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Increasing sequences of sets and preservation of properties

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Abstract. We deal with increasing sequences of sets: by considering sequences of sets in a given class, we study when the closure of the union belongs to the same class. We consider here several classes of bounded, closed convex sets.

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1 Introduction

From several decades, nested (decreasing) sequences of sets in some classes have received a lot of attention. For example, the nonemptiness of such intersections characterize interesting properties of the space or leads to define new ones. Increasing and decreasing will always be used here in the "weak" sense (not strictly).

Increasing sequences have received less attention: probably they are considered, in general, less interesting.
Unbounded sequences of balls are used to define the well known "Vlasov" property (see for example [6]).
Here instead we deal with bounded increasing sequences. We consider several properties, leading to some well known classes of sets; we study which of them are preserved when the union of sets in these classes is done.

A step in this direction has been done recently in [11].

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Let X be a Banach space with origin Θ and $A \neq \emptyset$ be a bounded subset of X. We denote by $cl(A)$, and $\delta(A)$ the *closure*, and the *diameter* of A, respectively. For $x \in X$, we set

$r(A, x) = \sup\{\|x - a\| \mid a \in A\}$.

Also, for $x \in X$ and $\rho \geq 0$, we denote by

$$B(x, \rho) = \{y \in X \mid \|x - y\| \leq \rho\}$$

the (closed) *ball centered at x having radius ρ.*

We shall consider sequences $(A_n)_{n \in N}$ which are *increasing*: $A_n \subseteq A_{n+1}$ for $n \in N$.

We say that a given class $\mathcal{C}$ of sets has property (P) if the following is true:

(P) If $A_n \in \mathcal{C}$ for every $n \in N, (A_n)_{n \in N}$ is increasing and $\bigcup_{n \in N} A_n$ is bounded, then $cl(\bigcup_{n \in N} A_n) \in \mathcal{C}$.

In Section 2 we consider a few simple facts. In Section 3, we consider a few classes of sets with a property, like constant width, diametrically maximal or some other defined in a somehow similar way. In Section 4 we consider admissible sets, where the situation seems to be less trivial, and another class of sets: for both classes (P) fails. Finally, in Section 5 (the last one) we summarize what we have done and we add a few comments.

For the sake of simplicity, we shall always assume that all sequences under consideration consist of bounded, closed and convex sets, *bcc* for short (containing at least 2 points).

2 A few simple facts

Some results involving increasing sequences of sets (mainly in finite dimensional spaces) were given in [8].
In [9], results concerning increasing sequences of cones in E^n were given.

Recall the following simple result (see for example [11, Proposition 4]):

(r) If the sequence $(A_n)_{n \in N}$ is increasing and $\bigcup_{n \in N} A_n$ is bounded, then $\delta(A) = \lim_{n \to \infty} \delta(A_n)$.

It is known that the class $\mathcal{B}$ of balls satisfies (P) (see [11, Proposition 7]).

It is also immediate to see that the class of convex sets, so the class $\mathcal{BCC}$ of *bcc* sets, satisfies (P).

It is simple to see that the class $\mathcal{K}$ of *compact* sets does not satisfy (P), as the following example shows.

Example 1. Let c_o be the space of real sequences converging to 0, with the *max* norm. Let:

$A_n = \{x = (x_i)_{n \in N} \mid 0 \leq x_i \leq 1 \text{ for } i = 1, 2, ..., n;\ x_i = 0 \text{ for } i > n\}.$

The sets A_n are compact. The closure of their union is the set of sequences, going to 0, with all components in [0, 1], which is not compact.

3 Constant width and related classes of sets

Given a *bcc* set A and $f \in S(X^*)$ (f a norm one functional), let

$w_f(A) = sup\{f(a_1) - f(a_2) \mid a_1, a_2 \in A\},$

and

$W(A) = sup\{w_f(A) \mid f \in S(X^*)\}.$

We say that A has *constant width*, (CW) for short, if $w_f(A)$ is constant for $f \in S(X^*)$. We indicate by $\mathcal{CW}$ the class consisting of these sets. Of course, balls are (CW) sets.

Theorem 1. *The class $\mathcal{CW}$ satisfies (P).*

Proof. Consider an increasing sequence $(A_n)_{n \in N}$ of (CW) sets, whose union is bounded. We recall that for every *bcc* set A we have (see for example [3, Proposition 4.5]): $\delta(A) = W(A)$.

Take $f \in S(X^*)$. For every $n \in N$ we have: $w_f(A_n) = W(A_n) = \delta(A_n)$. Moreover (use (r)) $w_f(A) \geq \lim_{n \to \infty} w_f(A_n) = \lim_{n \to \infty} \delta(A_n) = \delta(A) = W(A)$. Therefore $w_f(A) = W(A)$ for every $f \in S(X^*)$, and this concludes the proof. QED

Remark 1. Set:

$w(A) = inf\{w_f(A) \mid f \in S(X^*)\}$ (*minimal width* of A).

Clearly, if $(A_n)_{n \in N}$ is increasing and $A = cl(\bigcup_{n \in N} A_n)$ is bounded, then $w(A) \geq lim_{n \to \infty} w(A_n)$. Example 1 shows that we can have strict inequality: in fact, in that example, $w(A_n) = 0$ for every $n \in N$, while $w(A) = 1$.

We shall consider now larger classes of *bcc* sets, that have been considered in the literature: see for example [10]. They are larger and larger, since the properties defining them are weaker at any step. All of them contain the class $\mathcal{BCC}$. Consider

$\mathcal{CD}$: the class of *constant difference* sets; namely, of sets A satisfying:

(CD) $r(A, x) = \delta(A) + dist(x, A)$ for all $x \notin A$.

$\mathcal{DM}$: the class of *diametrically maximal* sets; namely, of sets A satisfying:

(DM) $\delta(A \cup \{x\}) > \delta(A)$ for all $x \notin A$.

Finally, in the last section, we shall consider the following class, larger than all the previous ones:

$\mathcal{M}$: the class of sets which are intersection of balls, also called *admissible* sets.

We start with the following result.

Theorem 2. *The class $\mathcal{CD}$ satisfies (P).*

Proof. Let $(A_n)_{n\in N}$ be an increasing sequence of (CD) sets with $A = cl(\bigcup_{n\in N} A_n)$ bounded. Assume (by contradiction) that A does not satisfy (CD): therefore there exist $\bar{x} \notin A$ and $\varepsilon > 0$ such that $r(A, \bar{x}) = \delta(A) + dist(\bar{x}, A) - \varepsilon$; set $dist(\bar{x}, A) = d$ (> 0). According to (CD) we have: $r(A_n, \bar{x}) = dist(\bar{x}, A_n) + \delta(A_n) \geq d + \delta(A_n)$ for all $n \in N$. Therefore $\delta(A) + d - \varepsilon = r(A, \bar{x}) \geq r(A_n, \bar{x}) \geq d + \delta(A_n)$.
Pass to the limit for $n \to \infty$: by applying (r) we obtain $\delta(A) + d - \varepsilon \geq d + \delta(A)$: this contradiction proves the theorem. QED

Concerning the class $\mathcal{DM}$, we give a direct proof of a "partial" result.

Proposition 1. *If X is uniformly convex, then $\mathcal{DM}$ satisfies (P).*

Proof. We recall (see for example [1, Proposition 3.1 a)]) that the sets A in $\mathcal{DM}$ can also be characterized in the following way:

(1) $A = \bigcap_{a\in A} B\big(a, \delta(A)\big)$.

Let $(A_n)_{n\in N}$ be an increasing sequence of (DM) sets with $\bigcup_{n\in N} A_n$ bounded. Assume (by contradiction) that A is not (DM). According to (1), there exists $\bar{x} \notin A$ such that $\bar{x} \in \bigcap_{a\in A} B\big(a, \delta(A)\big)$, or $\|\bar{x} - a\| \leq \delta(A)$ for every $a \in A$. Of course, (A) is *bcc*; set $\varepsilon = dist(\bar{x}, A)(> 0)$. Let $\bar{x}' \in A$ be the best approximation to $\bar{x}$ from A; namely, $\|\bar{x} - \bar{x}'\| = inf_{a\in A}\|\bar{x} - a\| = \varepsilon$. For every $a \in A$ we also have: $\|\bar{x}' - a\| \leq \delta(A)$.

Let $\delta_X(.)$ denote the modulus of uniform convexity of X: therefore $\|x - p\| \leq r$, $\|y - p\| \leq r$, $\|x - y\| \geq \varepsilon$ imply $\|\frac{x+y}{2} - p\| \leq r\big(1 - \delta_X(\varepsilon/r)\big)$. So we have (for $a \in A$):

(2) $\|\frac{\bar{x}+\bar{x}'}{2} - a\| \leq \big(1 - \delta_X(\varepsilon/\delta(A))\big) \cdot \delta(A)$.

Since $lim_{n\to\infty}\delta(A_n) = \delta(A)$, for n large enough we have:

$\delta(A) - \delta(A_n) < \delta_X(\varepsilon/\delta(A)) \cdot \delta(A)$:

by (2) and (1) this implies $\|\frac{\bar{x}+\bar{x}'}{2}-a\| < \delta(A_n)$ for all $a \in A_n \subseteq A$; thus $\frac{\bar{x}+\bar{x}'}{2} \in A_n$ (which is (DM)). But $\frac{\bar{x}+\bar{x}'}{2} \notin A$, so $\frac{\bar{x}+\bar{x}'}{2} \notin A_n \subseteq A$. This contradiction proves the result. QED

Previous result can be generalized by considering another class of sets.

More precisely, we introduce now two more classes of sets (based on conditions that look similar): they are both larger than $\mathcal{DM}$ (and smaller than $\mathcal{M}$); we shall see that they satisfy (P).

Given $x \in A$, we shall denote by $r'(A,x)$ the *sup* of the radii of balls centered at x and contained in A. By $r'(A)$ we will denote the inner radius of A, namely, $sup\{r'(A,x) \mid x \in A\}$.

We can consider the following property for a set A, called in [10] *constant radius*:

(CR) $\delta(A) = r(A,a) + r'(A,a)$ for all $a \in A$,

and the corresponding class:

$\mathcal{CR}$: the class of the sets A satisfying (CR), or (CR) sets.

We also consider the following "sum" property

(S) $\delta(A) = r(A) + r'(A)$,

and the corresponding class:

$\mathcal{S}$: the class of the sets A satisfying (S), or (S) sets.

Remark 2. It is simple to see that the following inequalities are always true:

$\delta(A) \geq r(A) + r'(A)$; $\delta(A) \geq r(A,a) + r'(A,a)$ for all $a \in A$.

Remark 3. The property (DM) implies both (CR) and (S), which are independent (see [2, examples 4 and 5]). The conditions (CR) and (S) together, for a set A, do not imply either that A is admissible: see the example in [10, p.827], or also Example 2 below.

Theorem 3. *The class $\mathcal{CR}$ satisfies (P).*

Proof. Let $(A_n)_{n\in N}$ be an increasing sequence of (CR) sets with $A = cl(\bigcup_{n\in N} A_n)$ bounded.

Let $x \in A$. Take $\varepsilon > 0$; there exists $\bar{x} \in \bigcup_{n\in N} A_n$ such that $\|x - \bar{x}\| < \varepsilon$, so $|r(A,\bar{x}) - r(A,x)| < \varepsilon$ and $|r'(A,\bar{x}) - r'(A,x)| < \varepsilon$. Let $\bar{x} \in A_n$ for all

n from some $\bar{n}$ onwards. According to (r), we can choose $n' \geq \bar{n}$ such that $\delta(A_{n'}) > \delta(A) - \varepsilon$. Since $A_{n'}$ is (CR) we have:

$r(A,\bar{x}) \geq r(A_{n'},\bar{x}) = \delta(A_{n'}) - r'(A_{n'},\bar{x}) > \delta(A) - \varepsilon - r'(A_{n'},\bar{x}) \geq \delta(A) - \varepsilon - r'(A,\bar{x})$. Therefore $r(A,x) > r(A,\bar{x}) - \varepsilon > \delta(A) - 2\varepsilon - r'(A,\bar{x})$; so, since $r'(A,x) > r'(A,\bar{x}) - \varepsilon$, we have $r(A,x) + r'(A,x) > \delta(A) - 3\varepsilon$. Since $\varepsilon > 0$ is arbitrary we have: $r(A,x) + r'(A,x) \geq \delta(A)$.

But the reverse inequality is always true, so we have equality: this proves the result. QED

Recall now the following definition (see for example [5, p.39]): X is said to have *normal structure*, (NS) for short, if for every *bcc* set A there exists $\bar{x} \in A$ such that $r(A,\bar{x}) < \delta(A)$. Recall that uniformly convex spaces have (NS) (see [5, p.53]).

The following result improves Proposition 1.

Theorem 4. *If X has (NS), then the class $\mathcal{DM}$ satisfies (P).*

Proof. Let $(A_n)_{n\in N}$ be an increasing sequence of (DM), thus also (CR) sets, with $A = cl(\bigcup_{n\in N} A_n)$ bounded. According to Theorem 3, A satisfies (CR). But X satisfies (NS): thus every *bcc* set satisfying (CR) has $r'(A,\bar{x}) = \delta(A) - r(A,\bar{x}) > 0$ for some $\bar{x} \in A$. So A has interior points, and it is known (see [10, p.45]) that every (CR) set with an interior point is (DM). This proves the result. QED

We conclude the section by indicating another "positive" result.

Theorem 5. *The class $\mathcal{S}$ satisfies (P).*

Proof. Let $(A_n)_{n\in N}$ be an increasing sequence of (S) sets with $A = cl(\bigcup_{n\in N} A_n)$ bounded. We trivially have $r(A) \geq \lim_{n\to\infty} r(A_n)$ and $r'(A) \geq \lim_{n\to\infty} r'(A_n)$ (while equalities do not always hold: see [11, Proposition 4]). Moreover $\delta(A) \geq r(A) + r'(A)$ always; therefore (S) implies:

$$\delta(A) \geq r(A) + r'(A) \geq \lim_{n\to\infty} \big(r(A_n) + r'(A_n)\big) = \lim_{n\to\infty} \delta(A_n).$$

But according to (r), $\delta(A) = \lim_{n\to\infty} \delta(A_n)$. Thus all these are equalities, and so $\delta(A) = r(A) + r'(A)$, that is the thesis. QED

4 Failure of property (P)

In this section we shall deal with two classes of sets that do not satisfy (P).

Consider the already defined class $\mathcal{M}$ of admissible sets; a set A belongs to $\mathcal{M}$ exactly when it satisfies the property:

(M) $\quad A = \bigcap\limits_{x \in X} B\big(x,\, r(A,x)\big).$

Note that in any case the set $\bigcap\limits_{x \in X} B\big(x, r(A,x)\big)$ is an intersection of balls, and it is the smallest set with this property containing A, usually denoted by A^a. We send to [12] for some facts concerning admissible sets.

The class of admissible sets has received much attention in the last decades. In fact, the spaces where every *bcc* set is admissible have been deeply studied, see for example [6]; this property for a space is called Mazur Intersection Property. Not so many examples of *bcc* sets which are not admissible appear in the literature.

The class $\mathcal{M}$ does not satisfy (P). The following example shows this.

Example 2. Let X be the space $C[0,1]$. Set

$A_n = \{f \in X \mid \; 0 \le f(x) \le 1 \text{ for } x \in [0,1];\; f(x) = 0 \text{ for } x \in [0, 1/n]\}.$

Clearly $A_n \subseteq A_{n+1}$ for every $n \in N$.

Set $A = \{f \in X \mid \; 0 \le f(x) \le 1 \text{ for } x \in [0,1];\; f(0) = 0\}$. We claim that $A = cl(\bigcup\limits_{n \in N} A_n)$.

Clearly $A_n \subseteq A$ for every $n \in N$, so $\; cl(\bigcup\limits_{n \in N} A_n) \subseteq A$.

Now take $f \in A$; given $\varepsilon > 0$, we will show that there exist $n \in N$ and $f_n \in A_n$ such that $\|f_n - f\| < \varepsilon$.

Let η such that $0 \le f(x) \le \varepsilon$ for $x \in [0, \eta]$. Take f_n as follows:

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \le x \le \eta/2; \\ f_n \;\; \text{is linear} & \text{in } [\eta/2, \eta]; \\ f(x) & \text{if } \eta \le x \le 1. \end{cases}$$

If $\frac{\eta}{2} \ge \frac{1}{n}$, we have $f_n \in A_n$ and $\|f_n - f\| \le \varepsilon$. Thus $A \subseteq cl(\bigcup\limits_{n \in N} A_n)$, and then equality holds.

We **claim** that A is not admissible; more precisely, we say that $A^a = \{f \in X \mid 0 \le f(x) \le 1 \text{ for } \; x \in [0,1]\}$.

Denote by A^α the set on the right hand side (which is clearly admissible and contains A_n for every $n \in N$).

Indicate by $\mathbf{1}$ the constant function whose value is 1 on the whole interval. We claim that the function $\mathbf{1}$ belongs to A^a (so, since the null function $\Theta \in A^a$, $A^a = A^\alpha$).

Take $f \in X$; set $r = r(A, f)$. We have $\|f - g\| \le r$ for every $g \in \bigcup_{n \in N} A_n$. Given $x \in (0,1]$, there exists $\bar{g}$ in some A_n $(n > 1/x)$ such that $\bar{g}(x) = 1$: thus $|f(x) - 1| \le r$ for every $x \in (0,1]$. This implies $\|f - \mathbf{1}\| \le r$, so $\mathbf{1} \in B\big(f, r(A,f)\big)$ for every $f \in X$. This proves the **claim** that A is not admissible.

Finally, we show that all A_n are admissible.
Fix $n \in N$. Let (for $m > n$):

$$g_{m,n}(x) = \begin{cases} -1 & \text{if } x \in [0,\, 1/n - 1/m]; \\ g_{m,n} \ \text{ is linear} & \text{in } [1/n - 1/m,\, 1/n] \ \big(g_{m,n}(x) = mx - m/n\big); \\ 0 & \text{if } x \in [1/n, 1]. \end{cases}$$

It is easy to see that $A_n = B(\mathbf{1},\, 1) \bigcap \big(\bigcap_{m>n} B(g_{m,n}, 1)\big)$.

This shows that $\mathcal{M}$ does not satisfy (P).

We give a second example concerning the failure of property (P).

We say that a set is *balanced* (or *symmetric*) if $tS \subseteq S$ for all $t \in [-1, 1]$. According to [4], we say that a set $S \subseteq X$ is a *star* if there exists $x_o \in S$ such that $S - x_0$ is balanced; in this case x_0 is called a *center* of S. We recall that every nested (decreasing) sequence of stars has nonempty intersection if and only if X is finite dimensional (see [4, Theorem 1]).

This class of sets, apart from the case when X is a Euclidean space, has seldom been considered; it is simple to see that for a bounded star, the center is necessarily unique (see for example [7, Section 4]).

Indicate by $\mathcal{ST}$ the class of stars. Next example shows that $\mathcal{ST}$ fails property (P).

Example 3. Let $X = c_0$. Consider the following *bcc* sets:
$A_n = \{x = (x_k)_{k \in N} \in c_0 \mid \ 1/k \le x_k \le 2 - 1/k \ \text{ for } k = 1, 2, ..., n; \ x_k = 1/k$ for $k > n\}$.

Clearly $(A_n)_{n \in N}$ is an increasing sequence of *bcc* and star sets. Their centers are the points $(x_{0,n})$ whose coordinates are:
$(x_{0,n})_k = 1$ for $k = 1, 2, ..., n$; $(x_{0,n})_k = 1/k$ for $k > n$.
The closure of their union is the *bcc* set:
$A = \{x = (x_k)_{k \in N} \in c_0 \mid 1/k \le x_k \le 2 - 1/k$ for every $k \in N\}$.

But A is not a star: in fact, the "candidate" center would be the sequence with all components equal to 1, which is not in c_0.

But we can also indicate a "positive" result concerning the class $\mathcal{ST}$.

Lemma 1. *Consider an increasing sequence $(A_n)_{n\in N}$ of stars, whose union is bounded; for every $n \in N$, let a_n be the center of A_n. If $(a_n)_{n\in N}$ has a converging subsequence, then $A = cl(\bigcup_{n\in N} A_n)$ is a star.*

Proof. We must prove that, if all assumptions are true, then $A = cl(\bigcup_{n\in N} A_n)$ is a star.

There exists a subsequence $(a_{n_k})_{k\in N}$ of $(a_n)_{n\in N}$ converging, say to a_0. Take $a \in A$; there is an increasing subsequence of $(n_k)_{k\in N}$, say n_{k_i} $(i \in N)$, such that for every $i \in N$ we can choose $b_i \in A_{n_{k_i}}$ satisfying: $\|b_i - a\| < 1/i$. Since the sequence $(A_n)_{n\in N}$ is increasing, it is enough to show that $cl(\bigcup_{i\in N} A_{n_{k_i}})$ is a star.

Then, for every $t \in [-1, 1]$ we have: $t(A_{n_{k_i}} - a_{n_{k_i}}) \subseteq A_{n_{k_i}} - a_{n_{k_i}}$; thus $t(b_i - a_{n_{k_i}}) + a_{n_{k_i}} \subseteq A_{n_{k_i}} \subseteq A$ for every $i \in N$. Letting $i \to \infty$, we obtain: $t(a - a_0) \subseteq A - a_0$. This shows that A is a star and that a_0 is its center. QED

Proposition 2. *Property (P) for bcc stars holds if X is finite dimensional.*

Proof. It is a consequence of previous lemma, together with the compactness of bounded closed sets in finite dimensional spaces. QED

5 Final remarks

In this paper we have considered several conditions; we list them and we indicate the implications among them.

ball $\Rightarrow$ (CW) $\Rightarrow$ (CD) $\Rightarrow$ (DM) $\Rightarrow$ (M) $\Rightarrow$ *bcc*.

Moreover: (DM) $\Rightarrow$ (CR) and (DM) $\Rightarrow$ (S); (M) $\not\Rightarrow$ (CR) and (M) $\not\Rightarrow$ (S); (CR) and (S) together $\not\Rightarrow$ (M) (see Remark 3).

The notion of star applies to sets that are not necessarily *bcc*; Example 3 shows that condition (P) fails also if we consider star, *bcc* sets.

Of course, a ball is a (*bcc*) star. A (CW) set is not necessarily a star: think for example at the classical Reuleaux triangle in the Euclidean plane; another example is the closure of the union in Example 1.

Note 1. Are there conditions on infinite dimensional spaces implying that the class of stars satisfy (P)? Is the existence of an inner product one of them?

Note 2. We have seen (Theorem 4) that in many spaces the class $\mathcal{DM}$ satisfies (P); we are not able to decide if it satisfies (P) in general.

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