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Duality between cuspidal butterflies and cuspidal S_1^- singularities on maximal surfaces

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Abstract. We give criteria for cuspidal butterflies and cuspidal S_1^- (cS_1^-) singularities in terms of the Weierstrass data for maximal surfaces, and also show non- existence of cuspidal S_1^+ (cS_1^+) singularities on maximal surfaces. Moreover, we show duality between these singularities considering the conjugate maximal surfaces each other.

Keywords: maximal surface, frontal, cuspidal butterfly, cuspidal S_k singularity

MSC 2000 classification: Primary 57R45, Secondary 53A10, 53B30

1 Introduction

There are classes of surfaces with singularities called frontals and (wave) fronts. Arnol'd and Zakalyukin showed that generic singularities of fronts in 3-space are cuspidal edges and swallowtails. Moreover, they showed that bifurcations in generic one-parameter families in 3-space are cuspidal lips/beaks, cuspidal butterflies and $D_4^{\pm}$ singularities (cf. [1, 2]). It is known that there are useful criteria for these singularities (see [17, 18, 22, 30, 31]). For frontals in 3-space, criteria for cuspidal cross caps and cuspidal $S_k^{\pm}$ singularities are also given in [11, 29]. Several new geometric properties for frontals and fronts are obtained by applying these criteria (cf. [6, 10, 12, 13, 15, 32]).

On the other hand, for maximal surfaces in 3-dimensional Lorentz space $\boldsymbol{R}^{2,1}$ of sigunature $(-++)$, a Weierstrass-type representation was stated by

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Kobayashi in [21], and was later refined to include certain singularities (called *maxfaces*) by Umehara and Yamada in [33]. The following fact is known.

Fact 29 ([33, Theorem 2.6])**.** Any maxface $f : \Sigma \subset \boldsymbol{C} \to \boldsymbol{R}^{2,1}$ can be represented as

$$f = \mathrm{Re}\left[\int (-2g, 1+g^2, i(1-g^2))\omega\right] \quad (i = \sqrt{-1}) \tag{1.1}$$

for a simply-connected domain Σ, where g is a meromorphic function, and $\omega = \hat{\omega}\ dz$ is a holomorphic 1-form such that $g^2\hat{\omega}$ is holomorphic on Σ, and $(1 + |g|^2)^2|\omega|^2 \neq 0$.

We call the pair (g, ω) appearing in Fact 29 the *Weierstrass data* of a maxface. In [33], criteria for cuspidal edges and swallowtails were obtained by using Weierstrass data (g, ω). Similarly, in [11] Fujimori, Saji, Umehara and Yamada showed that maxfaces have frontal singularities as generic singularities, and constructed criteria for cuspidal cross caps by using the Weierstrass data (g, ω). On the other hand, maxfaces have other singularities called cone-like singularities and fold singularities. Maxfaces with fold singularities are closely related to real anayitic extensions. For geometric properties about maxfaces with these two singularities, see [7, 8, 10, 19, 20]. Moreover, in [13, 14], singularities of spacelike constant mean curvature surfaces are studied.

In this paper, we give criteria for cuspidal butterflies and cuspidal S_1^- singularities on maxfaces by using the Weierstrass data. More precisely, we shall prove the following theorem.

Theorem 30. *Let $f : \Sigma \to \boldsymbol{R}^{2,1}$ be a maxface constructed from the Weierstrass data (g, ω), where Σ is a simply-connected domain of the complex plane $(\boldsymbol{C}, z)$, and let p be a singular point of f.*

(1) *f is $\mathcal{A}$-equivalent to a cuspidal butterfly at p if and only if $\mathrm{Re}[\varphi] \neq 0$, $\mathrm{Im}[\varphi] = 0$, $\mathrm{Re}[\phi] = 0$ and $\mathrm{Im}[\Phi] \neq 0$ at p.*

(2) *f is $\mathcal{A}$-equivalent to a cuspidal S_1^- singularity at p if and only if $\mathrm{Re}[\varphi] = 0$, $\mathrm{Im}[\varphi] \neq 0$, $\mathrm{Im}[\phi] = 0$ and $\mathrm{Re}[\Phi] \neq 0$ at p.*

Here, the functions φ, ϕ and Φ are defined by

$$\varphi := \frac{g_z}{g^2\hat{\omega}}, \quad \phi := \frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z, \quad \Phi := \frac{g}{g_z}\left\{\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right\}_z. \tag{1.2}$$

In particular, a maxface cannot have a cuspidal S_1^+ singularity.

In general, there are useful criteria for cuspidal butterflies and cuspidal S_1^- singularities on arbitrary surfaces as in [17, 29]. However, for general surfaces, it is difficult to find dualities between these two types of singularities, because there are some gaps between the criterion for cuspidal butterflies and one for cuspidal S_1^- singularities: We only need the data of "third" differentiation of surfaces in order to specify cuspidal butterflies. On the other hand, we need the data of "fifth" differentiation of surfaces in order to specify cuspidal S_1^- singularities. As in the above theorem, restricting to maxfaces, we can show the duality between cuspidal butterflies and cuspidal S_1^- singularities (see Corollary 37).

2 Preliminaries

2.1 Spacelike surfaces in the Lorentz 3-space

We recall some notions of the Lorentz 3-space.

Let $\boldsymbol{R}^3 = \{(x_1, x_2, x_3) \mid x_i \in \boldsymbol{R},\ 1 \le i \le 3\}$ be a 3-dimensional vector space. For any $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3) \in \boldsymbol{R}^3$, we define the *pseudo-inner product* by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3.$$

We denote $\boldsymbol{R}^{2,1} = (\boldsymbol{R}^3, \langle , \rangle)$ and call it the 3*-dimensional Lorentz space* (or *Lorentz* 3*-space*). If two vectors $\boldsymbol{x}$, $\boldsymbol{y} \in \boldsymbol{R}^{2,1}$ satisfy $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$, then we say that $\boldsymbol{x}$ and $\boldsymbol{y}$ are *pseudo-orthogonal*. Take a vector $\boldsymbol{x} \in \boldsymbol{R}^{2,1} \setminus \{\boldsymbol{0}\}$. Then $\boldsymbol{x}$ is said to be *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $= 0$ or < 0, respectively.

Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ be the pseudo-orthogonal basis of $\boldsymbol{R}^{2,1}$ such that $\boldsymbol{e}_1$ is a timelike unit vector, and let $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ be vectors in $\boldsymbol{R}^{2,1}$. Then we define the *pseudo-vector product* $\boldsymbol{x} \wedge \boldsymbol{y}$ as

$$\boldsymbol{x} \wedge \boldsymbol{y} = \begin{vmatrix} -\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

By the definition, one can check that $\langle \boldsymbol{x} \wedge \boldsymbol{y}, \boldsymbol{z} \rangle = \det(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ holds for $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z} \in \boldsymbol{R}^{2,1}$, where det is the canonical determinant of 3×3 matrices. Thus $\boldsymbol{x} \wedge \boldsymbol{y}$ is pseudo-orthogonal to both $\boldsymbol{x}$ and $\boldsymbol{y}$.

Let $f : \Sigma \to \boldsymbol{R}^{2,1}$ be an immersion, where Σ is a domain in $(\boldsymbol{R}^2; u, v)$. Then f is said to be a *spacelike* if the plane spanned by $\{f_u, f_v\}$ consists of spacelike vectors, where $f_u = \partial f / \partial u$ and $f_v = \partial f / \partial v$. In this case, we note that $f_u \wedge f_v$ is a timelike vector. We set a map $\nu : \Sigma \to H^2$ as

$$\nu(u, v) = \frac{f_u \wedge f_v}{|f_u \wedge f_v|}(u, v),$$

where $H^2 = \{\boldsymbol{x} \in \boldsymbol{R}^{2,1} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\}$ is the *hyperbolic* 2*-space* and $|\boldsymbol{x}| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$ for any $\boldsymbol{x} \in \boldsymbol{R}^{2,1}$. We call ν the *pseudo-unit normal vector to* f.

2.2 Frontals and fronts

We describe some notions of frontals and fronts. For more details, see [1, 2, 16, 32]. Let $f : \Sigma \to \boldsymbol{R}^3$ be a C^∞ map, where Σ is a domain in $\boldsymbol{R}^2$ with local coordinates u, v. We call the map f a *frontal* if there exists a unit vector field ν along f such that $\langle df(X_p), \nu(p)\rangle_{\mathrm{E}} = 0$ for any $p \in \Sigma$ and $X_p \in T_p\Sigma$, where $\langle \cdot, \cdot \rangle_{\mathrm{E}}$ means the canonical Euclidean inner product. A frontal f is said to be a *front* if the pair $L_f = (f, \nu) : \Sigma \to \boldsymbol{R}^3 \times S^2$ gives an immersion, where S^2 is the unit sphere in $\boldsymbol{R}^3$.

We fix a frontal f. A point $p \in \Sigma$ is called a *singular point* of f if $\operatorname{rank} df_p < 2$ holds. Let $S(f)$ denote the set of singular points of f. We now define a function $\lambda : \Sigma \to \boldsymbol{R}$ called the *signed area density function* as

$$\lambda(u, v) = \det(f_u, f_v, \nu)(u, v).$$

One can see that $S(f) = \lambda^{-1}(0)$. Take a singular point $p \in S(f)$. If $d\lambda(p) \neq 0$, then p is called a *non-degenerate singular point*. Assume that p is a non-degenerate singular point. Then by the implicit function theorem, there exist a neighborhood U of p and a regular curve $\gamma : (-\varepsilon, \varepsilon)t \mapsto \gamma(t) \in U(\subset \Sigma)$ $(\varepsilon > 0)$ such that $U \cap S(f)$ is locally parametrized by γ. We call this curve γ a *singular curve* and the direction of $\gamma' = d\gamma/dt$ the *singular direction*. In this case, there exists a vector field ξ on U such that ξ is tangent to $S(f)$, namely, ξ is parallel to γ' on $S(f)$. Moreover, a non-degenerate singular point p is a corank one singular point, namely, $\operatorname{rank} df_p = 1$, there exists a never vanishing vector field η on U such that $df_q(\eta) = \mathbf{0}$ for any $q \in U \cap S(f)$. This vector field η is called a *null vector field*.

Definition 31. Let $f_1, f_2 : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ be C^∞ map germs. Then f_1 and f_2 are $\mathcal{A}$*-equivalent* if there exist diffeomorphism germs $\theta : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^2, \mathbf{0})$ on the source and $\Theta : (\boldsymbol{R}^3, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ on the target such that $\Theta \circ f_1 = f_2 \circ \theta$ holds.

Definition 32. A *cuspidal edge* is a map germ $\mathcal{A}$-equivalent to the map germ $(u, v) \mapsto (u, v^2, v^3)$ at $\mathbf{0}$. A *swallowtail* is a map germ $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, 3v^4 + uv^2, 4v^3 + 2uv)$ at $\mathbf{0}$. A *cuspidal butterfly* is a map germ $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, 4v^5 + uv^2, 5v^4 + 2uv)$ at $\mathbf{0}$. A *cuspidal cross cap* is a map germ $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, v^2, uv^3)$ at $\mathbf{0}$. A *cuspidal $S_k^\pm$ singularity* (or *$cS_k^\pm$ singularity*, for short) $(k \geq 1)$ is a map germ $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, v^2, v^3(u^{k+1} \pm v^2))$ at $\mathbf{0}$. (See Figure 1.)

We note that cuspidal edges, swallowtails and cuspidal butterflies are fronts. On the other hand, cuspidal cross caps and $cS_k^\pm$ singularities are frontals but

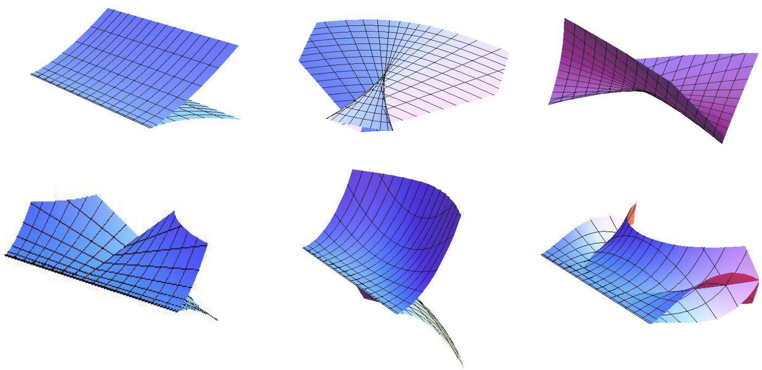

Figure 1. From top left to bottom right: cuspidal edge, swallowtail, cuspidal butterfly, cuspidal cross cap, cS_1^+ singularity and cS_1^- singularity.

not fronts. One can see that cuspidal cross caps and $cS_0^{\pm}$ singularities are $\mathcal{A}$-equivalent, and if k is even, cS_k^+ and cS_k^- are $\mathcal{A}$-equivalent to each other (cf. [29]). When we substitute $u = 0$ in the normal form (i.e. the standard parametrization as in Definition 32) of the $cS_k^{\pm}$ singularity $(u, v^2, v^3(u^{k+1} \pm v^2))$, it reduces to a $(2,5)$-cusp curve through $\mathbf{0}$, where $(2,5)$-*cusp curve* means a curve which is $\mathcal{A}$-equivalent to $t \mapsto (t^2, t^5, 0)$ (see Figure 2).

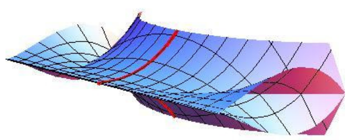

Figure 2. Surface with cS_1^- singularity. The red curve is a $(2,5)$-cusp curve on the surface through $\mathbf{0}$ which lies in the normal plane to $\hat{\gamma}$ at $\hat{\gamma}(0)$.

We define functions along the singular curve γ by

$$\delta(t) = \det(\gamma'(t), \eta(t)), \quad \psi(t) = \det(\hat{\gamma}'(t), \nu \circ \gamma(t), d\nu_{\gamma(t)}(\eta(t))) \tag{2.1}$$

for $t \in (-\varepsilon, \varepsilon)$, where $\hat{\gamma} = f \circ \gamma$ (cf. [11, 22]).

Fact 33 ([17, 22, 31])**.** Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, $p \in \Sigma$ be a non-degenerate singular point, γ a singular curve with $p = \gamma(0)$ and η a null vector field.

(1) f is $\mathcal{A}$-equivalent to a cuspidal edge at p if and only if $\delta(0) \neq 0$ holds.

(2) f is $\mathcal{A}$-equivalent to a swallowtail at p if and only if $\delta(0) = 0$ and $\delta'(0) \neq 0$ hold.

(3) f is $\mathcal{A}$-equivalent to a cuspidal butterfly at p if and only if $\delta(0) = \delta'(0) = 0$ and $\delta''(0) \neq 0$ hold.

Fact 34 ([11, 29])**.** Let $f : \Sigma \to \boldsymbol{R}^3$ be a frontal and $p \in \Sigma$ a non-degenerate singular point.

(1) f is $\mathcal{A}$-equivalent to a cuspidal cross cap at p if and only if $\delta(0) \neq 0$, $\psi(0) = 0$ and $\psi'(0) \neq 0$ hold.

(2) f is $\mathcal{A}$-equivalent to a $cS^{\pm}_{k-1}$ singularity ($k \geq 2$) at p if and only if the following conditions (a)-(d) hold:

(a) $\delta(0) \neq 0$.

(b) There exists a curve $c : (-\varepsilon, \varepsilon) \to \Sigma$ such that $c'(0)$ is parallel to $\eta(0)$, $\hat{c}'(0) = \boldsymbol{0}$, $\hat{c}''(0) \neq \boldsymbol{0}$ and there exists $\ell \in \boldsymbol{R}$ satisfying $\hat{c}'''(0) = \ell\hat{c}''(0)$ and $A := \det(\hat{\gamma}', \hat{c}'', 3\hat{c}^{(5)} - 10\ell\hat{c}^{(4)})(0) \neq 0$, where $\hat{c} = f \circ c$.

(c) $\psi(0) = \psi'(0) = \cdots = \psi^{(k-1)}(0) = 0$ and $B := \psi^{(k)}(0) \neq 0$.

(d) If k is even, the sign $\pm$ of the $cS^{\pm}_{k-1}$ singularity coincides with the sign of the product $AB = \det(\hat{\gamma}', \hat{c}'', 3\hat{c}^{(5)} - 10\ell\hat{c}^{(4)})(0) \cdot \psi^{(k)}(0)$. Here, we choose η and t so that $c'(0)$ points the same direction as the null vector $\eta(0)$ and that $(\gamma', \eta)(0)$ is positively oriented.

We note that the conditions in Facts 33 and 34 do not depend on the choice of coordinates on the source, the choice of η, on the choice of ν nor on the choice of coordinates on the target. We note that criteria for $(2, 5)$-cuspidal edges $(u, v) \mapsto (u, v^2, v^5)$ are given in [14, Theorem 4.1].

Lemma 35. Let $f : \Sigma \to \boldsymbol{R}^3$ be a frontal, p a non-degenerate singular point and γ a singular curve through p. Let ξ and η be C^∞ vector fields on a neighborhood of p such that ξ is tangent to γ and η is a null vector for each point on the singular curve. Suppose that the pair (ξ, η) is positively oriented along γ, namely, $\det(\xi, \eta) > 0$ along γ. Moreover, assume that $\langle \xi f(p), \eta\eta f(p)\rangle_{\mathrm{E}} = 0$ and $\eta\eta\eta f(p) = \boldsymbol{0}$. Then there exists a curve $c(t)$ having the properties in (2) (b) in Fact 34, and the real number A in Fact 34 is expressed as $A = \det(\xi f, \eta\eta f, \eta^5 f)(p)$, where $\eta^k f$ means a k-times derivative $\eta \cdots \eta f$. In particular, if $\eta^5 f(p) \neq 0$, A does not vanish.

Proof. Without loss of generality, we can consider $p = (0, 0)$. It is known that there exisits the following local coordinate system $(U; u, v)$ around p (cf. [27, 32]): the singular curve is the u-axis and the null vector η is $\eta = \partial_v$. In this case, ξ can be taken as $\xi = \partial_u$, and hence (ξ, η) is positively oriented. Under this local coordinate system, $f_{uv}(p) = \mathbf{0}$ holds because $f_v = \mathbf{0}$ on the u-axis. We now consider the set Λ defined by

$$\Lambda = \{f(u,v) \,|\, \langle f(u,v), f_u(p)\rangle_{\mathrm{E}} = 0\}.$$

Since $f_u(p) \neq \mathbf{0}$, there exists a C^∞ curve $c(t) = (u(t), t)$ such that

$$\langle \hat{c}(t), f_u(p)\rangle_{\mathrm{E}} = 0, \tag{2.2}$$

where $\hat{c} = f \circ c$. We show that this curve c satisfies the properties in (2) (b) in Fact 34.

Differentiating of (2.2), we have

$$\langle u'(v) f_u(u(v), v), f_u(p)\rangle_{\mathrm{E}} + \langle f_v(u(v), v), f_u(p)\rangle_{\mathrm{E}} = 0,$$

where $u' = du/dv$. Since $\eta f(p) = f_v(p) = \mathbf{0}$, we have $u'(0) = 0$. Thus the differential of $c(v) = (u(v), v)$ by v is parallel to η at $v = 0$. Moreover, we see that

$$\begin{aligned}
&u''(0)\langle f_u(p), f_u(p)\rangle_{\mathrm{E}} + \langle f_{vv}(p), f_u(p)\rangle_{\mathrm{E}} = 0,\\
&u'''(0)\langle f_u(p), f_u(p)\rangle_{\mathrm{E}} + u''(0)\langle f_{uv}(p), f_u(p)\rangle_{\mathrm{E}} + \langle f_{vvv}(p), f_u(p)\rangle_{\mathrm{E}} = 0
\end{aligned}$$

hold. Thus $u''(0) = u'''(0) = 0$ hold by $\langle f_{vv}(p), f_u(p)\rangle_{\mathrm{E}} = 0$, $f_{uv}(p) = f_{vvv}(p) = \mathbf{0}$. On the other hand, differentials of $\hat{c}$ satisfies $\hat{c}' = \mathbf{0}$, $\hat{c}'' = f_{vv} \neq \mathbf{0}$, $\hat{c}''' = \mathbf{0}$ at $v = 0$, and a real number ℓ satisfying $\hat{c}'''(0) = \ell \hat{c}''(0)$ is $\ell = 0$. This implies that the curve c satisfies the properties in (2) (b) in Fact 34.

The 5-time derivative of $\hat{c}$ is

$$\hat{c}^{(5)} = u^{(5)} f_u + f_{vvvvv}$$

at $v = 0$. Thus the real number A in Fact 34 (2) (b) can be written as

$$\begin{aligned}
A &= \det(\hat{\gamma}', \hat{c}'', 3\hat{c}^{(5)} - 10\ell \hat{c}^{(4)})(0) = \det(f_u, f_{vv}, 3 f_{vvvvv})(p)\\
&= 3\det(\xi f, \eta\eta f, \eta^5 f)(p).
\end{aligned}$$

Therefore we may take $A = (\xi f, \eta\eta f, \eta^5 f)(p)$, and hence we have the assertion. $\boxed{QED}$

We suppose that the pair (ξ, η) satisfies the conditions of Lemma 35. If we take another pair of vector fields $(\bar{\xi}, \bar{\eta})$ as

$$\bar{\xi} = a_1(u,v)\xi + a_2(u,v)\eta, \quad \bar{\eta} = b_1(u,v)\xi + b_2(u,v)\eta,$$

where $a_i, b_i : U \to \boldsymbol{R}$ $(i = 1, 2)$ are C^∞ functions and $a_2 = b_1 = 0$ on the set of singular point of f. Assume that $a_1 b_2 > 0$ at p and $(\bar{\xi}, \bar{\eta})$ also satisfies the condition of Lemma 35. Under this settings, we show the sign of $\det(\xi f, \eta\eta f, \eta^5 f)(p)$ coinsides with the sign of $\det(\bar{\xi} f, \bar{\eta}\bar{\eta} f, \bar{\eta}^5 f)(p)$.

By direct computations, we have

$$\bar{\xi} f(p) = a_1(p)\xi f(p), \quad \bar{\eta} f(p) = b_2(p)\eta f(p) = \mathbf{0},$$
$$\bar{\eta}\bar{\eta} f(p) = b_2(p)(\eta b_2(p)\xi f(p) + b_2(p)\eta\eta f(p))$$

holds. Since $(\bar{\xi}, \bar{\eta})$ satisfies $\langle \bar{\xi} f, \bar{\eta}\bar{\eta} f\rangle_{\mathrm{E}}(p) = 0$, we have $\eta b_1(p) = 0$. By this assumption and $\eta\eta\eta f(p) = \mathbf{0}$, we have

$$\bar{\eta}\bar{\eta}\bar{\eta} f(p) = b_2(p)^2(\eta\eta b_1(p)\xi f(p) + 3\eta b_2(p)\eta\eta f(p)) = \mathbf{0}.$$

Since $\xi f(p)$ and $\eta\eta f(p)$ are linearly independent, it follows that $\eta\eta b_1(p) = \eta b_2(p) = 0$. Similarly, we also obtain

$$\begin{aligned} \bar{\eta}^5 f(p) &= b_2(p)^3 \left\{ b_2(p)\eta^4 b_1(p) + \eta^3 b_1(p)\xi b_1(p) \right\} \xi f(p) \\ &+ 5 b_2(p)^4 \eta^3 b_2(p)\eta\eta f(p) + b_2(p)^5 \eta^5 f(p) \end{aligned}$$

using the conditions $b_1(p) = \eta b_1(p) = \eta\eta b_1(p) = \eta b_2(p) = 0$ and $\eta f(p) = \eta^3 f(p) = \xi\eta f(p) = \mathbf{0}$. So we have

$$\det(\bar{\xi} f, \bar{\eta}\bar{\eta} f, \bar{\eta}^5 f)(p) = (a_1(p) b_2(p)) b_2(p)^6 \det(\xi f, \eta\eta f, \eta^5 f)(p).$$

Thus, the sign of $\det(\xi f, \eta\eta f, \eta^5 f)(p)$ coinsides with the sign of $\det(\bar{\xi} f, \bar{\eta}\bar{\eta} f, \bar{\eta}^5 f)(p)$. This implies that we don't need to mind the choice of the pair (ξ, η) if (ξ, η) satisfies the conditions of Lemma 35.

3 Proof of Theorem 30

In this section, we shall prove Theorem 30 in Section 1. Before the proof, we introduce the related results given in [11, 33] as follows:

Fact 36 ([11, 33])**.** Let $f : \Sigma \to \boldsymbol{R}^{2,1}$ be a maxface constructed from the Weierstrass data (g, ω), where Σ is a simply-connected domain of the complex plane $(\boldsymbol{C}, z)$, and let p be a singular point of f.

(1) f has a non-degenerate singular point at p if and only if $g_z \neq 0$ at p.

(2) f is a front at p if and only if $\mathrm{Re}[\varphi] \neq 0$ at p.

(3) f is $\mathcal{A}$-equivalent to a cuspidal edge at p if and only if $\mathrm{Re}[\varphi] \neq 0$ and $\mathrm{Im}[\varphi] \neq 0$ at p.

(4) f is $\mathcal{A}$-equivalent to a swallowtail at p if and only if $\mathrm{Re}[\varphi] \neq 0$, $\mathrm{Im}[\varphi] = 0$ and $\mathrm{Re}[\phi] \neq 0$ at p.

(5) f is $\mathcal{A}$-equivalent to a cuspidal cross cap at p if and only if $\mathrm{Re}[\varphi] = 0$, $\mathrm{Im}[\varphi] \neq 0$ and $\mathrm{Im}[\phi] \neq 0$ at p.

Here, the functions φ and ϕ are same as in (1.2).

We show Theorem 30.

Proof of Theorem 30 (1). Now we prove the criterion for cuspidal butterflies by using a similar calculation as in the proof of the above Fact 36 (see [11, 33]).

By (1.1), we see that

$$f_z = \frac{\hat{\omega}}{2}\left(-2g, 1+g^2, i(1-g^2)\right), \quad f_{\bar{z}} = \frac{\bar{\hat{\omega}}}{2}\left(-2\bar{g}, 1+\bar{g}^2, -i(1-\bar{g}^2)\right). \tag{3.1}$$

We now identify $\boldsymbol{R}^{2,1}$ with $\boldsymbol{R}^3$. Then using the Euclidean inner product $\langle \cdot, \cdot \rangle_{\mathrm{E}}$, we can define the following Euclidean unit normal vector $\boldsymbol{n}$ to f as

$$\boldsymbol{n} := \frac{1}{\sqrt{(1+|g|^2)^2 + 4|g|^2}}\left(1+|g|^2, 2\,\mathrm{Re}[g], 2\,\mathrm{Im}[g]\right). \tag{3.2}$$

By (3.1) and (3.2), the signed area density function λ of f is given by

$$\lambda = \langle f_u \times f_v, \boldsymbol{n} \rangle_{\mathrm{E}} = (|g|^2 - 1)|\hat{\omega}|^2\sqrt{(1+|g|^2)^2 + 4|g|^2}, \tag{3.3}$$

where we used identifications $2f_z = f_u - if_v$ and $2f_{\bar{z}} = f_u + if_v$ (cf. [11, 33]). We notice by (3.3) that the singular set of f is

$$S(f) = \left\{ z \in \Sigma \mid |g(z)|^2 = 1 \right\}.$$

We also define the null direction η as

$$\eta = \frac{i}{g\hat{\omega}}\partial_z - \frac{i}{\overline{g\hat{\omega}}}\partial_{\bar{z}}. \tag{3.4}$$

Assume that f is a front at p and p is a non-degenerate singular point, namely, $\mathrm{Re}[\varphi]$ as in (1.2) does not vanish at p. Then we apply Fact 33 (3). In this case, there exists a singular curve $\gamma(t)$ passing through $\gamma(0) = p$ with $|g(\gamma(t))| = 1$. Then one can choose a suitable parametrization of $\gamma(t)$ such that $\xi(t) = \gamma'(t)$, where $\xi(t)$ is

$$\xi = i\overline{\left(\frac{g_z}{g}\right)}\partial_z - i\left(\frac{g_z}{g}\right)\partial_{\bar{z}}. \tag{3.5}$$

By (3.4) and (3.5), the function $\delta(t) = \det(\xi, \eta)$ can be written as

$$\delta = \operatorname{Im}[\varphi]$$

on $\gamma(t)$. Differentiating this, we have $\delta' = -|g_z|^2 \operatorname{Re}[\phi]$ on $\gamma(t)$, where $\phi = \frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z$. Under the assumption that $\delta(0) = \delta'(0) = 0$ hold, it follows that $\delta'' = -|g_z|^4 \operatorname{Im}[\Phi]$ at $t = 0$, where $\Phi = \frac{g}{g_z}\left\{\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right\}_z$. By non-degeneracy, Fact 33 (3) and Fact 36, it follows that $\delta(0) = \delta'(0) = 0$ and $\delta''(0) \neq 0$ if and only if $\operatorname{Im}[\varphi] = \operatorname{Re}[\phi] = 0$ and $\operatorname{Im}[\Phi] \neq 0$ at p. Thus we have the conclusion. $\boxed{QED}$

Proof of Theorem 30 (2). In the proof of Theorem 30 (1), we already checked the condition (2)(a):

$$\delta = \operatorname{Im}[\varphi] \neq 0 \tag{3.6}$$

at $p(= \gamma(0))$. We check the condition (2)(c). By (2.1), we have

$$\psi = -|\hat{\omega}|^2 \operatorname{Im}[\varphi]\operatorname{Re}[\varphi],\ \psi' = |g_z|^2|\hat{\omega}|^2 \operatorname{Im}[\varphi]\operatorname{Im}[\phi],\ \psi'' = |g_z|^4|\hat{\omega}|^2 \operatorname{Im}[\varphi]\operatorname{Re}[\Phi]$$

at $p(= \gamma(0))$. Thus, the condition (2)(a) and (2)(c) hold if and only if $\operatorname{Im}[\varphi] \neq 0$, $\operatorname{Re}[\varphi] = \operatorname{Im}[\phi] = 0$ and $\operatorname{Re}[\Phi] \neq 0$ at $p = \gamma(0)$.

Now assuming the conditions (2)(a) and (2)(c), we prove that (2)(b) is trivial. Differentiating f by the singular direction ξ as in (3.5) and by the null direction η as in (3.4) on $\gamma(t)$, we obtain

$$\begin{aligned}
\xi f &= \left(2\operatorname{Re}\left[-i\hat{\omega}g\overline{\left(\frac{g_z}{g}\right)}\right], 2\operatorname{Re}\left[\frac{i\hat{\omega}}{2}\overline{\left(\frac{g_z}{g}\right)}(1+g^2)\right], -2\operatorname{Re}\left[\frac{\hat{\omega}}{2}\overline{\left(\frac{g_z}{g}\right)}(1-g^2)\right]\right),\\
\eta f &= \left(0, 2\operatorname{Re}\left[\frac{i}{2g}(1+g^2)\right], 2\operatorname{Re}\left[\frac{-1}{2g}(1-g^2)\right]\right),\\
\eta^2 f &= \left(0, -2\operatorname{Re}\left[\frac{\varphi}{2}(g-g^{-1})\right], 2\operatorname{Re}\left[\frac{i\varphi}{2}(g-g^{-1})\right]\right),\\
\eta^3 f &= \left(0, 2\operatorname{Re}\left[\frac{-i}{2}\left\{\phi\varphi(g-g^{-1}) + \varphi^2(g+g^{-1})\right\}\right],\right.\\
&\qquad\qquad \left. 2\operatorname{Re}\left[\frac{-1}{2}\left\{\phi\varphi(g+g^{-1}) + \varphi^2(g-g^{-1})\right\}\right]\right).
\end{aligned}$$

By using these equations and condition (2) (c) in Fact 34, namely, $\varphi + \bar{\varphi} = \phi - \bar{\phi} = 0$ at p, we get

$$\langle \xi f(p), \eta\eta f(p)\rangle_{\mathrm{E}} = 0 \quad \text{and} \quad \eta^3 f(p) = \mathbf{0}.$$

Continuing to differentiate f by η, we also have

$$\eta^4 f = \left(0, 2\,\mathrm{Re}\left[\frac{1}{2}\left\{\varphi(\Phi\varphi + \phi^2 + \varphi^2)(g - g^{-1}) + 3\varphi^2\phi(g + g^{-1})\right\}\right],\right.$$
$$\left. 2\,\mathrm{Re}\left[\frac{-i}{2}\left\{\varphi(\Phi\varphi + \phi^2 + \varphi^2)(g + g^{-1}) + 3\varphi^2\phi(g - g^{-1})\right\}\right]\right),$$

$$\eta^5 f = \Big(0,$$
$$2\,\mathrm{Re}\left[\frac{i}{2}\varphi\left\{(\Gamma\varphi^2 + 4\varphi\phi\Phi + \phi^3 + 6\varphi^2\phi)(g - g^{-1}) + (4\varphi^2\Phi + 7\varphi\phi^2 + \varphi^3)(g + g^{-1})\right\}\right],$$
$$2\,\mathrm{Re}\left[\frac{1}{2}\varphi\left\{(\Gamma\varphi^2 + 4\varphi\phi\Phi + \phi^3 + 6\varphi^2\phi)(g + g^{-1}) + (4\varphi^2\Phi + 7\varphi\phi^2 + \varphi^3)(g - g^{-1})\right\}\right]\Big),$$

where we set $\Gamma := \frac{g}{g_z}\left[\frac{g}{g_z}\left\{\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right\}_z\right]_z$. It follows that $\eta^5 f$ does not vanish at p. Thus we can apply Lemma 35 for maxfaces, and

$$A = \det(\xi f, \eta\eta f, \eta^5 f)(p). \tag{3.7}$$

By these functions, we have

$$A = 32i\varphi(p)^5|\hat{\omega}(p)|^2\,\mathrm{Re}[\Phi(p)]$$

at p, and condition (c) implies $A \neq 0$. Hence the condition (2)(b) follows if the conditions (2)(a) and (2)(c) are satisfied.

To end of proof, we should check the condition (2)(d). It follows that $A = 32i\varphi(p)^5|\hat{\omega}(p)|^2\,\mathrm{Re}[\Phi(p)]$ and $B = -i\varphi(p)|g_z(p)|^4|\hat{\omega}(p)|^2\,\mathrm{Re}[\Phi(p)]$ hold. Thus

$$AB = 32\varphi(p)^6|\hat{\omega}(p)|^4|g_z(p)|^4\,\mathrm{Re}[\Phi(p)]^2 < 0 \tag{3.8}$$

holds since $\varphi(p) \in i\boldsymbol{R}$ by the conditions (2)(a) and (2)(c). Therefore we have the assertion. $\boxed{QED}$

By (3.8), we see that a maxface f cannot have cS_1^+ singularities.

Let (g, ω) be a Weierstrass data of a maxface f in $\boldsymbol{R}^{2,1}$. Then we call $\tilde{f}$ a *conjugate maxface* if the Weierstrass data of $\tilde{f}$ is given by $(g, i\omega)$ (cf. [11]). It is known that a dualty between swallowtails and cuspidal cross caps on maxfaces holds ([11, Corollary 2.5]). By Theorem 30, we obtain a new duality between singularities on maxfaces.

Corollary 37. Let $f : \Sigma \to \boldsymbol{R}^{2,1}$ be a maxface with a Weierstrass data (g, ω). Then f at p is a cuspidal butterfly (resp. cuspidal S_1^- singularity) if and only if its conjugate $\tilde{f}$ at p is a cuspidal S_1^- singularity (resp. cuspidal butterfly).

4 Examples

Here we show examples of maxfaces with cuspidal butterflies or cS_1^- singularities.

Maxface with cuspidal butterflies. Define Weierstrass data as follows:

$$g(z) = -e^z + \frac{1}{\sqrt{2}}, \quad \omega = \hat{\omega} dz = -\frac{i}{2} e^{-z} dz.$$

This data was first found in [3] as the data of a special case of maximal Thomsen-type families. Maximal Thomsen families are defined as both maxfaces in $\boldsymbol{R}^{2,1}$ and affine minimal surfaces (see [26]).

This surface is periodic in the v-direction, where $z = u+iv$ and for simplicity we restrict the domain to $(u, v) \in \boldsymbol{R} \times [-\pi, \pi)$. Then, we have

$$S(f) = \left\{ (u, v) \in \boldsymbol{R} \times [-\pi, \pi) \,\middle|\, e^u = \frac{1}{\sqrt{2}} \cos v + \sqrt{1 - \frac{1}{2} \sin^2 v} \right\}.$$

We notice that at $p_1 = (\log(1/\sqrt{2}), \pm\pi/2)$

$$\varphi(p_1) = \frac{g_z}{g^2 \hat{\omega}}(p_1) = \pm 1, \quad \phi(p_1) = \frac{g}{g_z} \left(\frac{g_z}{g^2 \hat{\omega}} \right)_z (p_1) = 2i,$$
$$\Phi(p_1) = \frac{g}{g_z} \left\{ \frac{g}{g_z} \left(\frac{g_z}{g^2 \hat{\omega}} \right)_z \right\}_z (p_1) = \mp(2 \pm 2i),$$

where the sign $\pm$ corresponds to one of p_1. Thus this surface have cuspidal butterflies at p_1.

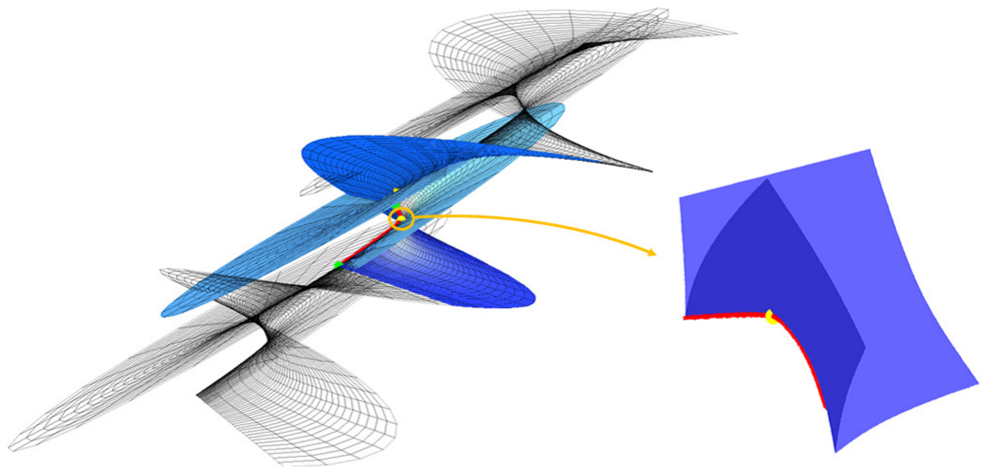

Figure 3. A special case of a maximal Thomsen surface. It has cuspidal butterflies at yellow points, swallowtails at green points and cuspidal edges on the red curve.

Maxface with cuspidal S_1^-. Consider the conjugate surface of the above example:

$$g(z) = -e^z + \frac{1}{\sqrt{2}}, \quad \omega = \hat{\omega} dz = \frac{1}{2} e^{-z} dz.$$

This data gives the conjugate surface of a special case of the maximal Thomsen-type family. In [3] and [25], it was found that the resulting surfaces of this data have planar curvature lines in both the u and v directions, called maximal Bonnet-type surfaces.

This conjugate surface is also periodic in the v-direction, and for simplicity we restrict the domain to $(u, v) \in \boldsymbol{R} \times [-\pi, \pi)$. We also notice that at $p_1 = (\log(1/\sqrt{2}), \pm\pi/2)$

$$\varphi(p_1) = \frac{g_z}{g^2 \hat{\omega}}(p_1) = \pm i, \quad \phi(p_1) = \frac{g}{g_z} \left(\frac{g_z}{g^2 \hat{\omega}} \right)_z (p_1) = -2,$$
$$\Phi(p_1) = \frac{g}{g_z} \left\{ \frac{g}{g_z} \left(\frac{g_z}{g^2 \hat{\omega}} \right)_z \right\}_z (p_1) = \mp i(2 \pm 2i),$$

where the sign $\pm$ corresponds to one of p_1. Thus this conjugate surface have cuspidal S_1^- at p_1.

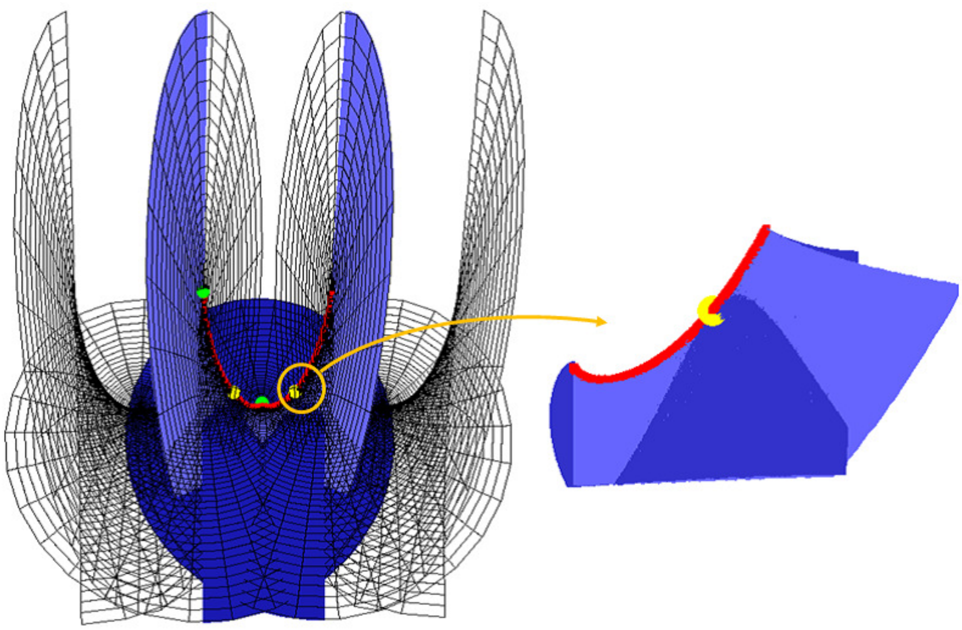

Figure 4. A special case of a maximal Bonnet-type surface. It has cS_1^- singularities at yellow points, swallowtails at green points and cuspidal edges on the red curve.

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