Note di Matematica
Note Mat. **38** (2018) no. 2, 21–33.
ISSN 1123-2536, e-ISSN 1590-0932
doi:10.1285/i15900932v38n2p21

Certain results on $N(k)$-paracontact metric manifolds

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Received: 23.2.2018; accepted: 12.7.2018.

Abstract. The purpose of this paper is to study conformally flat $N(k)$-paracontact metric manifolds. Also we study η-Ricci solitons on three-dimensional $N(k)$-paracontact metric manifolds. Finally, we study gradient Ricci solitons on three-dimensional contractible $N(k)$-paracontact metric manifolds.

Keywords: $N(k)$-paracontact, conformally flat, η-Ricci soliton, gradient Ricci soliton, para-Sasakian manifold.

MSC 2000 classification: primary 53B30, 53C15, 53C25, 53C50, secondary 53D10, 53D15

1 Introduction

In 1985, Kaneyuki and Williams [20] introduced the notion of paracontact geometry. After that many authors [1, 2, 13, 16] contribute to study paracontact geometry. A systematic study of paracontact metric manifolds was given by Zamkovoy [31]. More recently, Cappelletti-Montano et al. [10] introduced a new type of paracontact geometry, so-called paracontact metric (k, μ)-spaces, where k and μ are some real constants. Also Martin-Molina [23, 24] obtained some classification theorems on paracontact metric (k, μ)-spaces and constructed some examples (see also [5]).

The conformal curvature tensor C is invariant under conformal transformations and vanishes identically for three-dimensional manifolds. Using this fact several authors [14, 21, 22] studied various types of three-dimensional manifolds.

A Ricci soliton is a natural generalization of an Einstein metric [3]. In a pseudo-Riemannian manifold M a Ricci soliton is a triplet (g, V, λ), with g, a pseudo-Riemannian metric, V a vector field (called the potential vector field) and λ a real constant such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1}$$

where $\pounds_V$ is the Lie derivative with respect to V and S is the Ricci tensor of type $(0, 2)$. The Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero or positive, respectively. The compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t} g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. For details, we refer to Chow and Knopf [12].

If the complete vector field V is the gradient of a potential function $-f$, then g is said to be a gradient Ricci soliton and equation (1.2) takes the form

$$\text{Hess}f = S + \lambda g, \tag{2}$$

where Hess$f$ denotes the Hessian of a smooth function f on M and defined by Hess$f = \nabla\nabla f$.

A Ricci soliton on a compact manifold has constant curvature in dimension 2 (Hamilton [18]) and also in dimension 3 (Ivey [19]). It is well known [29] that a Ricci soliton on a compact manifold is a gradient Ricci soliton. Ricci solitons have been studied by several authors such as Sharma [7], Cho and Kimura [11], De and Mandal [15], Ghosh [17] and many others. A complete classification of Ricci solitons of non-reductive homogeneous 4-spaces was given by Calvaruso and Zaeim [7].

For a manifold M an η-Ricci soliton (g, V, λ, ν), with g, a pseudo-Riemannian metric is defined by [11]

$$\pounds_V g + 2S + 2\lambda g + 2\nu.\eta \otimes \eta = 0, \tag{3}$$

where $\pounds_V$ is the Lie derivative with respect to the potential vector filed V, S is the Ricci tensor of type $(0, 2)$, λ and ν are real scalars. In particular, if $\nu = 0$, then an η-Ricci soliton (g, V, λ, ν) reduces to a Ricci soliton (g, V, λ).

A Riemannian or, pseudo-Riemannian manifold of dimension n is said to be special manifold [28] with the associated symmetric tensor B, denoted by $(\psi B)_n$, if the $(0, 4)$-type tensor R' satisfies the condition

$$R'(X, Y, Z, W) = B(X, W)B(Y, Z) - B(Y, W)B(X, Z), \tag{4}$$

where $B(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$, for some constants a and b.

Now we recall some useful lemmas.

Lemma 1. *[14, Proposition 2.2] A three-dimensional $N(k)$-paracontact metric manifold is a manifold of constant curvature k if and only if the scalar curvature $r = 6k$.*

Lemma 2. *[28, Theorem 5.4] A three-dimensional para-Sasakian η-Ricci soliton $M^3(g, \xi, \lambda, \mu)$ is a $(\psi B)_3$ with associated symmetric tensor B given by*

$$B(X,Y) = \sqrt{\frac{(\mu - \lambda)}{2}}g(X,Y) - \mu\sqrt{\frac{2}{(\mu - \lambda)}}\eta(X)\eta(Y).$$

In [26], Olszak proved that any contact metric manifold of constant curvature and of dimension > 3 has constant curvature 1 and is Sasakian. Also it is known [25] that every Sasakian conformally flat manifold of dimension > 3 is of constant curvature 1.

In this circumstances, it is interesting to study conformally flat $N(k)$-paracontact manifolds of dimension > 3 and also for dimension three.

The present paper is organized as follows. In Section 2, we present some basic results of $N(k)$-paracontact metric manifolds. Section 3 is devoted to study conformally flat $N(k)$-paracontact metric manifolds and we prove that there does not exist any conformally flat $N(k)$-paracontact metric manifold M of dimension > 3. The next section deals with η-Ricci solitons on three-dimensional $N(k)$-paracontact metric manifolds. Finally, we study gradient Ricci solitons on three-dimensional contractible $N(k)$-paracontact metric manifolds.

2 Preliminaries

A smooth odd dimensional manifold M^n $(n > 1)$ is said to an almost paracontact manifold if it admits a $(1,1)$-type tensor field ϕ, a vector field ξ and a 1-form η satisfying the following conditions [20]
(i) $\phi^2 = I - \eta \otimes \xi$,
(ii) $\phi(\xi) = 0, \ \eta \circ \phi = 0, \ \eta(\xi) = 1$, and
(iii) on each fibre of $\mathcal{D} = \ker(\eta)$ the tensor field ϕ induces an almost paracomplex structure, that is, the eigendistributions $\mathcal{D}_\phi^+$ and $\mathcal{D}_\phi^-$ of ϕ corresponding to the respective eigenvalues 1 and -1 have same dimension n. Here ξ is called the Reeb vector field.
An almost paracontact manifold M is said to be an *almost paracontact metric manifold* if there is a pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{5}$$

for all $X, Y \in \chi(M)$, and (ϕ, ξ, η, g) is said to be an *almost paracontact metric structure.* Here the signature of g is necessarily $(n+1, n)$.

An almost paracontact structure is said to be a *paracontact* structure if $\Phi(X,Y) = d\eta(X,Y)$ [31], the fundamental 2-form is defined by $\Phi(X,Y) = g(X,\phi Y)$; almost paracontact structure is said to be *normal* [31] if the $(1,2)$-type torsion tensor $N_\phi = [\phi,\phi] - 2d\eta \otimes \xi = 0$, where $[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X,\phi Y]$.
Any almost paracontact metric manifold (M^n,ϕ,ξ,η,g) admits (at least, locally) a ϕ-basis [31], and thus in particular a three dimensional almost paracontact metric manifold, any (local) pseudo-orthonormal basis of $\ker(\eta)$ gives a ϕ-basis, up to sign. If $\{e_2, e_3\}$ is a (local) pseudo-orthonormal basis of $\ker(\eta)$, with e_3, time-like, so by (5) vector field $\phi e_2 \in \ker(\eta)$ is time-like and orthogonal to e_2. Therefore, $\phi e_2 = \pm e_3$ and $\{\xi, e_2, \pm e_3\}$ is a ϕ-basis [6]. For a paracontact metric manifold we can easily get a symmetric, trace-free $(1,1)$-tensor $h = \frac{1}{2}\pounds_\xi \phi$ satisfying the following formulas [9, 31]

$$\phi h + h\phi = 0, \quad h\xi = 0, \tag{6}$$

$$\nabla_X \xi = -\phi X + \phi h X, \tag{7}$$

for all $X \in \chi(M)$, where ∇ is the Levi-Civita connection of the pseudo-Riemannian manifold.

Furthermore, if ξ is a Killing vector field (or equivalently h vanishes identically) then the manifold (M^n,ϕ,ξ,η,g) is called a K-paracontact manifold.

The following definition follows from Cappelletti-Montano and Di Terlizzi [9]:

Definition 1. A paracontact metric manifold is said to be a paracontact (k,μ)-manifold if the curvature tensor R satisfies

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{8}$$

for all vector fields $X, Y \in \chi(M)$ and k, μ are real constants.

If $\mu = 0$, then the paracontact metric (k,μ)-manifold reduces to an $N(k)$-paracontact metric manifold and hence for an $N(k)$-paracontact metric manifold we get the following formula

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y), \tag{9}$$

for all vector fields $X, Y \in \chi(M)$ and k is a real constant. In an $N(k)$-paracontact metric manifold of dimension $(2n+1)$ the Ricci operator is given by

$$\begin{aligned} QY =& 2(1-n)Y + 2(n-1)hY \\ &+ [2(n-1) + 2nk]\eta(Y)\xi, \qquad \text{for } k \neq -1. \end{aligned} \tag{10}$$

Replacing Y by ξ in the above equation we have

$$Q\xi = 2nk\xi. \tag{11}$$

Making use of (11), we get

$$(\nabla_X \eta)Y = g(X, \phi Y) - g(hX, \phi Y). \tag{12}$$

The following results hold for an $N(k)$-paracontact metric manifold $(M^3, \phi, \xi, \eta, g)$ [4, 14, 27]

$$QX = \Big(\frac{r}{2} - k\Big)X + \Big(-\frac{r}{2} + 3k\Big)\eta(X)\xi, \quad k \neq -1, \tag{13}$$

$$S(X,Y) = \Big(\frac{r}{2} - k\Big)g(X,Y) + \Big(3k - \frac{r}{2}\Big)\eta(X)\eta(Y), \tag{14}$$

$$Q\xi = 2k\xi, \tag{15}$$

$$S(X,\xi) = 2k\eta(X), \tag{16}$$

$$\begin{aligned} R(X,Y)Z \;=\; & \Big(\frac{r}{2} - 2k\Big)\{g(Y,Z)X - g(X,Z)Y\} + \Big(-\frac{r}{2} + 3k\Big)\{\eta(Y)\eta(Z)X \\ & -\eta(X)\eta(Z)Y + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi\}, \end{aligned} \tag{17}$$

where X, Y, Z are any vector fields on M, Q denotes the Ricci operator of M, r is the scalar curvature of M. We have the following result due to Cappelletti-Montano et al. [10, p.686].

Lemma 3. *A paracontact metric (k, μ)-manifold of dimension three is Einstein if and only if $k = \mu = 0$.*

Though any paracontact metric (k, μ)-manifold of dimension three is Einstein if and only if $k = \mu = 0$, it always admits some compatible Einstein metrics [8].

3 Conformally flat $N(k)$-paracontact metric manifolds

In this section we study conformally flat $N(k)$-paracontact metric manifolds. A pseudo-Riemannian manifold M of dimension $(2n + 1)$ is conformally flat if and only if the Weyl conformal curvature tensor filed C defined by

$$\begin{aligned} C(X,Y)Z \;=\; & R(X,Y)Z - \frac{1}{2n-1}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ & -g(X,Z)QY\} + \frac{r}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{18}$$

vanishes for $n > 1$, and

$$(\nabla_X Q)Y - (\nabla_Y Q)X = \frac{1}{4}\{(Xr)Y - (Yr)X\} \tag{19}$$

for $n = 1$. The Weyl conformal curvature tensor field vanishes identically for $n = 1$.
Thus for a conformally flat manifolds M for $n > 1$, $C = 0$ implies

$$\begin{aligned} R(X,Y)Z &= \frac{1}{2n-1}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY\} - \frac{r}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{20}$$

Putting $Y = Z = \xi$ in (18) and using (15), (15) and (16), we get

$$\begin{aligned} R(X,\xi)\xi &= \frac{1}{2n-1}\{2nkX - 2nk\eta(X)\xi + QX - 2nk\eta(X)\xi\} \\ &\quad -\frac{r}{2n(2n-1)}\{X - \eta(X)\xi\}. \end{aligned} \tag{21}$$

Form the above equation, we get the following,

$$QX = \left\{\frac{r}{2n} - k\right\}X - \left\{\frac{r}{2n} - 2nk - k\right\}\eta(X)\xi. \tag{22}$$

Taking covariant derivative of the above equation along and arbitrary vector field Y, we obtain

$$\begin{aligned} (\nabla_Y Q)X &= \frac{Y(r)}{2n}X - \frac{Y(r)}{2n}\eta(X)\xi - \left\{\frac{r}{2n} - 2nk - k\right\}\{(\nabla_Y\eta)(X)\xi \\ &\quad +\eta(X)\nabla_Y\xi\}. \end{aligned} \tag{23}$$

Now using (12) in (21), we obtain

$$\begin{aligned} (\nabla_Y Q)X &= \frac{Y(r)}{2n}\{X - \eta(X)\xi\} - \left\{\frac{r}{2n} - 2nk - k\right\}\{g(\phi hY,X)\xi - g(\phi Y,X)\xi \\ &\quad -\eta(X)\phi Y + \eta(X)\phi hY\}. \end{aligned} \tag{24}$$

Interchanging X and Y in (22), we obtain

$$\begin{aligned} (\nabla_X Q)Y &= \frac{X(r)}{2n}\{Y - \eta(Y)\xi\} - \left\{\frac{r}{2n} - 2nk - k\right\}\{g(\phi hX,Y)\xi - g(\phi X,Y)\xi \\ &\quad -\eta(Y)\phi X + \eta(Y)\phi hX\}. \end{aligned} \tag{25}$$

Since $C = 0$, we have $divC = 0$ or equivalently,

$$(\nabla_X Q)Y - (\nabla_Y Q)X = \frac{1}{4n}\{(X(r))Y - (Y(r))X\}. \tag{26}$$

Thus from (22), (23) and (24), we have

$$\frac{X(r)}{2n}\{Y-\eta(Y)\xi\}-\frac{Y(r)}{2n}\{X-\eta(X)\xi\}-\Big\{\frac{r}{2n}-2nk-k\Big\}\{2g(X,\phi Y)\xi$$
$$-\eta(Y)\phi hX+\eta(X)\phi Y-\eta(X)\phi hY\}=\frac{1}{4n}\{(Xr)Y-(Yr)X\}. \tag{27}$$

Contracting Y in (25), we get

$$(n-1)X(r)+\xi r\eta(X)=0. \tag{28}$$

Substituting $X=\xi$ in the above equations yields,

$$\xi r=0. \tag{29}$$

Now using this in (26), we get $Xr=0$ and hence r is constant. Thus by (25), we have

$$\Big(\frac{r}{2n}-2nk-k\Big)\,2g(X,\phi Y)=0. \tag{30}$$

Making use of (28) in (20), we obtain

$$QX=2nkX, \qquad \text{for all } X\in\chi(M). \tag{31}$$

Taking trace of the above equation we have

$$r=2n(2n+1)k. \tag{32}$$

Applying (29) and (32) in (18), we finally get

$$R(X,Y)Z=k\{g(Y,Z)X-g(X,Z)Y\}, \tag{33}$$

for any vector fields X,Y and $Z\in\chi(M)$. But as a consequence of Corollary 4.14 and Corollary 5.12 of [10] we conclude that a $N(k)$-paracontact metric manifold of constant curvature can not exist for dimension >3. Thus we can state the following:

Theorem 1. *There does not exist any conformally flat $N(k)$-paracontact metric manifold M of dimension >3.*

Now we study conformally flat $N(k)$-paracontact metric manifolds for dimension three. Thus from (17) we have

$$(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)=\frac{1}{4}\{(Xr)g(Y,Z)-(Yr)g(X,Z)\}. \tag{34}$$

Taking the covariant derivative of (15) along an arbitrary vector field Y and using (11) and (12) we have

$$\begin{aligned}(\nabla_Y S)(X,Z) &= \frac{dr(Y)}{2}\{g(X,Z)-\eta(X)\eta(Z)\}+\Big(3k-\frac{r}{2}\Big)\{g(Y,\phi X)\eta(Z)\\ &\quad -g(hY,\phi X)\eta(Z)-\eta(X)g(\phi Y,Z)+\eta(X)g(\phi hY,Z)\}. \quad (35)\end{aligned}$$

Interchanging X and Y in (35) we infer that

$$\begin{aligned}(\nabla_X S)(Y,Z) &= \frac{dr(X)}{2}\{g(Y,Z)-\eta(Y)\eta(Z)\}+\Big(3k-\frac{r}{2}\Big)\{g(X,\phi Y)\eta(Z)\\ &\quad -g(hX,\phi Y)\eta(Z)-\eta(Y)g(\phi X,Z)+\eta(Y)g(\phi hX,Z)\}. \quad (36)\end{aligned}$$

Making use of (35) and (36) in (34) we have

$$\begin{aligned}&\frac{X(r)}{2}\{g(Y,Z)-\eta(Y)\eta(Z)\}-\frac{Y(r)}{2}\{g(X,Z)-\eta(X)\eta(Z)\}\\ &+\Big(3k-\frac{r}{2}\Big)\{2g(X,\phi Y)\eta(Z)-g(\phi X,Z)\eta(Y)\\ &+g(\phi hX,Z)\eta(Y)+g(\phi Y,Z)\eta(X)-g(\phi hY,Z)\eta(X)\}\\ &=\frac{1}{4}\{X(r)g(Y,Z)-Y(r)g(X,Z)\}. \quad (37)\end{aligned}$$

Substituting ξ for both X and Z in the above equation it follows that

$$Y(r)=0, \quad (38)$$

for all vector fields Y. Applying (38) in (37) we obtain

$$\begin{aligned}&\Big(3k-\frac{r}{2}\Big)\{2g(X,\phi Y)\eta(Z)-g(\phi X,Z)\eta(Y)\\ &+g(\phi hX,Z)\eta(Y)+g(\phi Y,Z)\eta(X)-g(\phi hY,Z)\eta(X)\}=0. \quad (39)\end{aligned}$$

Putting ϕY for Y in (39) yields

$$\begin{aligned}&\Big(3k-\frac{r}{2}\Big)\{2g(X,Y)\eta(Z)+g(Y,Z)\eta(X)\\ &+g(hY,Z)\eta(X)-3\eta(X)\eta(Y)\eta(Z)\}=0. \quad (40)\end{aligned}$$

Let $\{e_i\}$, $i=1,2,3$ be a ϕ-basis of the tangent space at each point of the manifold. Then substituting $Y=Z=e_i$ in (40) and taking summation over i, $1\le i\le 3$, we get

$$\Big(3k-\frac{r}{2}\Big)\eta(X)=0. \quad (41)$$

This gives $r=6k$ (since $\eta(X)$ does not vanish for all vector fields X), which implies by Lemma 1 that the manifold is of constant curvature k.
This leads to the following:

Theorem 2. *A conformally flat $N(k)$-paracontact metric manifold M of dimension three is of constant curvature k.*

4 η-Ricci solitons on $N(k)$-paracontact metric manifolds

This section is devoted to study η-Ricci solitons on three-dimensional $N(k)$-paracontact metric manifolds. Now the equation (31) implies that

$$(\pounds_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\nu\eta(X)\eta(Y) = 0. \tag{42}$$

For an $N(k)$-paracontact metric manifold, we have

$$(\pounds_\xi g)(X,Y) = 2g(\phi hX,Y). \tag{43}$$

Thus the equation (32) reduces to

$$S(X,Y) = -g(\phi hX,Y) - \lambda g(X,Y) - \nu\eta(X)\eta(Y). \tag{44}$$

Substituting ϕX for X in (34) we infer that

$$S(\phi X,Y) = g(hX,Y) - \lambda(\phi X,Y). \tag{45}$$

Replacing X by ϕX in (15) yields

$$S(\phi X,Y) = \left(\frac{r}{2} - k\right) g(\phi X,Y). \tag{46}$$

Equating the right hand sides of (45) and (46) we have

$$g(hX,Y) = \left(\frac{r}{2} - k + \lambda\right) g(\phi X,Y). \tag{47}$$

Interchanging X and Y in (47) we get

$$g(hY,X) = \left(\frac{r}{2} - k + \lambda\right) g(\phi Y,X). \tag{48}$$

Adding (47) and (48) we obtain $g(hX,Y) = 0$, that is, $h = 0$. Now we know that $h = 0$ holds if and only if ξ is a Killing vector field and consequently M is a K-paracontact metric manifold. For three-dimensional case, a K-paracontact metric manifold is a paraSasakian manifold. Thus M is a paraSasakian manifold. Using Lemma 2 we get the following result.

Theorem 3. *If a three-dimensional $N(k)$-paracontact metric manifold admits a η-Ricci soliton whose potential vector field is the Reeb vector field ξ, then the manifold is a $(\psi B)_3$ with associated symmetric tensor B given by*

$$B(X,Y) = \sqrt{\frac{(\nu-\lambda)}{2}}\, g(X,Y) - \nu\sqrt{\frac{2}{(\nu-\lambda)}}\,\eta(X)\eta(Y),$$

for all vector fields $X,Y \in \chi(M)$.

5 Gradient Ricci solitons on three-dimensional contractible $N(k)$-paracontact metric manifolds

In this section we study gradient Ricci solitons on three-dimensional contractible $N(k)$-paracontact metric manifolds. Let M be a three-dimensional contractible $N(k)$-paracontact metric manifold and g a gradient Ricci soliton. Then the equation (2) can be put in of the form

$$\nabla_Y Df = QY + \lambda Y, \tag{49}$$

for all vector fields $Y \in \chi(M)$, where D denotes the gradient operator of the pseudo-metric g.
From (49), we have

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X. \tag{50}$$

Making use of (14) in (50) we obtain

$$\begin{aligned} R(X,Y)Df &= \frac{dr(X)}{2}\{Y - \eta(Y)\xi\} - \frac{dr(Y)}{2}\{X - \eta(X)\xi\} \\ &\quad -\Big(\frac{r}{2} - 3k\Big)\{2g(X,\phi Y)\xi + \eta(X)\phi Y - \eta(Y)\phi X \\ &\quad +\eta(Y)\phi hX - \eta(X)\phi hY\}. \end{aligned} \tag{51}$$

Substituting ξ for X in (51) we infer that

$$R(\xi,Y)Df = \frac{dr(\xi)}{2}\{Y - \eta(Y)\xi\} - \Big(\frac{r}{2} - 3k\Big)\{\phi Y - \phi hY\}. \tag{52}$$

Taking inner product of (52) with ξ yields

$$g(R(\xi,Y)Df,\xi) = 0. \tag{53}$$

Again from (17) it follows that

$$g(R(\xi,Y)Z,\xi) = k\{g(Y,Z) - \eta(Y)\eta(Z)\}. \tag{54}$$

Thus from (53) and (54) we get

$$k\{g(Y,Z) - \eta(Y)\eta(Z)\} = 0. \tag{55}$$

`Case 1:` Let us suppose that $k = 0$. From Lemma 3 we conclude that the manifold is an Einstein manifold and hence, being three-dimensional, a space of

constant curvature.
`Case 2`: Let $g(Y,Z) - \eta(Y)\eta(Z) = 0$. From which it follows that

$$g(Y, Df) = \xi f \eta(Y),$$

that is,

$$Df = (\xi f)\xi. \tag{56}$$

Again using (11) and (56) we obtain

$$g(\nabla_Y Df, X) = Y(\xi f)\eta(X) - (\xi f)\{g(\phi X, Y) - g(\phi hY, X)\}. \tag{57}$$

Noticing (49) and (57) we infer that

$$S(X,Y) + \lambda g(X,Y) = Y(\xi f)\eta(X) - (\xi f)\{g(\phi X, Y) - g(\phi hY, X)\}. \tag{58}$$

Putting $X = \xi$ and using (15) yields

$$Y(\xi f) = (\lambda + 2k)\eta(Y). \tag{59}$$

Therefore we have from (59) and (57)

$$g(\nabla_Y Df, X) = (\lambda + 2k)\eta(X)\eta(Y) - (\xi f)\{g(\phi X, Y) - g(\phi hY, X)\}. \tag{60}$$

Applying Poincaré's lemma: On a contractible manifold, all closed forms are exact. Therefore $d^2 f(X,Y) = 0$, for all $X, Y \in \chi(M)$. Thus we have

$$XY(f) - YX(f) - [X,Y]f = 0,$$

from which we have,

$$Xg(\text{grad}f, Y) - Yg(\text{grad}f, X) - g(\text{grad}f, [X,Y]) = 0.$$

This is equivalent to

$$\nabla_X g(\text{grad}f, Y) - g(\text{grad}f, \nabla_X Y) - \nabla_Y g(\text{grad}f, X) + g(\text{grad}f, \nabla_Y X) = 0.$$

Since $\nabla g = 0$, the above equation implies

$$g(\nabla_X \text{grad}f, Y) - g(\nabla_Y \text{grad}f, X) = 0,$$

that is, $g(\nabla_X Df, Y) = g(\nabla_Y Df, X)$. Applying this in (60), we have (ξf) $g(\phi X, Y) = 0$, that is, $(\xi f)d\eta(X,Y) = 0$. Since $d\eta \neq 0$, it follows that $\xi f = 0$. Consequently from (56) we obtain $Df = 0$. Hence from (49) we have

$$S(X,Y) = -\lambda g(X,Y).$$

Substituting ξ for both X and Y in the above equation we obtain

$$\lambda = -2k. \tag{61}$$

Thus we obtain $S(X,Y) = 2kg(X,Y)$, that is, the manifold becomes Einstein and hence, being three-dimensional, a space of constant curvature.

Combining the above two cases we have the following:

Theorem 4. *If the pseudo-Riemannian metric g of a three-dimensional contractible $N(k)$-paracontact metric manifold is a gradient Ricci soliton, then the manifold is of constant curvature.*

Acknowledgements. The authors are thankful to the referee for his/her valuable suggestions towards the improvement of the paper.

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