrovic’s inequality and the Holder’s inequality. Upon some parameter variations, we recover some known results as special cases of the established results.

Keywords: \((p,k)\)-gamma function, \((p,k)\)-digamma function, \((p,k)\)-polygamma functions, inequality

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1 Introduction

In [15], the authors introduced a two-parameter deformation of the classical gamma function which is called the \((p,k)\)-gamma function. It is defined for \(p \in \mathbb{N}, k > 0\) and \(x > 0\) as

\[
\Gamma_{p,k}(x) = \int_0^p t^{x-1} \left(1 - \frac{t^k}{pk}\right)^p \, dt,
\]

or

\[
\Gamma_{p,k}(x) = \frac{(p+1)!k^{p+1}(pk)^{x-1}}{x(x+k)(x+2k)\ldots(x+pk)},
\]

and satisfies the properties

\[
\Gamma_{p,k}(x + k) = \frac{p^k x}{x + pk + k} \Gamma_{p,k}(x), \tag{1}
\]

\[
\Gamma_{p,k}(k) = 1.
\]
Closely related to the \((p,k)\)-gamma function is the \((p,k)\)-digamma function which is defined as follows.

\[
\psi_{p,k}(x) = \frac{d}{dx} \ln \Gamma_{p,k}(x) = \frac{1}{k} \ln(pk) - \sum_{n=0}^{p} \frac{1}{nk + x} = \frac{1}{k} \ln(pk) - \int_{0}^{\infty} \frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}} e^{-xt} \, dt. \tag{2}
\]

By this definition, the functional equation (1) gives

\[
\psi_{p,k}(x + k) - \psi_{p,k}(x) = \frac{1}{x} - \frac{1}{x + pk + k}. \tag{4}
\]

Also, the \((p,k)\)-polygamma function is defined for \(v \in \mathbb{N}\) as

\[
\psi_{p,k}^{(v)}(x) = \frac{d^v}{dx^v} \psi_{p,k}(x) = (-1)^{v+1} v! \sum_{n=0}^{p} \frac{1}{(nk + x)^{v+1}} = (-1)^{v+1} \int_{0}^{\infty} \frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}} t^v e^{-xt} \, dt, \tag{5}
\]

where, \(\psi_{p,k}^{(0)}(x) \equiv \psi_{p,k}(x)\). By successive differentiations of (4), one obtains

\[
\psi_{p,k}^{(v)}(x + k) - \psi_{p,k}^{(v)}(x) = \frac{(-1)^{v} v!}{x^{v+1}} - \frac{(-1)^{v} v!}{(x + pk + k)^{v+1}} \tag{7}
\]

for \(v \in \mathbb{N}_0\), where \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\) and \(\mathbb{N} = \{1, 2, 3, \ldots\}\). Also, it can be deduced from (5) and (6) that:

(a) \(\psi_{p,k}(x)\) is increasing,

(b) \(\psi_{p,k}^{(v)}(x)\) is positive and decreasing if \(v \in \{2n + 1 : n \in \mathbb{N}_0\}\),

(c) \(\psi_{p,k}^{(v)}(x)\) is negative and increasing if \(v \in \{2n : n \in \mathbb{N}\}\).

Furthermore, the \((p,k)\)-gamma and \((p,k)\)-polygamma functions satisfy the limit relations

\[
\Gamma_{p,k}(x) \xrightarrow{p \to \infty} \Gamma_k(x) \quad \text{as} \quad k \to 1 \quad \Gamma_p(x) \xrightarrow{p \to \infty} \Gamma(x) \quad \text{as} \quad k \to 1
\]

and

\[
\psi_{p,k}^{(v)}(x) \xrightarrow{p \to \infty} \psi_{k}^{(v)}(x) \quad \text{as} \quad k \to 1 \quad \psi_{p}^{(v)}(x) \xrightarrow{p \to \infty} \psi_{k}^{(v)}(x)
\]
where, $\Gamma(x)$ and $\psi^{(v)}(x)$ are the ordinary gamma and polygamma functions; $\Gamma_p(x)$ and $\psi^{(v)}_p(x)$ are the $p$-gamma and $p$-polygamma functions (see [8],[9]); and $\Gamma_k(x)$ and $\psi^{(v)}_k(x)$ are the $k$-gamma and $k$-polygamma functions [5].

The $(p,k)$-gamma and $(p,k)$-polygamma functions have been studied in various ways. See for instance the recent works [6], [11], [12], [13], [14], [16], [17], [19] and [20].

In the present work, our goal is to establish some inequalities involving the $(p,k)$-gamma and $(p,k)$-polygamma functions. Among other tools, we shall make use of the mean value theorem, the Hermite-Hadamard’s inequality, Petrovic’s inequality and the Holder’s inequality. Upon some parameter variations, we recover some known results as special cases of the established results. We present our results in the following section.

2 Main Results

**Theorem 1.** Let $p \in \mathbb{N}$ and $k > 0$. Then for $0 \leq x \leq k$, the inequality

$$
\left( \frac{pk}{p+2} \right) e^{(x-k)\psi_{p,k}(2k)} \leq \Gamma_{p,k}(x+k) \leq \left( \frac{pk}{p+2} \right) e^{(x-k)\psi_{p,k}(x+k)},
$$

(8)

is satisfied. Equality holds if and only if $x = k$.

**Proof.** The situations where $x = 0$ and $x = k$ are obvious. Consider the function $\ln \Gamma_{p,k}(x+k)$ on the interval $(x,k)$. Then by the classical mean value theorem, there exists a $\lambda \in (x,k)$ such that

$$
\frac{\ln \Gamma_{p,k}(2k) - \ln \Gamma_{p,k}(x+k)}{k-x} = \psi_{p,k}(\lambda + k).
$$

Since $\psi_{p,k}(z)$ is increasing for all $z > 0$, we have

$$
\psi_{p,k}(x+k) < \frac{\ln \Gamma_{p,k}(2k) - \ln \Gamma_{p,k}(x+k)}{k-x} < \psi_{p,k}(2k),
$$

which implies that

$$(k-x)\psi_{p,k}(x+k) < \ln \left( \frac{pk}{p+2} \right) - \ln \Gamma_{p,k}(x+k) < (k-x)\psi_{p,k}(2k).$$

This further implies that

$$
\ln \left( \frac{pk}{p+2} \right) + (k-x)\psi_{p,k}(2k) < \ln \Gamma_{p,k}(x+k) < \ln \left( \frac{pk}{p+2} \right) + (k-x)\psi_{p,k}(x+k),
$$

and by taking exponents, we obtain inequality (8).
**Remark 1.** By letting $k = 1$ and $p \to \infty$ in Theorem 1, we obtain
\[
e^{(1-x)(1-\gamma)} \leq \Gamma(x + 1) \leq e^{(1-x)\psi(x+1)},
\]
as presented in [10] for $0 \leq x \leq 1$ where $\gamma$ is the Euler-Mascheroni constant.

**Theorem 2.** Let $p \in \mathbb{N}$ and $k > 0$. Then the inequality
\[
e^{\psi_{p,k}(k)} < [\Gamma_{p,k}(x + k)]^{\frac{1}{x}} < e^{\psi_{p,k}(x + k)},
\]holds for all $x > 0$.

*Proof.* Consider the function $\ln \Gamma_{p,k}(x)$ on the interval $(k, x + k)$. Then by the mean value theorem, there exists a $\lambda \in (k, x + k)$ such that
\[
\frac{\ln \Gamma_{p,k}(k + x) - \ln \Gamma_{p,k}(k)}{x} = \psi_{p,k}(\lambda).
\]
Similarly, since $\psi_{p,k}(z)$ is increasing for all $z > 0$, we have
\[
\psi_{p,k}(k) < \frac{\ln \Gamma_{p,k}(x + k)}{x} < \psi_{p,k}(x + k),
\]
which upon taking exponents, gives inequality (9). \[QED\]

By applying the Hermite-Hadamard inequality, we obtain a similar inequality as shown in the following theorem.

**Theorem 3.** Let $p \in \mathbb{N}$ and $k > 0$. Then the inequality
\[
e^{\frac{1}{2}(\psi_{p,k}(k) + \psi_{p,k}(x + k))} < [\Gamma_{p,k}(x + k)]^{\frac{1}{x}} < e^{\psi_{p,k}(\frac{x}{2} + k)},
\]holds for all $x > 0$.

*Proof.* It is seen from (5) and (6) that, the function $\psi_{p,k}(z)$ is concave for all $z > 0$. Then by applying the Hermite-Hadamard inequality on the interval $(k, x + k)$, we obtain
\[
\frac{1}{2}(\psi_{p,k}(k) + \psi_{p,k}(x + k)) \leq \frac{1}{x} \int_{k}^{x + k} \psi_{p,k}(z) \, dz < \psi_{p,k}(\frac{x}{2} + k)
\]
which gives
\[
\frac{1}{2}(\psi_{p,k}(k) + \psi_{p,k}(x + k)) < \ln [\ln \Gamma_{p,k}(x + k)]^{\frac{1}{x}} < \psi_{p,k}(\frac{x}{2} + k).
\]
Then, by taking exponents, we obtain inequality (10). \[QED\]
Remark 2. Clearly, $\psi_{p,k}(\frac{x}{2} + k) < \psi_{p,k}(x + k)$ and also, by the arithmetic-geometric mean inequality, we have

$$\frac{1}{2}(\psi_{p,k}(k) + \psi_{p,k}(x + k)) > \sqrt{\psi_{p,k}(k)\psi_{p,k}(x + k)} > \sqrt{(\psi_{p,k}(k))^2} = \psi_{p,k}(k).$$

Thus, (10) is sharper than (9).

Remark 3. By letting $k = 1$ and $p \to \infty$ in (9), we obtain

$$e^{-\gamma} < \Gamma(x + 1)^{\frac{1}{2}} < e^{\psi(x + 1)},$$

which is the same as what was established in Theorem 4.2 of [10].

Remark 4. Also, by letting $k = 1$ and $p \to \infty$ in (10), we obtain

$$e^{\frac{1}{2}(-\gamma + \psi(x + 1))} < \Gamma(x + 1)^{\frac{1}{2}} < e^{\psi(x + 1)}.$$

The following lemma is known in the literature as Petrovic’s inequality for convex functions (see [1]).

Lemma 1. Suppose that $f : I \subseteq [0, \infty) \to \mathbb{R}$ is a convex function. If $x_i \in I$ for $i = 1, 2, \ldots, n$ and $x_1 + x_2 + \cdots + x_n \in I$, then

$$f(x_1) + f(x_2) + \cdots + f(x_n) \leq f(x_1 + x_2 + \cdots + x_n) + (n - 1)f(0),$$

with equality if and only if $n = 1$ or $x_1 = x_2 = \cdots = x_n = 0$.

Theorem 4. Let $p \in \mathbb{N}$, $k > 0$ and $x_i > 0$ for $i = 1, 2, \ldots, n$. Then the inequality

$$\frac{\Gamma_{p,k}(x_1)\Gamma_{p,k}(x_2)\cdots\Gamma_{p,k}(x_n)}{\Gamma_{p,k}(x_1 + x_2 + \cdots + x_n)} \leq \frac{\left(\frac{\Gamma_{p,k}(x_1 + x_2 + \cdots + x_n)}{(x_1 + x_2 + \cdots + x_n)^{pk}}\right)}{\left(\frac{\Gamma_{p,k}(x_1)\Gamma_{p,k}(x_2)\cdots\Gamma_{p,k}(x_n)}{(x_1 + x_2 + \cdots + x_n)}\right)},$$

holds.

Proof. It has been shown in [15, Theorem 2.1] that the function $f(x) = \ln \Gamma_{p,k}(x + k)$ is convex on $I = (0, \infty)$. Now let $x_i \in (0, \infty)$ for $i = 1, 2, \ldots, n$. Then by Lemma 1, we obtain

$$\ln \Gamma_{p,k}(x_1 + k) + \ln \Gamma_{p,k}(x_2 + k) + \cdots + \ln \Gamma_{p,k}(x_n + k) \leq \ln \Gamma_{p,k}(x_1 + x_2 + \cdots + x_n + k),$$

since $f(0) = 0$. That is,

$$\Gamma_{p,k}(x_1 + k)\Gamma_{p,k}(x_2 + k)\cdots\Gamma_{p,k}(x_n + k) \leq \Gamma_{p,k}(x_1 + x_2 + \cdots + x_n + k).$$
Then by (1) we obtain
\[
\frac{(pk)^n(x_1,x_2\ldots x_n)}{(x_1 + pk + k)(x_2 + pk + k)\ldots(x_n + pk + k)} \Gamma_{p,k}(x_1)\Gamma_{p,k}(x_2)\ldots \Gamma_{p,k}(x_n)
\leq \frac{pk(x_1 + x_2 + \cdots + x_n)}{(x_1 + x_2 + \cdots + x_n) + pk} \Gamma_{p,k}(x_1 + x_2 + \cdots + x_n),
\]
which when rearranged, gives inequality (11). \[QED\]

**Remark 5.** By letting \(k = 1\) and \(p \to \infty\) in (11), we obtain
\[
\frac{\Gamma(x_1)\Gamma(x_2)\ldots\Gamma(x_n)}{\Gamma(x_1 + x_2 + \cdots + x_n)} \leq \frac{x_1 + x_2 + \cdots + x_n}{x_1 x_2 \ldots x_n}
\]
which gives an upper bound for the beta function of \(n\) variables [2].

**Remark 6.** In particular, let \(n = 2, x_1 = x\) and \(x_2 = y\) in (11). Then we obtain
\[
\frac{\Gamma_{p,k}(x)\Gamma_{p,k}(y)}{\Gamma_{p,k}(x + y)} \leq \frac{x + y}{xy} \cdot \frac{(x + pk + k)(y + pk + k)}{pk(x + y + pk + k)} \tag{12}
\]
which gives an upper bound for the \((p,k)\)-beta function. Moreover, if \(k = 1\) and \(p \to \infty\) in (12), the we obtain
\[
\frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} \leq \frac{x + y}{xy}, \tag{13}
\]
for \(x > 0\) and \(y > 0\).

**Theorem 5.** Let \(p \in \mathbb{N}\) and \(k > 0\). Then the inequality
\[
\frac{\Gamma_{p,k}(x)\Gamma_{p,k}(y)}{\Gamma_{p,k}(x + y)} \leq \left( \frac{1}{xy} \right)^\frac{k}{2} \left( \frac{x + pk + k}{pk} \cdot \frac{y + pk + k}{pk} \right)^\frac{k}{2} \tag{14}
\]
holds for \(x \geq k\) and \(y \geq k\), with equality if and only if \(x = y = k\).

**Proof.** Consider the \((p,k)\)-beta function, \(B_{p,k}(x,y) = \frac{\Gamma_{p,k}(x)\Gamma_{p,k}(y)}{\Gamma_{p,k}(x+y)}\) for \(x \geq k\) and \(y \geq k\). With no loss of generality, let \(y\) be fixed. Then logarithmic differentiation yields
\[
\frac{\partial}{\partial x} B_{p,k}(x,y) = B_{p,k}(x,y) \left( \psi_{p,k}(x) - \psi_{p,k}(x + y) \right) < 0.
\]
Thus, \(B_{p,k}(x,y)\) is decreasing in \(x\). Then for \(x \geq k\), we have
\[
\frac{\Gamma_{p,k}(x)\Gamma_{p,k}(y)}{\Gamma_{p,k}(x + y)} \leq \frac{\Gamma_{p,k}(k)\Gamma_{p,k}(y)}{\Gamma_{p,k}(k + y)} = \frac{\Gamma_{p,k}(y)}{\frac{pk}{y+pk+k}\Gamma_{p,k}(y)} = \frac{y + pk + k}{pk y}. \tag{15}
\]
Likewise, fixing \(x\) yields
\[
\frac{\Gamma_{p,k}(x)\Gamma_{p,k}(y)}{\Gamma_{p,k}(x+y)} \leq \frac{x+pk+k}{pkx}.
\] (16)

Now, (15) and (16) imply
\[
\frac{\Gamma_{p,k}(x)\Gamma_{p,k}(y)}{\Gamma_{p,k}(x+y)} \leq \left(\frac{x+pk+k}{pkx} \cdot \frac{y+pk+k}{pkx}\right)^\frac{1}{2},
\]
which completes the proof. 

\[\text{QED}\]

**Remark 7.** By letting \(k = 1\) and \(p \to \infty\) in (14), we obtain
\[
\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \leq \left(\frac{1}{xy}\right)^\frac{1}{2},
\] (17)
for \(x \geq 1\) and \(y \geq 1\). However, this is weaker than the main result of [7]. Additional information on inequalities of type (13) and (17) can be found in [3], [4], and the related references therein.

**Theorem 6.** Let \(p \in \mathbb{N}\), \(k > 0\), \(u \geq 0\), \(s \in \{2^n : n \in \mathbb{N}_0\}\) and \(r \in \{2^n + 1 : n \in \mathbb{N}_0\}\). Then the inequalities
\[
(k-u)\psi_{p,k}^{(s+1)}(x+k) \leq \psi_{p,k}^{(s)}(x+k) - \psi_{p,k}^{(s)}(x+u) \leq (k-u)\psi_{p,k}^{(s+1)}(x+u),
\] (18)
\[
(k-u)\psi_{p,k}^{(r+1)}(x+k) \leq \psi_{p,k}^{(r)}(x+k) - \psi_{p,k}^{(r)}(x+u) \leq (k-u)\psi_{p,k}^{(r+1)}(x+k),
\] (19)
are valid for all \(x > 0\). Equality holds if and only if \(u = k\).

**Proof.** The case where \(u = k\) is obvious. Now, let \(0 \leq u < k\) and for \(s \in \{2^n : n \in \mathbb{N}_0\}\), consider the function \(\psi_{p,k}^{(s)}(x)\) on the interval \((x+u, x+k)\). Then by the mean value theorem, there exist a \(\lambda \in (x+u, x+k)\) such that
\[
\frac{\psi_{p,k}^{(s)}(x+k) - \psi_{p,k}^{(s)}(x+u)}{k-u} = \psi_{p,k}^{(s+1)}(\lambda).
\]
Since \(\psi_{p,k}^{(s+1)}(z)\) is decreasing for all \(z > 0\), we obtain
\[
\psi_{p,k}^{(s+1)}(x+k) < \frac{\psi_{p,k}^{(s)}(x+k) - \psi_{p,k}^{(s)}(x+u)}{k-u} < \psi_{p,k}^{(s+1)}(x+u),
\]
which gives inequality (18). By the same procedure, the case for \(u > k\) yields the same result. Hence we omit the details.
Similarly, let $0 \leq u < k$ and for $r \in \{2n + 1 : n \in \mathbb{N}_0\}$, consider the function $\psi_{p,k}^{(r)}(x)$ on the interval $(x + u, x + k)$. The mean value theorem gives

$$\frac{\psi_{p,k}^{(r)}(x + k) - \psi_{p,k}^{(r)}(x + u)}{k - u} = \psi_{p,k}^{(r+1)}(\delta),$$

for $\delta \in (x + u, x + k)$. Since $\psi_{p,k}^{(r+1)}(z)$ is increasing for all $z > 0$, we have

$$\psi_{p,k}^{(r+1)}(x + u) < \frac{\psi_{p,k}^{(r)}(x + k) - \psi_{p,k}^{(r)}(x + u)}{k - u} < \psi_{p,k}^{(r+1)}(x + k),$$

which gives inequality (19). The case for $u > k$ gives the same result and so, we omit the details. \hfill \Box

**Corollary 1.** Let $p \in \mathbb{N}$, $k > 0$, $s \in \{2n : n \in \mathbb{N}_0\}$ and $r \in \{2n+1 : n \in \mathbb{N}_0\}$. Then the inequalities

$$k\psi_{p,k}^{(s+1)}(x + k) < \frac{s!}{x^{s+1}} - \frac{s!}{(x + pk + k)^{s+1}} < k\psi_{p,k}^{(s+1)}(x), \quad (20)$$

$$k\psi_{p,k}^{(r+1)}(x) < \frac{r!}{(x + pk + k)^{r+1}} - \frac{r!}{x^{r+1}} < k\psi_{p,k}^{(r+1)}(x + k), \quad (21)$$

are valid for all $x > 0$.

**Proof.** By letting $u = 0$ in Theorem 6, we obtain

$$k\psi_{p,k}^{(s+1)}(x + k) < \psi_{p,k}^{(s)}(x + k) - \psi_{p,k}^{(s)}(x) < k\psi_{p,k}^{(s+1)}(x), \quad (22)$$

$$k\psi_{p,k}^{(r+1)}(x) < \psi_{p,k}^{(r)}(x + k) - \psi_{p,k}^{(r)}(x) < k\psi_{p,k}^{(r+1)}(x + k). \quad (23)$$

Then by applying (7) to (22) and (23), we obtain respectively (20) and (21). \hfill \Box

**Remark 8.** By letting $k = 1$ and $p \to \infty$ in Theorem 6 and Corollary 1, we obtain the results of Theorem 5.4 and Corollary 5.5 of [10].

**Remark 9.** If $k = 1$, $s = 0$ and $r = 1$, then (20) and (21) reduces to

$$\psi_p^\prime(x + 1) < \frac{1}{x} - \frac{1}{x + p + 1} < \psi_p^\prime(x),$$

$$\psi_p^\prime(x) < \frac{1}{(x + p + 1)^2} - \frac{1}{x^2} < \psi_p^\prime(x + 1),$$

where, $\psi_p(x)$ is the $p$-digamma function.
Lemma 2. Let $p \in \mathbb{N}$, $k > 0$ and $r \in \{2n + 1 : n \in \mathbb{N}_0\}$. Then, the inequality
\[ \psi_{p,k}^{(r)}(x) \psi_{p,k}^{(r+2)}(x) - [\psi_{p,k}^{(r+1)}(x)]^2 \geq 0, \]
holds for $x > 0$.

Proof. See Corollary 2.3 of [15]. \[Q.E.D.\]

Lemma 3. Let $p \in \mathbb{N}$, $k > 0$ and $r \in \{2n + 1 : n \in \mathbb{N}_0\}$. Then, the function
\[ \psi_{p,k}^{(m+1)}(x) \psi_{p,k}^{(m)}(x) \]
is increasing on $(0, \infty)$.

Proof. Direct differentiation yields
\[ \left( \frac{\psi_{p,k}^{(r+1)}(x)}{\psi_{p,k}^{(r)}(x)} \right)' = \frac{\psi_{p,k}^{(r)}(x) \psi_{p,k}^{(r+2)}(x) - [\psi_{p,k}^{(r+1)}(x)]^2}{[\psi_{p,k}^{(r)}(x)]^2} \geq 0, \]
which follows directly from Lemma 2. \[Q.E.D.\]

Theorem 7. Let $p \in \mathbb{N}$, $k > 0$, $u \geq 0$ and $r \in \{2n + 1 : n \in \mathbb{N}_0\}$. Then the inequality
\[ \exp \left\{ (k - u) \frac{\psi_{p,k}^{(r+1)}(x + u)}{\psi_{p,k}^{(r)}(x + u)} \right\} \leq \frac{\psi_{p,k}^{(r)}(x + k)}{\psi_{p,k}^{(r)}(x + u)} \leq \exp \left\{ (k - u) \frac{\psi_{p,k}^{(r+1)}(x + k)}{\psi_{p,k}^{(r)}(x + k)} \right\}, \]
valid for all $x > 0$. Equality holds if and only if $u = k$.

Proof. Suppose that $0 \leq u < k$ and consider the function $\ln \psi_{p,k}^{(r)}(x)$ on the interval $(x + u, x + k)$. Then the mean value theorem yields
\[ \frac{\ln \psi_{p,k}^{(r)}(k + x) - \ln \psi_{p,k}^{(r)}(x + u)}{k - u} = \frac{\psi_{p,k}^{(r+1)}(c)}{\psi_{p,k}^{(r)}(c)}, \]
where, $c \in (x + u, x + k)$. Then, since $\frac{\psi_{p,k}^{(r+1)}(z)}{\psi_{p,k}^{(r)}(z)}$ is increasing for all $z > 0$, we have
\[ \frac{\psi_{p,k}^{(r+1)}(x + u)}{\psi_{p,k}^{(r)}(x + u)} < \frac{1}{k - u} \frac{\psi_{p,k}^{(r)}(x + k)}{\psi_{p,k}^{(r)}(x + u)} < \frac{\psi_{p,k}^{(r+1)}(x + k)}{\psi_{p,k}^{(r)}(x + k)}, \]
which upon taking exponents, gives inequality (24). The case where $u > k$ gives the same result by following a similar procedure. \[Q.E.D.\]
Remark 10. By letting \( k = 1 \) and \( p \to \infty \) in (24), we recover inequality (8.1) of [10]. However, we noticed that, the proof of inequality (8.2) of [10] has a drawback. The expression \( \ln \psi(2n)(x + 1) - \ln \psi(2n)(x + \lambda) \) is not defined since \( \psi(i)(x) < 0 \) for even values of \( i \).

Theorem 8. Let \( p \in \mathbb{N}, k > 0 \) and \( r \in \{2n + 1 : n \in \mathbb{N}_0\} \). Then the inequality

\[
\psi_{p,k}^{(r)}(x + y + z) - \psi_{p,k}^{(r)}(x + y)\psi_{p,k}^{(r)}(x + z) > 0,
\]

holds for positive real numbers \( x, y \) and \( z \).

Proof. For positive real numbers \( x, y \) and \( z \), let \( Q \) be defined as

\[
Q(x) = \frac{\psi_{p,k}^{(r)}(x + z)}{\psi_{p,k}^{(r)}(x)}. \]

Then

\[
Q'(x) = \psi_{p,k}^{(r+1)}(x + y + z) - \psi_{p,k}^{(r+1)}(x + y)\psi_{p,k}^{(r)}(x + z) > 0,
\]

since \( \frac{\psi_{p,k}^{(r+1)}(x)}{\psi_{p,k}^{(r)}(x)} \) is increasing. Hence \( Q(x) \) is increasing. Therefore, \( Q(x+y) > Q(x) \) which gives inequality (25). \( \square \)

Theorem 9. Let \( p \in \mathbb{N}, k > 0, r \in \{2n+1 : n \in \mathbb{N}_0\} \) and \( s \in \{2n : n \in \mathbb{N}_0\} \). Then the inequalities

\[
\psi_{p,k}^{(r)}(x + y) < \psi_{p,k}^{(r)}(x) + \psi_{p,k}^{(r)}(y), \tag{26}
\]

\[
\psi_{p,k}^{(s)}(x + y) > \psi_{p,k}^{(s)}(x) + \psi_{p,k}^{(s)}(y), \tag{27}
\]

holds for \( x > 0 \) and \( y > 0 \).

Proof. Let \( G \) be defined for \( x > 0, y > 0 \) and \( r \in \{2n + 1 : n \in \mathbb{N}_0\} \) as

\[
G(x, y) = \psi_{p,k}^{(r)}(x + y) - \psi_{p,k}^{(r)}(x) - \psi_{p,k}^{(r)}(y).
\]

Without loss of generality, let \( y \) be fixed. Then

\[
\frac{\partial}{\partial x} G(x, y) = \psi_{p,k}^{(r+1)}(x + y) - \psi_{p,k}^{(r+1)}(x) > 0,
\]
Inequalities concerning the \((p,k)\)-gamma function

since \(\psi^{(r)}_{p,k}(x)\) is increasing for \(v \in \{2n : n \in \mathbb{N}_0\}\). Thus, \(G(x, y)\) is increasing in \(x\). Furthermore, by using representation (6), we obtain

\[
\lim_{x \to \infty} G(x, y) = -\int_0^\infty \frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}} t^{r+1} e^{-yt} < 0.
\]

Therefore, \(G(x, y) < \lim_{x \to \infty} G(x, y) < 0\) which gives (26). The proof of (27) follows a similar procedure. Hence we omit the details.

\[\text{QED}\]

Remark 11. Theorem 9 generalizes Theorem 2.2 and Theorem 2.4 of [18].

**Theorem 10.** Let \(p \in \mathbb{N}, k > 0, r \in \{2n+1 : n \in \mathbb{N}_0\}, a > 1\) and \(\frac{1}{a} + \frac{1}{b} = 1\). Then the inequality

\[
\psi^{(r)}_{p,k}(x + y) \leq \left(\psi^{(r)}_{p,k}(x)\right)^{\frac{1}{a}} \left(\psi^{(r)}_{p,k}(y)\right)^{\frac{1}{b}} \tag{28}
\]

is valid for \(x > 0\) and \(y > 0\).

**Proof.** By utilizing the Hölder’s inequality for finite sums, we obtain

\[
\psi^{(r)}_{p,k}(x + y) = \sum_{n=0}^{p} \frac{r!}{(nk + x + y)^{r+1}}
\]

\[
= \sum_{n=0}^{p} \frac{(r!)^{\frac{1}{a}} (r!)^{\frac{1}{b}}}{(nk + x + y)^{\frac{r+1}{a+b}}} (nk + x + y)^{\frac{r+1}{a+b}}
\]

\[
\leq \sum_{n=0}^{p} \frac{(r!)^{\frac{1}{a}}}{(nk + x)^{\frac{r+1}{a}}} \frac{1}{(nk + y)^{\frac{r+1}{b}}}
\]

\[
\leq \left( \sum_{n=0}^{p} \frac{r!}{(nk + x)^{r+1}} \right)^{\frac{1}{a}} \left( \sum_{n=0}^{p} \frac{r!}{(nk + y)^{r+1}} \right)^{\frac{1}{b}}
\]

\[
= \left(\psi^{(r)}_{p,k}(x)\right)^{\frac{1}{a}} \left(\psi^{(r)}_{p,k}(y)\right)^{\frac{1}{b}},
\]

which completes the proof.

\[\text{QED}\]

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References