Polyharmonic maps into the Euclidean space

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Abstract. We study polyharmonic \((k\text{-harmonic})\) maps between Riemannian manifolds with finite \(j\)-energies \((j = 1, \ldots, 2k - 2)\). We show that if the domain is complete and the target is the Euclidean space, then such a map is harmonic.

Keywords: harmonic map, polyharmonic map, Chen’s conjecture, generalized Chen’s conjecture

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Introduction

This paper is an extension of our previous work ([25]) to polyharmonic maps. Harmonic maps play a central role in geometry; they are critical points of the energy functional \(E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 v_g\) for smooth maps \(\varphi\) of \((M, g)\) into \((N, h)\). The Euler-Lagrange equations are given by the vanishing of the tension filed \(\tau(\varphi)\). In 1983, J. Eells and L. Lemaire [6] extended the notion of harmonic map to polyharmonic map, which are, by definition, critical points of the \(k\)-energy \((k \geq 2)\)

\[
E_k(\varphi) = \frac{1}{2} \int_M |(d + \delta)^k \varphi|^2 v_g.
\]

(0.1)

After G.Y. Jiang [15] studied the first and second variation formulas of \(E_2\) \((k = 2)\), extensive studies in this area have been done (for instance, see [2], [4], [18], [19], [22], [26], [28], [12], [13], [14], etc.). Notice that harmonic maps are always polyharmonic by definition.
For harmonic maps, it is well known that:
If a domain manifold \((M, g)\) is complete and has non-negative Ricci curvature, and the sectional curvature of a target manifold \((N, h)\) is non-positive, then every energy finite harmonic map is a constant map (cf. [29]).

In our previous paper, we showed that

**Theorem 1.** [25] Let \((M, g)\) be a complete Riemannian manifold, and the curvature of \((N, h)\) is non-positive. Then,

1. every biharmonic map \(\varphi : (M, g) \rightarrow (N, h)\) with finite energy and finite bienergy must be harmonic.
2. In the case \(\text{Vol}(M, g) = \infty\), every biharmonic map \(\varphi : (M, g) \rightarrow (N, h)\) with finite bienergy is harmonic.

Now, in this paper, we want to extend it to \(k\)-harmonic maps \((k \geq 2)\). Indeed, we will show

**Theorem 2. Theorems 4 and 6** Let \((M, g)\) be a complete Riemannian manifold, and \((N, h)\), the \(n\)-dimensional Euclidean space. Then,

1. every \(k\)-harmonic map \(\varphi : (M, g) \rightarrow (N, h)\) \((k \geq 2)\) with finite \(j\)-energies for all \(j = 1, 2, \ldots, 2k - 2\), must be harmonic.
2. In the case of \(\text{Vol}(M, g) = \infty\), every \(k\)-harmonic map \(\varphi : (M, g) \rightarrow (N, h)\) with finite \(j\)-energy for all \(j = 2, 4, \ldots, 2k - 2\), is harmonic.

Theorem 2 gives an affirmative answer to the generalized B.Y. Chen’s conjecture (cf. [4]) on \(k\)-harmonic maps \((k \geq 2)\) under the \(L^2\)-conditions.

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1 Preliminaries and statement of main theorem

In this section, we prepare materials for the first variational formula for the biharmonic maps. Let us recall the definition of a harmonic map \(\varphi : (M, g) \rightarrow (N, h)\), of a compact Riemannian manifold \((M, g)\) into another Riemannian manifold \((N, h)\), which is an extremal of the energy functional defined by

\[
E(\varphi) = \int_M e(\varphi) v_g,
\]
where $e(\varphi) := \frac{1}{2}|d\varphi|^2$ is called the energy density of $\varphi$. That is, for any variation $\{\varphi_t\}$ of $\varphi$ with $\varphi_0 = \varphi$,

$$
\frac{d}{dt} \bigg|_{t=0} E(\varphi_t) = -\int_M h(\tau(\varphi), V) v_g = 0, \quad (1.1)
$$

where $V \in \Gamma(\varphi^{-1}TN)$ is a variation vector field along $\varphi$ which is given by $V(x) = \frac{d}{dt}|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N$, $(x \in M)$, and the tension field is given by $\tau(\varphi) = \sum_{i=1}^m B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN)$, where $\{e_i\}_{i=1}^m$ is a locally defined frame field on $(M, g)$, and $B(\varphi)$ is the second fundamental form of $\varphi$ defined by

$$
B(\varphi)(X,Y) = (\tilde{\nabla}d\varphi)(X,Y) = (\tilde{\nabla}_X d\varphi)(Y) = \tilde{\nabla}_X (d\varphi(Y)) - d\varphi(\nabla_X Y), \quad (1.2)
$$

for all vector fields $X,Y \in \mathfrak{X}(M)$. Here, $\nabla$, and $\nabla^N$, are the Levi-Civita connections of $(M,g)$, $(N,h)$, respectively, and $\tilde{\nabla}$, and $\tilde{\nabla}$ are the induced ones on $\varphi^{-1}TN$, and $T^*M \otimes \varphi^{-1}TN$, respectively. By (2), $\varphi$ is harmonic if and only if $\tau(\varphi) = 0$.

The second variation formula is given as follows. Assume that $\varphi$ is harmonic. Then,

$$
\frac{d^2}{dt^2} \bigg|_{t=0} E(\varphi_t) = \int_M h(J(V), V) v_g, \quad (1.3)
$$

where $J$ is an elliptic differential operator, called the Jacobi operator acting on $\Gamma(\varphi^{-1}TN)$ given by

$$
J(V) = \tilde{\Delta}V - R(V), \quad (1.4)
$$

where $\tilde{\Delta}V = \tilde{\nabla}^\nabla \nabla V = -\sum_{i=1}^m \{\tilde{\nabla}_e_i \nabla_{e_i} V - \nabla_{\nabla_e_i e_i} V\}$ is the rough Laplacian and $R$ is a linear operator on $\Gamma(\varphi^{-1}TN)$ given by $R(V) = \sum_{i=1}^m R^N(V, d\varphi(e_i))d\varphi(e_i)$, and $R^N$ is the curvature tensor of $(N,h)$ given by $R^N(U, V) = \nabla^N_U \nabla^N_V - \nabla^N_V \nabla^N_U - \nabla_{[V,U]}^N$ for $U, V \in \mathfrak{X}(N)$.


$$
E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g, \quad (1.5)
$$

where $|V|^2 = h(V,V), V \in \Gamma(\varphi^{-1}TN)$. The first variation formula of the bienergy functional is given by

$$
\frac{d}{dt} \bigg|_{t=0} E_2(\varphi_t) = -\int_M h(\tau_2(\varphi), V) v_g. \quad (1.6)
$$
Here,
\[
\tau_2(\varphi) := J(\tau(\varphi)) = \Delta(\tau(\varphi)) - R(\tau(\varphi)),
\]
which is called the bitension field of \( \varphi \), and \( J \) is given in (5).

A smooth map \( \varphi \) of \((M, g)\) into \((N, h)\) is said to be biharmonic if \( \tau_2(\varphi) = 0 \).

Now let us recall the definition of the \( k \)-energy \( E_k(\varphi) \) \((k \geq 2)\):

**Definition 1.** The \( k \)-energy \( E_k(\varphi) \) \((k \geq 2)\) is defined formally ([7]) by
\[
E_k(\varphi) := \frac{1}{2} \int_M |(d + \delta)^k \varphi|^2 v_g
\]
for every smooth map \( \varphi \in C^\infty(M, N) \). Then, it is given ([12], p. 270, Lemma 40) by the following formula:
\[
E_k(\varphi) = \begin{cases} 
\frac{1}{2} \int_M |W^{\ell}_\varphi|^2 v_g & \text{(if } k \text{ is even, say } 2\ell) \\
\frac{1}{2} \int_M |\nabla W^{\ell}_\varphi|^2 v_g & \text{(if } k \text{ is odd, say } 2\ell + 1) 
\end{cases}
\]
\[
(1.8)
\]
Here, \( W^{\ell}_\varphi \) is given as, by definition,
\[
W^{\ell}_\varphi := \sum_{\ell_1} \cdots \sum_{\ell_{\ell-1}} \tau(\varphi) \in \Gamma(\varphi^{-1}TN).
\]
\[
(1.9)
\]
For \( k = 1 \), that is, \( \ell = 0 \), we define \( W^0_\varphi = \varphi \), also.

Then, the definition and the first variation formula for the \( k \)-energy \( E_k \) are given as follows:

**Definition 2.** \( k \)-harmonic map For each \( k = 2, 3, \cdots \), and a smooth map \( \varphi : (M, g) \rightarrow (N, h) \), is \( k \)-harmonic if
\[
\left. \frac{d}{dt} \right|_{t=0} E_k(\varphi_t) = 0
\]
for every smooth variation \( \varphi_t : M \rightarrow N \ (-\varepsilon < t < \varepsilon) \) with \( \varphi_0 = \varphi \).

Then, we have ([12], p.269, Theorem 39)

**Theorem 3.** The first variation formula of the \( k \)-energy Assume that \((N, h) = (\mathbb{R}^n, h_{\mathbb{R}^n})\) is the \( n \)-dimensional Euclidean space. For every \( k = 2, 3, \cdots \), it holds that
\[
\left. \frac{d}{dt} \right|_{t=0} E_k(\varphi_t) = -\int_M \langle \tau_k(\varphi), V \rangle v_g, 
\]
\[
(1.12)
\]
where $V$ is a variation vector field given by $V(x) = \frac{d}{dt} \big|_{t=0} \varphi_t(x) \in T_{\varphi(x)} N$ ($x \in M$). The $k$-tension field $\tau_k(\varphi)$ is given by

$$\tau_k(\varphi) = J(W^{k-1}_\varphi) = \Delta(W^{k-1}_\varphi), \quad (1.13)$$

where $W^{k-1}_\varphi = \Delta \cdots \Delta \tau(\varphi) \in \Gamma(\varphi^{-1}TN)$.

Thus, $\varphi : (M, g) \to (N, h)$ is $k$-harmonic if and only if $\Delta^{k-1} \tau(\varphi) = 0$ which is equivalent to $W^{k}_\varphi = 0$.

Notice that the formula (14) of the $k$-tension field $\tau_k(\varphi)$ coincides with the $k$-tension field in Theorems 2.2 and 2.3 in [21] in the case that the target space $(N, h)$ is the $n$-dimensional Euclidean space $(N, h) = (\mathbb{R}^n, h_{\mathbb{R}^n})$ because of $R^N \equiv 0$.

Here, we denote by $\nabla W^{\ell}_\varphi = \nabla \varphi = d\varphi$ for $\ell = 0$, and $k = 2\ell + 1 = 1$,

$$E_1(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 v_g.$$  

Then, we can state our main theorem.

**Theorem 4. Main theorem**  Assume that the domain manifold $(M, g)$ is a complete Riemannian manifold, and the target space $(N, h)$ is the $n$-dimensional Euclidean space. Let $\varphi : (M, g) \to (N, h)$ be a $k$-harmonic map ($k \geq 2$). Assume that

1. $E_j(\varphi) < \infty$ for all $j = 2, 4, \ldots, 2k - 2$, and  
2. either  
   $E_j(\varphi) < \infty$ for all $j = 1, 3, \ldots, 2k - 3$, or  
   $\text{Vol}(M, g) = \infty$.

Then, $\varphi : (M, g) \to (N, h)$ is harmonic.

In the case of the $n$-dimensional Euclidean space $(N, h) = (\mathbb{R}^n, h_{\mathbb{R}^n})$, Theorem 4 and the following Theorem 5 are natural extensions of our previous theorem in [25] which is:

**Theorem 5.** Assume that $(M, g)$ is complete and the sectional curvature of $(N, h)$ is non-positive.

1. Every biharmonic map $\varphi : (M, g) \to (N, h)$ with finite energy $E(\varphi) < \infty$ and finite bienergy $E_2(\varphi) < \infty$, is harmonic.
2. In the case $\text{Vol}(M, g) = \infty$, every biharmonic map $\varphi : (M, g) \to (N, h)$ with finite bienergy $E_2(\varphi) < \infty$, is harmonic.
2 The iteration proposition.

By virtue of (10), we have to notice the the energy conditions in (1) and (2) of Theorem 4: Indeed, the condition which $E_j(\varphi) < \infty$ for all $j = 2, 4, \cdots, 2k - 2$ in (1) of Theorem 4 is equivalent to that
\[
\int_M |W^j_\varphi|^2 v_g < \infty \quad (j = 1, 2, \cdots, k - 1),
\]
and the condition which $E_j(\varphi) < \infty$ for all $j = 1, 3, \cdots, 2k - 3$ in (2) of Theorem 4 is equivalent to that
\[
\int_M |\nabla W^j_\varphi|^2 v_g < \infty \quad (j = 0, 1, \cdots, k - 2).
\]
Therefore, to show Theorem 4, we only have to prove the following theorem:

**Theorem 6.** Assume that the domain manifold $(M, g)$ is a complete Riemannian manifold, and the target space $(N, h)$ is the $n$-dimensional Euclidean space. Let $\varphi : (M, g) \to (N, h)$ be a $k$-harmonic map.

Assume that

(1) $\int_M |W^j_\varphi|^2 v_g < \infty$ for all $j = 1, 2, \cdots, k - 1$, and

(2) either
\[
\int_M |\nabla W^j_\varphi|^2 v_g < \infty \quad (j = 0, 1, \cdots, k - 2),
\]
or
\[
\text{Vol}(M, g) = \infty.
\]
Then, $\varphi : (M, g) \to (N, h)$ is harmonic.

To prove Theorem 6 whose proof will be given in the next section, we need the following iteration proposition:

**Proposition 1.** the iteration method Let $(M, g)$ be a complete Riemannian manifold, and $(N, h)$, an arbitrary Riemannian manifold. Let $\varphi : (M, g) \to (N, h)$ be an arbitrary $C^\infty$ map satisfying that for some $j \geq 2$,
\[
W^j_\varphi = 0. \tag{2.3}
\]
If we assume the following two conditions:

\[
\begin{cases}
(1) & \int_M |W^{j-1}_\varphi|^2 v_g < \infty, \text{ and } \\
(2) & \text{either } \int_M |\nabla W^{j-2}_\varphi|^2 v_g < \infty \text{ or } \text{Vol}(M, g) = \infty,
\end{cases} \tag{2.4}
\]
then, we have
\[ W_j^{j-1} = 0. \tag{2.5} \]

**Remark 1.** Under the assumptions (16), if we have \( W_k = 0 \) for some \( k \geq 2 \), then we have automatically, \( W_1 = \tau(\varphi) = 0 \), i.e., \( \varphi \) is harmonic.

In this section, we give a proof of Proposition 1 which consists of four steps.

*(The first step)* For a fixed point \( x_0 \in M \), and for every \( 0 < r < \infty \), we first take a cut-off \( C^\infty \) function \( \eta \) on \( M \) (for instance, see [16]) satisfying that
\[
\begin{cases}
0 \leq \eta(x) \leq 1 \quad (x \in M), \\
\eta(x) = 1 \quad (x \in B_r(x_0)), \\
\eta(x) = 0 \quad (x \notin B_{2r}(x_0)), \\
|\nabla \eta| \leq \frac{2}{r} \quad (x \in M).
\end{cases}
\tag{2.6}
\]

*(The second step)* Notice that (17) is equivalent to that
\[ \Delta W_j^{j-1} = 0 \tag{2.7} \]
because of \( W_j = \Delta W_j^{j-1} \).

Then, we have
\[
0 = \int_M \langle \eta^2 W_j^{j-1}, \Delta W_j^{j-1} \rangle v_g \\
= \int_M \sum_{i=1}^m \langle \nabla e_i (\eta^2 W_j^{j-1}), \nabla e_i W_j^{j-1} \rangle v_g \\
= \int_M \eta^2 \sum_{i=1}^m |\nabla e_i W_j^{j-1}|^2 v_g + 2 \int_M \sum_{i=1}^m \eta e_i(\eta) \langle W_j^{j-1}, \nabla e_i W_j^{j-1} \rangle v_g. \tag{2.8}
\]

By moving the second term in the last equality of (22) to the left hand side, we have
\[
\int_M \eta^2 \sum_{i=1}^m |\nabla e_i W_j^{j-1}|^2 = -2 \int_M \sum_{i=1}^m \langle \eta \nabla e_i W_j^{j-1}, e_i(\eta) W_j^{j-1} \rangle v_g \\
= -2 \int_M \sum_{i=1}^m \langle S_i, T_i \rangle v_g, \tag{2.9}
\]
where we put \( S_i := \eta \nabla e_i W_j^{j-1} \), and \( T_i := e_i(\eta) W_j^{j-1} \) (\( i = 1 \cdots m \)).
Now let recall the following inequality:

\[ \pm 2 \langle S_i, T_i \rangle \leq \varepsilon |S_i|^2 + \frac{1}{\varepsilon} |T_i|^2 \]  

(2.10)

for all positive \( \varepsilon > 0 \) because of the inequality \( 0 \leq |\sqrt{\varepsilon} S_i \pm \frac{1}{\sqrt{\varepsilon}} T_i|^2 \). Therefore, for (24), we obtain

\[ -2 \int_M \sum_{i=1}^{m} \langle S_i, T_i \rangle \, v_g \leq \varepsilon \int_M \sum_{i=1}^{m} |S_i|^2 \, v_g + \frac{1}{\varepsilon} \int_M \sum_{i=1}^{m} |T_i|^2 \, v_g. \]  

(2.11)

If we put \( \varepsilon = \frac{1}{2} \), we obtain, by (23) and (25),

\[ \int_M \eta^2 \sum_{i=1}^{m} |\nabla e_i W_j^{j-1}|^2 \, v_g \leq \frac{1}{2} \int_M \sum_{i=1}^{m} \eta^2 |\nabla e_i W_j^{j-1}|^2 \, v_g + 2 \int_M \sum_{i=1}^{m} e_i(\eta)^2 |W_j^{j-1}|^2 \, v_g. \]  

(2.12)

Thus, by (26) and (20), we obtain

\[ \int_M \eta^2 \sum_{i=1}^{m} |\nabla e_i W_j^{j-1}|^2 \, v_g \leq 4 \int_M |\nabla \eta|^2 |W_j^{j-1}|^2 \, v_g \leq 16 \int_M |W_j^{j-1}|^2 \, v_g. \]  

(2.13)

(The third step) By definition of \( \eta \) in the first step, (27) turns out that

\[ \int_{B_r(x_0)} |\nabla W_j^{j-1}|^2 \, v_g \leq \frac{16}{r^2} \int_M |W_j^{j-1}|^2 \, v_g. \]  

(2.14)

Here, recall our assumption that \( (M, g) \) is complete and non-compact, and (1) \( \int_M |W_j^{j-1}|^2 \, v_g < \infty \). When we tend \( r \to \infty \), the right hand side of (26) goes to zero, and the left hand side of (26) goes to \( \int_M |\nabla W_j^{j-1}|^2 \, v_g \). Thus, we obtain

\[ 0 \leq \int_M |\nabla W_j^{j-1}|^2 \, v_g \leq 0, \]

which implies that

\[ \nabla W_j^{j-1} = 0 \]  

(2.15)

everywhere on \( M \).
(The fourth step) (a) In the case that $\int_M |\nabla W^j_{\varphi} - 2\varphi|^2 v_g < \infty$, let us define a smooth 1-form $\alpha$ on $M$ by

$$\alpha(X) := \langle W^j_{\varphi} - 1, \nabla_X W^j_{\varphi} - 2 \varphi \rangle \quad (X \in \mathfrak{X}(M)).$$

Then, we have:

$$\text{div}(\alpha) = -|W^j_{\varphi} - 1|^2.$$  \hfill (2.17)

Because we have

$$\text{div}(\alpha) = m \sum_{i=1}^m (\nabla_{e_i} \alpha)(e_i)$$

$$= \sum_{i=1}^m \left\{ e_i \alpha(e_i) - \alpha(\nabla_{e_i} e_i) \right\}$$

$$= \sum_{i=1}^m \left\{ e_i \left( \langle W^j_{\varphi} - 1, \nabla_{e_i} W^j_{\varphi} - 2 \rangle \right) - \langle W^j_{\varphi} - 1, \nabla_{\nabla_{e_i} e_i} W^j_{\varphi} - 2 \rangle \right\}$$

$$= \sum_{i=1}^m \left\{ \langle \nabla_{e_i} W^j_{\varphi} - 1, \nabla_{e_i} W^j_{\varphi} - 2 \rangle + \langle W^j_{\varphi} - 1, \nabla_{e_i} \nabla_{e_i} W^j_{\varphi} - 2 \rangle \right. - \left. \langle W^j_{\varphi} - 1, \nabla_{\nabla_{e_i} e_i} W^j_{\varphi} - 2 \rangle \right\}$$

$$= \langle W^j_{\varphi} - 1, -\Delta W^j_{\varphi} - 2 \rangle \quad (\text{because of (29) and definition of } \overline{\Delta})$$

$$= -|W^j_{\varphi} - 1|^2,$$  \hfill (2.18)

which is (31).

Furthermore, we have

$$\int_M |\alpha| v_g < \infty.$$  \hfill (2.19)

Because we have, by definition of $\alpha$ in (30),

$$\int_M |\alpha| v_g = \int_M |\langle W^j_{\varphi} - 1, \nabla W^j_{\varphi} - 2 \rangle| v_g$$

$$\leq \left( \int_M |W^j_{\varphi} - 1|^2 v_g \right)^{\frac{1}{2}} \left( \int_M |\nabla W^j_{\varphi} - 2|^2 v_g \right)^{\frac{1}{2}}$$

$$< \infty$$  \hfill (2.20)

because of our assumptions $\int_M |W^j_{\varphi} - 1|^2 v_g < \infty$ and $\int_M |\nabla W^j_{\varphi} - 2|^2 v_g < \infty$. Thus, we can apply Gaffney’s theorem to this $\alpha$ (cf. [10], and Theorem 4.1 in
Appendix in [25]). We obtain
\[ 0 = \int_M \text{div}(\alpha) v_g = -\int_M |W_{\varphi}^{-1}|^2 v_g, \tag{2.21} \]
which implies that $W_{\varphi}^{-1} = 0$.

(b) In the case that $\text{Vol}(M,g) = \infty$, we first notice that $|W_{\varphi}^{-1}|^2$ is constant on $M$, say $C_0$. Because for every $X \in \mathfrak{X}(M)$, we have
\[ X |W_{\varphi}^{-1}|^2 = 2 \langle \nabla_X W_{\varphi}^{-1}, W_{\varphi}^{-1} \rangle = 0 \tag{2.22} \]
due to (29). Then, due to the assumption (1) of Proposition 1, and the above, we obtain
\[ \infty > \int_M |W_{\varphi}^{-1}|^2 v_g = C_0 \int_M v_g = C_0 \text{Vol}(M,g). \tag{2.23} \]
By our assumption that $\text{Vol}(M,g) = \infty$, (37) implies that $C_0 = 0$. We obtain $W_{\varphi}^{-1} \equiv 0$. We obtain Proposition 1.

Proof of Theorem 6. We apply Proposition 1 to our map $\varphi : (M,g) \to (N,h)$, then the iteration procedure works well since $\varphi$ is $k$-harmonic, i.e., $W_{\varphi}^k = 0$. Then, we have $W_{\varphi}^{k-1} = 0$, and then we have $W_{\varphi}^{k-2} = 0$, etc. Finally, we obtain $\tau(\varphi) = W_{\varphi}^1 = 0$. Thus, $\varphi : (M,g) \to (N,h)$ is harmonic. We obtain Theorem 6.

References


