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Almost semi-braces and the Yang-Baxter equation

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Abstract. In this note we find new set-theoretic solutions of the Yang-Baxter equation through almost left semi-braces, a new structure that is a generalization of left semi-braces.

Keywords: Quantum Yang-Baxter equation, set-theoretical solution, semi-brace.

MSC 2000 classification: 16T25, 16Y99, 16N20, 81R50

Introduction

In order to find new solutions of the Yang-Baxter equation, Drinfeld [5] asked the question of finding the so-called set-theoretic solutions on an arbitrary non-empty set. We recall that, if X is a non-empty set, a function $r : X \times X \to X \times X$ is called a *set-theoretic solution* of the Yang-Baxter equation if

$$r_1 r_2 r_1 = r_2 r_1 r_2$$

where $r_1 := r \times id_X$ and $r_2 := id_X \times r$.
After the seminal papers of Etingof, Schedler and Soloviev [6] and of Gateva-Ivanova and M. Van den Bergh [7], many papers about this subject appeared and many links to different topics pointed out. In this context Rump [9] introduced braces, a generalization of radical rings. As reformulated by Cedó, Jespers and Okniński [4], a *left brace* is a set B with two operations $+$ and $\circ$ such that $(B, +)$ is an abelian group, $(B, \circ)$ is a group and

$$a \circ (b + c) + a = a \circ b + a \circ c$$

holds for all $a, b, c \in B$. Recently Guarnieri and Vendramin [8] introduced skew braces, a generalization of braces. A *skew left brace* is a set B with two operations $+$ and $\circ$ such that $(B, +)$ and $(B, \circ)$ are groups and

$$a \circ (b + c) = a \circ b - a + a \circ c$$

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holds for all $a, b, c \in B$. Note that, recently, Brzeziński [1] introduced a generalization of skew braces, the skew trusses. More precisely, a *skew left truss* is a set B with two operations $+$ and $\circ$ and a function $\sigma : B \to B$ such that $(B, +)$ is a group and $(B, \circ)$ is a semigroupsand

$$a \circ (b + c) = a \circ b - \sigma(a) + a \circ c$$

holds for all $a, b, c \in B$. Let us note that the skew left trusses are related to circle algebras, structures introduced by Catino and Rizzo in [3].
A generalization of skew braces that is useful to find set-theoretic solutions of the Yang-Baxter equation is the semi-brace, introduced by Catino, Colazzo and Stefanelli [2]. A *left semi-brace* is a set B with two operations $+$ and $\circ$ such that $(B, +)$ is a left cancellative semigroup, $(B, \circ)$ is a group and

$$a \circ (b + c) = a \circ b + a \circ (a^- + c)$$

holds for all $a, b, c \in B$, where we denote by a^- the inverse of a with respect to $\circ$.

In this note we introduce a new structure, the almost semi-brace, a generalization of semi-brace. More precisely, an *almost left semi-brace* is a set B with two operations $+$ and $\circ$ and a map $\iota : B \to B$ satisfying such that $(B, +)$ is a left cancellative semigroup, $(B, \circ)$ is a group and

$$a \circ (b + c) = a \circ b + a \circ (\iota(a) + c)$$

holds for all $a, b, c \in B$. Then we show that given any almost left semi-brace B, the function $r : B \times B \to B \times B$ given by

$$r(a, b) = (\ a \circ (\iota(a) + b),\ \ (\iota(a) + b)^- \circ b\ )$$

is a set-theoretic solution.

1 Basic results

Recall that a semigroup $(B, +)$ is said to be *left cancellative* if $a + b = a + c$ implies that $b = c$, for all $a, b, c \in B$. Note that in a left cancellative semigroups every idempotent is a left identity.

Definition 1. Let B be a set with two operations $+$ and $\circ$ such that $(B, +)$ is a left cancellative semigroup, $(B, \circ)$ is a group and there exists a function $\iota : B \to B$ such that, for all $a, b \in B$,

$$\iota(a \circ b) = b^- \circ \iota(a), \quad (\iota(a) + b) \circ \iota(1) = \iota(a) + b \circ \iota(1) \tag{1.1}$$

where b^- is the inverse of b with respect $\circ$ and 1 is the identity of $(B, \circ)$.
We say that $(B,\, +,\, \circ,\, \iota)$ is an *almost left semi-brace* if

$$a \circ (b + c) = a \circ b + a \circ (\iota(a) + c), \tag{1.2}$$

for all $a, b, c \in B$.

If $(B, +, \circ)$ is a left semi-brace, then it is an almost semi-brace with $\iota(a) = a^-$, for every $a \in B$, and viceversa.
Examples of almost left semi-braces can be obtained by any group. In fact, if $(E, \circ)$ is a group, then $(E, +, \circ, \iota)$, where $a + b = b$ for all $a, b \in E$ and $\iota : E \to E, a \mapsto a^- \circ e$ with e a fixed element of E, is an almost semi-brace.

Definition 2. Let $(B_1, +_1, \circ_1, \iota_1)$ and $(B_2, +_2, \circ_2, \iota_2)$ almost left semi-braces. A function $f : B_1 \to B_2$ is a *homomorphism of almost left semi-braces* if f is a semigroup homomorphism from $(B_1, +_1)$ to $(B_2, +_2)$, f is a group homomorphism from $(B_1, \circ_1)$ to $(B_2, \circ_2)$ and, $f\iota_1 = \iota_2 f$.

Note that a semi-brace $(B, +, \circ)$ reviewed as almost semi-brace can not be isomorphic to an almost semi-brace $(B, +, \circ, \iota_B)$ with $\iota_B(1) \neq 1$. Indeed such isomorphism f have to satisfy $\iota_B(1) = \iota_B f(1) = f(1) = 1$.

Proposition 1. *Let $(B,\, +,\, \circ,\, \iota)$ be an almost left semi-brace. Then, $\iota(1)$ is a left identity of $(B, +)$ and, $\iota(a) = a^- \circ \iota(1)$ for every $a \in B$. Moreover, the function ι is bijective.*

Proof. By (1.2) we have

$$\iota(1) + \iota(1) = 1 \circ (\iota(1) + \iota(1)) = 1 \circ \iota(1) + 1 \circ (\iota(1) + \iota(1)) = \iota(1) + \iota(1) + \iota(1)$$

and by left cancellativity $\iota(1) = \iota(1) + \iota(1)$. Thus, $\iota(1)$ is a left identity of $(B, +)$.
Now, if $a \in B$, by (1.1) we have $\iota(a) = \iota(1 \circ a) = a^- \circ \iota(1)$.
Finally, ι is bijective. In fact, if $a, b \in B$ and $\iota(a) = \iota(b)$, then $a^- \circ \iota(1) = b^- \circ \iota(1)$, so $a = b$. Moreover, if $b \in B$, then $\iota(\iota(1) \circ b^-) = b \circ \iota(1)^- \circ \iota(1) = b$.

QED

We close this section with a pair of results that are useful for the next section.

Proposition 2. *Let $(B,\, +,\, \circ,\, \iota)$ be an almost left semi-brace and $a \in B$. Then, the function*

$$\lambda_a : B \to B,\ b \mapsto a \circ (\iota(a) + b)$$

is an automorphism of the semigroup $(B, +)$ and $\lambda_a^{-1} = \lambda_{a^-}$. Moreover, the function λ from the group $(B, \circ)$ into the group of the automorphisms of $(B, +)$ given by $\lambda(b) = \lambda_b$, for every $b \in B$, is a homomorphism.

Proof. Let $a, b, c \in B$. Then

$$\begin{aligned}\lambda_a(b+c) &= a\circ(\iota(a)+b+c) = a\circ(\iota(a)+b)+a\circ(\iota(a)+c)\\ &= \lambda_a(b)+\lambda_a(c)\end{aligned}$$

Moreover,

$$\begin{aligned}\lambda_{a\circ b}(c) &= a\circ b\circ(\iota(a\circ b)+c) = a\circ(b\circ\iota(a\circ b)+b\circ(\iota(b)+c))\\ &= a\circ(b\circ b^-\circ\iota(a)+b\circ(\iota(b)+c)) = a\circ(1\circ\iota(a)+b\circ(\iota(b)+c)\\ &= \lambda_a\lambda_b(c)\end{aligned}$$

Finally, $\lambda_1(c) = 1\circ(\iota(1)+c) = \iota(1)+c = c$, for every $c \in B$, and so $\lambda_a\lambda_{a^-} = \lambda_{a\circ a^-} = id_B = \lambda_{a^-\circ a} = \lambda_{a^-}\lambda_a$.
Therefore, λ is a homomorphism from the group $(B, \circ)$ into the group $Aut(B, +)$ of the automorphisms of $(B, +)$.

QED

Proposition 3. *Let $(B, +, \circ, \iota)$ be an almost left semi-brace and let*

$$\rho_b : B \to B,\ a \mapsto (\iota(a)+b)^-\circ b$$

for every $b \in B$. Then the function ρ from the group $(B, \circ)$ into the monoid B^B of the functions of B into itself given by $\rho(b) = \rho_b$, for every $b \in B$, is a semigroup antihomomorphism.

Proof. Let $a, b, c \in B$. Then

$$\begin{aligned}\rho_{b\circ c}(a) &= (\iota(a)+b\circ c)^-\circ b\circ c = (b^-\circ(\iota(a)+b\circ c))^-\circ c\\ &= (b^-\circ(\iota(a)+b\circ(\iota(1)+c))^-\circ c\\ &= (b^-\circ(\iota(a)+b\circ\iota(1)+b\circ(\iota(b)+c))^-\circ c\\ &= (b^-\circ(\iota(a)+\iota(b^-)+b\circ(\iota(b)+c))^-\circ c\\ &= (b^-\circ(\iota(a)+\iota(b^-))+b^-\circ(\iota(b^-)+b\circ(\iota(b)+c))^-\circ c\\ &= (b^-\circ(\iota(a)+\iota(b^-))+b^-\circ(b\circ\iota(1)+b\circ(\iota(b)+c))^-\circ c\\ &= (b^-\circ(\iota(a)+\iota(b^-))+b^-\circ b\circ(\iota(1)+c))^-\circ c\\ &= (b^-\circ(\iota(a)+\iota(b^-))+c)^-\circ c\\ &= (b^-\circ(\iota(a)+b\circ\iota(1))+c)^-\circ c\\ &= (b^-\circ\iota(1\circ(\iota(a)+b)^-)+c)^-\circ c\\ &= (\iota((\iota(a)+b)^-\circ b)+c)^-\circ c = (\iota(\rho_b(a))+c)^-\circ c\\ &= \rho_c\rho_b(a)\end{aligned}$$

QED

2 Almost semi-braces and solutions

In this section we obtain a left non-degenerate solution of the Yang-Baxter equation from every almost semi-braces.

Theorem 1. *Let* $(B, +, \circ, \iota)$ *be an almost semi-brace. Then the function* $r : B \times B \to B \times B$ *defined by*

$$r(a,b) = (\, a \circ (\iota(a) + b), \ \ (\iota(a) + b)^{-} \circ b\,)$$

is a left non-degenerate solution of the Yang-Baxter equation.

Proof. First for all, we remark that if $a, b \in B$, then

$$\lambda_a(b) \circ \rho_b(a) = a \circ b. \tag{2.1}$$

Now, let $a, b, c \in B$ and set

$$\begin{aligned} (t_1, t_2, t_3) : &= r_1 r_2 r_1(a,b,c) \\ &= (\lambda_{\lambda_a(b)}\lambda_{\rho_b(a)}(c),\ \rho_{\lambda_{\rho_b(a)}(c)}(\lambda_a(b)),\ \rho_c\rho_b(a)) \end{aligned}$$

and

$$\begin{aligned} (s_1, s_2, s_3) : &= r_2 r_1 r_2(a,b,c) \\ &= (\lambda_a\lambda_b(c),\ \lambda_{\rho_{\lambda_b(c)}(a)}(\rho_c(b)),\ \rho_{\rho_c(b)}\rho_{\lambda_b(c)}(a)). \end{aligned}$$

Then we have $t_1 \circ t_2 \circ t_3 = s_1 \circ s_2 \circ s_3$. In fact, by (2.1), we have that

$$\begin{aligned} t_1 \circ t_2 \circ t_3 &= \lambda_{\lambda_a(b)}\lambda_{\rho_b(a)}(c) \circ \rho_{\lambda_{\rho_b(a)}(c)}(\lambda_a(b)) \circ \rho_c\rho_b(a) \\ &= \lambda_a(b) \circ \lambda_{\rho_b(a)}(c) \circ \rho_c\rho_b(a) \\ &= \lambda_a(b) \circ \rho_b(a) \circ c \\ &= a \circ b \circ c \end{aligned}$$

and similarly

$$\begin{aligned} s_1 \circ s_2 \circ s_3 &= \lambda_a\lambda_b(c) \circ \lambda_{\rho_{\lambda_b(c)}(a)}(\rho_c(b)) \circ \rho_{\rho_c(b)}\rho_{\lambda_b(c)}(a) \\ &= \lambda_a\lambda_b(c) \circ \rho_{\lambda_b(c)}(a) \circ \rho_c(b) \\ &= a \circ \lambda_b(c) \circ \rho_c(b) \\ &= a \circ b \circ c \end{aligned}$$

Moreover, by Proposition 2 and (2.1),

$$t_1 = \lambda_{\lambda_a(b)}\lambda_{\rho_b(a)}(c) = \lambda_{\lambda_a(b)\circ\rho_b(a)}(c) = \lambda_{a\circ b}(c) = \lambda_a\lambda_b(c) = s_1$$

and by Proposition 3 and (2.1),

$$s_3 = \rho_{\rho_c(b)}\rho_{\lambda_b(c)}(a) = \rho_{\lambda_b(c)\circ\rho_c(b)}(a) = \rho_{b\circ c}(a) = \rho_c\rho_b(a) = t_3$$

Hence $t_2 = s_2$, since $(B, \circ)$ is a group. Therefore r is a solution of the Yang-Baxter equation. Furthermore, r is left non-degenerate, since λ_b is bijective, for every $b \in B$, by Proposition 2.

QED

Remark. If $(B_1, +, \circ, \iota_1)$ and $(B_2, +, \circ, \iota_2)$ are isomorphic almost semi-brace, then the solutions r_1 and r_2 associated respectively to B_1 and B_2 are isomoprhic (in the sense of [4], p. 105). In fact, if $f : B_1 \to B_2$ is an almost left semi-brace isomorphism, by the equality $f\iota_1 = \iota_2 f$, we obtain $(f \times f)r_1 = r_2(f \times f)$.

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