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Hermite-Hadamard type inequalities for a new class of harmonically convex functions

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Abstract. In this paper, we introduce a new class of harmonically convex functions. Several interesting Hermite-Hadamard type inequalities are established. Results obtained are extensions and generalizations of known results in literature. Applications to special means of real numbers are also given.

Keywords: Harmonically convex functions, harmonically ϕ_{h-s} convex functions, Hermite-Hadamard inequality, fractional integral

MSC 2000 classification: primary 26A51, 26D15, secondary 26A33, 26D10

1 Introduction and Preliminaries

The role played by inequalities in mathematics cannot be undermined. Infact, most mathematical inequalities are basic tools for constructing analytic proofs of many important theorems. Over the years, the study of convex inequalities has has steadily gained the attention of many researchers. Also, many classes of convex functions have been introduced to extend several known inequalities in literature; see [1],[4], [11] and the references therein. An important extension of convex function, the class of h-convex functions, was introduced

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by Varosanec in [12]. This was further generalized in [8] when the ϕ_{h-s} convex function was introduced by the authors. In [4], the class of harmonically convex functions was introduced and was significantly extended in [1] by the class of harmonically h-convex functions.
In this paper, we extend the class of harmonically h-convex functions and then establish some Hermite-Hadamard type inequalities using a new class of harmonically convex function.

Theorem 1 (Hermite-Hadamard inequality, [2])**.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

Definition 1. [1, 4] Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be:

(i). harmonically convex if for every $x, y \in I$, $t \in [0,1]$,

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t)f(x) + tf(x)$$

(ii). harmonically Breckner s-convex where $0 < s \leq 1$, if for every $x, y \in I$, $t \in [0,1]$,

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t)^s f(x) + t^s f(y)$$

(iii). harmonically s-Godunova-Levin of the second kind if for every $x, y \in I$, $t \in (0,1)$ and $s \in [0,1]$,

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \frac{1}{(1-t)^s} f(x) + \frac{1}{t^s} f(y)$$

(iv). harmonically Godunova-Levin of the second kind if for every $x, y \in I$, $t \in (0,1)$,

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \frac{1}{(1-t)} f(x) + \frac{1}{t} f(y)$$

(v). harmonically P-function if for every $x, y \in I$, $t \in [0,1]$,

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq f(x) + f(y).$$

Definition 2. [1] Let $h : [0,1] \subseteq J \to \mathbb{R}$ be a non-negative function. A function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ is said to be harmonically h-convex if for all $x, y \in I$ and $t \in (0,1)$,

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq h(1-t)f(x) + h(t)f(y).$$

2 The class of harmonically ϕ_{h-s} convex functions

We introduce the following new concept in order to unify the classes of harmonically convex functions given in the previous section.

Definition 3. Let $h : [0,1] \subseteq J \to (0,\infty)$, $s \in [0,1]$, $t \in (0,1)$ and ϕ be a given real valued function. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval, then $f : I \to \mathbb{R}$ is an harmonically ϕ_{h-s} convex if for all $x, y \in I$,

$$f\left(\frac{\phi(x)\phi(y)}{t\phi(x) + (1-t)\phi(y)}\right) \leq \left(\frac{h(t)}{t}\right)^{-s} f(\phi(y)) + \left(\frac{h(1-t)}{1-t}\right)^{-s} f(\phi(x)). \quad (2.1)$$

Remark 1. We discuss the special cases of the harmonically ϕ_{h-s} convex function. Denote by $\mathcal{H}SX(\phi_{h-s}, I)$, $\mathcal{H}SX(h, I)$, $\mathcal{H}SX(I)$, $\mathcal{H}Q_s(I)$, $\mathcal{H}Q(I)$ and $\mathcal{H}P(I)$ the class of all harmonically ϕ_{h-s} convex functions, harmonically h-convex functions, harmonically convex functions, harmonically s-Godunova-Levin functions, harmonically Godunova-Levin functions and harmonically P-functions respectively.

(i). Then for $\phi(x) = x$, $h(t) \leq t$, one has the following relations;

$$\begin{aligned} \mathcal{H}P(I) = \mathcal{H}Q_0(I) &= \mathcal{H}SX(\phi_{h-0}, I) \subseteq \mathcal{H}SX(\phi_{h-s_1}, I) \\ &\subseteq \mathcal{H}SX(\phi_{h-s_2}, I) \subseteq \mathcal{H}SX(\phi_{h-1}, I) \end{aligned}$$

where $0 \leq s_1 \leq s_2 \leq 1$.

(ii). Let ϕ be the identity function. Observe that

(i). if $s = 0$, then $f \in \mathcal{H}P(I)$

(ii). if $h(t) = 1$ and $s = 1$, then $f \in \mathcal{H}SX(I)$

(iii). if $h(t) = t^{\frac{s}{s+1}}$, then $f \in \mathcal{H}SX(h, I)$

(iv). if $h(t) = 1$, then f is harmonically Breckner s-convex

(v). if $h(t) = t^2$, then $f \in \mathcal{H}Q_s(I)$

(vi). if $h(t) = t^2$ and $s = 1$, then $f \in \mathcal{H}Q(I)$.

(iii). For $t = \frac{1}{2}$, we obtain the Jensen's type harmonically ϕ_{h-s} convex function or the harmonically-arithmetically (HA) ϕ_{h-s} convex function

$$f\left(\frac{2\phi(x)\phi(y)}{\phi(x)+\phi(y)}\right) \leq \left(2h\left(\frac{1}{2}\right)\right)^{-s}(f(\phi(x))+f(\phi(y))). \tag{2.2}$$

Example 1. (i). For $h(t) \leq t^{1-\frac{1}{s}}$ and $\phi(x) = x$, all known examples of harmonically convex functions are harmonically ϕ_{h-s} convex.

(ii). Let $I = [a, b] \subseteq \mathbb{R}\backslash\{0\}$. Consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ defined by $g(x) = f\left(\frac{1}{x}\right)$, then $f \in \mathcal{H}SX(\phi_{h-s}, I)$ if and only if $g \in SX(\phi_{h-s}, J)$ where $\phi(x) = x$, $I = [a, b]$, $J = \left[\frac{1}{b}, \frac{1}{a}\right]$ and $SX(\phi_{h-s}, J)$ is the class of all ϕ_{h-s} convex functions on J.

(iii). Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval, $\phi : I \to I$ and $f : I \to \mathbb{R}$ be a function,

a. if $I \subset (0, \infty)$ and $f \in SX(\phi_{h-s}, I)$ where f is nondecreasing on I then $f \in \mathcal{H}SX(\phi_{h-s}, I)$.

b. if $I \subset (0, \infty)$, $f \in \mathcal{H}SX(\phi_{h-s}, I)$ where f is nonincreasing on I, then $f \in SX(\phi_{h-s}, I)$.

c. if $I \subset (-\infty, 0)$, $f \in \mathcal{H}SX(\phi_{h-s}, I)$ where f is nondecreasing on I then $f \in SX(\phi_{h-s}, I)$.

d. if $I \subset (-\infty, 0)$, $f \in SX(\phi_{h-s}, I)$ where f is nonincreasing on I, then $f \in \mathcal{H}SX(\phi_{h-s}, I)$.

Definition 4. The functions $f, g : \mathbb{R} \to \mathbb{R}$ are said to be similarly ordered if for every $x, y \in \mathbb{R}$, $(f(x) - f(y))(g(x) - g(y)) \geq 0$.

Proposition 1. *Let $f, g \in \mathcal{H}SX(\phi_{h-s}, I)$. If f and g are similarly ordered and $H(t, s) = \left(\frac{h(t)}{t}\right)^{-s} + \left(\frac{h(1-t)}{1-t}\right)^{-s} \leq 1$, then the product $fg \in \mathcal{H}SX(\phi_{h-s}, I)$.*

Proof. Since $f, g \in \mathcal{H}SX(\phi_{h-s}, I)$, then

$$\begin{aligned}
& f\left(\frac{\phi(x)\phi(y)}{t\phi(x)+(1-t)\phi(y)}\right) g\left(\frac{\phi(x)\phi(y)}{t\phi(x)+(1-t)\phi(y)}\right) \\
\leq\ & \left(\frac{h(t)}{t}\right)^{-2s} f(\phi(y))g(\phi(y)) + \left(\frac{h(1-t)}{1-t}\right)^{-2s} f(\phi(x))g(\phi(x)) \\
& + \left(\frac{h(t)}{t}\frac{h(1-t)}{1-t}\right)^{-s} \Big(f(\phi(y))g(\phi(x)) + f(\phi(x))g(\phi(y))\Big) \\
\leq\ & \left(\frac{h(t)}{t}\right)^{-2s} f(\phi(y))g(\phi(y)) + \left(\frac{h(1-t)}{1-t}\right)^{-2s} f(\phi(x))g(\phi(x))
\end{aligned}$$

$$
\begin{aligned}
& + \left(\frac{h(t)}{t}\frac{h(1-t)}{1-t}\right)^{-s}\Big(f(\phi(x))g(\phi(x)) + f(\phi(y))g(\phi(y))\Big) \\
= \;& H(t,s)\left(f(\phi(y))g(\phi(y))\left(\frac{h(t)}{t}\right)^{-s} + f(\phi(x))g(\phi(x))\left(\frac{h(1-t)}{1-t}\right)^{-s}\right) \\
\leq \;& \left(\frac{h(t)}{t}\right)^{-s} f(\phi(y))g(\phi(y)) + \left(\frac{h(1-t)}{1-t}\right)^{-s} f(\phi(x))g(\phi(x)).
\end{aligned}
$$

QED

3 Hermite-Hadamard type inequalities for $\mathcal{H}SX(\phi_{h-s}, I)$

Theorem 2. *Let $f \in \mathcal{H}SX(\phi_{h-s}, I)$. Suppose that $f \in L[a,b]$ where $a, b \in I$ with $a < b$ and ϕ is the identity function, then*

$$
\frac{1}{2^{1-s}\left(h\left(\frac{1}{2}\right)\right)^{-s}} f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx \leq (f(a)+f(b))\int_0^1 \left(\frac{h(t)}{t}\right)^{-s} dt. \tag{3.1}
$$

Proof. Since $f \in \mathcal{H}SX(\phi_{h-s}, I)$, then by setting $t = \frac{1}{2}$ we obtain (2.2). Set $x = \frac{ab}{ta+(1-t)b}$ and $y = \frac{ab}{(1-t)a+tb}$, then

$$
f\left(\frac{2ab}{a+b}\right) \leq \left(2h\left(\frac{1}{2}\right)\right)^{-s}\left(f\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{(1-t)a+tb}\right)\right) \tag{3.2}
$$

Integrating (3.2) with respect to t over $(0,1)$, we get

$$
\begin{aligned}
f\left(\frac{2ab}{a+b}\right) \;&\leq\; \left(2h\left(\frac{1}{2}\right)\right)^{-s}\left(\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt + \int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)dt\right) \\
&\leq\; \left(2h\left(\frac{1}{2}\right)\right)^{-s}\int_0^1 \left(\left(\frac{h(t)}{t}\right)^{-s} f(b) + \left(\frac{h(1-t)}{1-t}\right)^{-s} f(a)\right. \\
&\qquad \left. + \left(\frac{h(t)}{t}\right)^{-s} f(a) + \left(\frac{h(1-t)}{1-t}\right)^{-s} f(b)\right) dt \\
&=\; 2\left(2h\left(\frac{1}{2}\right)\right)^{-s}(f(a)+f(b))\int_0^1 \left(\frac{h(t)}{t}\right)^{-s} dt.
\end{aligned}
$$

But,

$$
\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt = \int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)dt = \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx.
$$

Thus, we have that

$$f\left(\frac{2ab}{a+b}\right) \leq 2\left(2h\left(\frac{1}{2}\right)\right)^{-s} \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx$$

and

$$2\left(2h\left(\frac{1}{2}\right)\right)^{-s} \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx \leq 2\left(2h\left(\frac{1}{2}\right)\right)^{-s} (f(a)+f(b))\int_0^1 \left(\frac{h(t)}{t}\right)^{-s} dt.$$

This gives (3.1), hence the proof. QED

Theorem 3. *Let $f, g \in \mathcal{H}SX(\phi_{h-s}, I)$ be two non-negative functions where $a, b \in I$ with $a < b$. If $fg \in L[a, b]$, then*

$$\begin{aligned}\frac{\phi(a)\phi(b)}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)} \frac{f(x)g(x)}{x^2}\,dx \quad &\leq \quad M(\phi(a),\phi(b))\int_0^1 \left(\frac{h(t)}{t}\right)^{-2s} dt \\ &\quad +N(\phi(a),\phi(b))\int_0^1 \left(\frac{h(t)h(1-t)}{t(1-t)}\right)^{-s} dt\end{aligned}$$

where

$$M(\phi(a),\phi(b)) = f(\phi(a))g(\phi(a)) + f(\phi(b))g(\phi(b))$$
$$N(\phi(a),\phi(b)) = f(\phi(b))g(\phi(a)) + f(\phi(a))g(\phi(b)).$$

Proof. Since $f, g \in \mathcal{H}SX(\phi_{h-s}, I)$, then

$$f\left(\frac{\phi(a)\phi(b)}{t\phi(a)+(1-t)\phi(b)}\right) \leq \left(\frac{h(t)}{t}\right)^{-s} f(\phi(b)) + \left(\frac{h(1-t)}{1-t}\right)^{-s} f(\phi(a)) \quad (3.3)$$

$$g\left(\frac{\phi(a)\phi(b)}{(1-t)\phi(b)+t\phi(a)}\right) \leq \left(\frac{h(t)}{t}\right)^{-s} g(\phi(b)) + \left(\frac{h(1-t)}{1-t}\right)^{-s} g(\phi(a)) \quad (3.4)$$

Multiplying (3.3) by (3.4) and integrating with respect to t over $(0, 1)$, we obtain

$$\begin{aligned}&\int_0^1 f\left(\frac{\phi(a)\phi(b)}{t\phi(a)+(1-t)\phi(b)}\right) g\left(\frac{\phi(a)\phi(b)}{t\phi(a)+(1-t)\phi(b)}\right) dt \\ &\leq (f(\phi(b))g(\phi(b)) + f(\phi(a))g(\phi(a)))\int_0^1 \left(\frac{h(t)}{t}\right)^{-2s} dt \\ &+(f(\phi(b))g(\phi(a)) + f(\phi(a))g(\phi(b)))\int_0^1 \left(\frac{h(t)}{t}\frac{h(1-t)}{1-t}\right)^{-s} dt\end{aligned}$$

By using the substitution $x = \frac{\phi(a)\phi(b)}{t\phi(a)+(1-t)\phi(b)}$, we have

$$\begin{aligned}\frac{\phi(a)\phi(b)}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)}\frac{f(x)g(x)}{x^2}\,dx &\leq M(\phi(a),\phi(b))\int_0^1\left(\frac{h(t)}{t}\right)^{-2s} \\ &+N(\phi(a),\phi(b))\int_0^1\left(\frac{h(t)h(1-t)}{t(1-t)}\right)^{-s}\,dt.\end{aligned}$$

QED

Corollary 1. *Under the conditions of Theorem 3, suppose that f and g are similarly ordered and* $\left(\frac{h(t)}{t}\right)^{-s}+\left(\frac{h(1-t)}{1-t}\right)^{-s}\leq 1$, *then*

$$\frac{\phi(a)\phi(b)}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)}\frac{f(x)g(x)}{x^2}\,dx \leq 2M(\phi(a),\phi(b))\int_0^1\left(\frac{h(t)}{t}\right)^{-s}\,dt.$$

Remark 2. Theorem 2 reduces to Theorem 2.4 in [4] when $h(t) = s = 1$. By setting $\phi(x) = x$, $h(t) = t^{\frac{1}{2}}, s = 1$ in Theorem 3, we obtain Theorem 3.6 in [1]. By setting $\phi(x) = x$, $h(t) = t^{\frac{s}{s+1}}$ in Theorem 2, we obtain Theorem 3.2 in [1] and by applying Remark 1(ii) accordingly, we obtain Corollaries 3.3 – 3.5 in [1]. Corollary 1 reduces to Theorem 3.7 in [1] when $\phi(x) = x$.

4 Inequalities for $\mathcal{H}SX(\phi_{h-s}, I)$ via fractional integration

Let $f \in L[a,b]$, the Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and J^{α}_{b-} of order $\alpha > 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt, \quad x > a \text{ and}$$

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt, \quad x < b$$

respectively where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}\,dt$ and $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$. Fractional integral reduces to the classical integral for $\alpha = 1$.

Hermite-Hadamard type inequalities have been proved for fractional integrals which naturally extends the classical integrals (see for example, [3], [5], [9], [10], [13], [14]). Infact, M. Z. Sarikaya et. al. [9] proved the following Hermite-Hadamard inequalities for convex functions via fractional integrals.

Theorem 4. *Let $f : I \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequality for fractional integrals holds.*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left(J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\right) \leq \frac{f(a)+f(b)}{2}.$$

In this section, we establish Hermite-Hadamard type inequalities for harmonically ϕ_{h-s} via fractional integrals.

Theorem 5. *Let $f : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a function such that $f \in L[a,b]$ where $a, b \in I$ with $a < b$. If $f \in \mathcal{H}SX(\phi_{h-s}, I)$ and $\phi : I \to I$, then the following holds.*

$$\begin{aligned}
&\left(2h\left(\frac{1}{2}\right)\right)^s f\left(\frac{2ab}{a+b}\right)\\
&\leq \left(\frac{ab}{b-a}\right)^\alpha \Gamma(\alpha+1)\left(J^\alpha_{1/a-}(f\circ g)(1/b) + J^\alpha_{1/b+}(f\circ g)(1/a)\right)\\
&\leq 2((f\circ\phi)(a) + (f\circ\phi)(b))\int_0^1 t^{\alpha-1}\left(\left(\frac{h(t)}{t}\right)^{-s} + \left(\frac{h(1-t)}{1-t}\right)^{-s}\right) dt
\end{aligned}$$

where $g(x) = \phi(\frac{1}{x})$.

Proof. Set $\phi(x) = \frac{ab}{tb+(1-t)a}$ and $\phi(y) = \frac{ab}{ta+(1-t)b}$ in (2.2), then

$$f\left(\frac{2ab}{a+b}\right) \leq \left(2h\left(\frac{1}{2}\right)\right)^{-s}\left(f\left(\frac{ab}{tb+(1-t)a}\right) + f\left(\frac{ab}{ta+(1-t)b}\right)\right) \quad (4.1)$$

Multiplying both sides of (4.1) by $t^{\alpha-1}$ and integrating the result with respect to t over $(0,1)$,

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)\\
\leq\ &\left(2h\left(\frac{1}{2}\right)\right)^{-s}\alpha\left(\int_0^1 t^{\alpha-1}f\left(\frac{ab}{tb+(1-t)a}\right)dt\right.\\
&\left.+\int_0^1 t^{\alpha-1}f\left(\frac{ab}{ta+(1-t)b}\right)dt\right)\\
=\ &\left(2h\left(\frac{1}{2}\right)\right)^{-s}\left(\frac{ab}{b-a}\right)^\alpha \alpha\left(\int_{\frac{1}{b}}^{\frac{1}{a}}\left(u-\frac{1}{b}\right)^{\alpha-1} f\left(\frac{1}{u}\right)du\right.\\
&\left.+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-v\right)^{\alpha-1} f\left(\frac{1}{v}\right)dv\right)\\
=\ &\left(2h\left(\frac{1}{2}\right)\right)^{-s}\left(\frac{ab}{b-a}\right)^\alpha \Gamma(\alpha+1)\left(J^\alpha_{1/a-}(f\circ g)(1/b) + J^\alpha_{1/b+}(f\circ g)(1/a)\right).
\end{aligned}$$

This proves the first inequality. Since $f \in \mathcal{H}SX(\phi_{h-s}, I)$, then for $a, b \in I$,

$$f\left(\frac{ab}{tb+(1-t)a}\right) + f\left(\frac{ab}{ta+(1-t)b}\right) \\ \leq \left(\frac{h(t)}{t}\right)^{-s}(f\circ\phi)(a) + \left(\frac{h(1-t)}{1-t}\right)^{-s}(f\circ\phi)(b) \\ + \left(\frac{h(1-t)}{1-t}\right)^{-s}(f\circ\phi)(a) + \left(\frac{h(t)}{t}\right)^{-s}(f\circ\phi)(b).$$

The remaining part of the proof follows by multiplying both sides of the last inequality by $t^{\alpha-1}$ and integrating the result with respect to t over $(0,1)$.

QED

Corollary 2. *[6] Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for fractional integral holds*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left(J^{\alpha}_{1/a-}(f\circ g)(1/b) + J^{\alpha}_{1/b+}(f\circ g)(1/a)\right) \\ \leq \frac{f(a)+f(b)}{2} \quad \textit{with } \alpha > 0.$$

Proof. The result follows by setting $h(t) = 1$, $s = 1$ and $\phi(x) = x$ in Theorem 5.

QED

5 Application to special means of real numbers

We recall the following definitions of some special means of two non-negative real numbers which are quite important for numerical approximations and computations.

Definition 5. (1) The Arithmetic mean $A = A(a,b) := \frac{a+b}{2}$.

(2) The Geometric mean $G = G(a,b) := \sqrt{ab}$.

(3) The Harmonic mean $H = H(a,b) := \frac{2ab}{a+b}$.

(4) The Logarithmic mean $L = L(a,b) := \frac{b-a}{\ln b - \ln a}$.

(5) The p-logarithmic mean $L_p = L_p(a,b) := \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $p \neq -1, 0$.

Proposition 2. *Let $h(t) = t^{1-\frac{1}{2s}}$, $s \in (0,1)$, $\phi(x) = x$ and $0 < a < b$, then* *(i).* $\frac{1}{\sqrt{2}}H(a,b) \leq \frac{G^2(a,b)}{L(a,b)} \leq \frac{4}{3}A(a,b)$.

(ii). $\frac{1}{\sqrt{2}}H^2(a,b) \leq G^2(a,b) \leq \frac{4}{3}A(a^2,b^2)$.

(iii). $\frac{1}{\sqrt{2}}H^n(a,b) \leq \frac{p+3}{p+1}G^2L_p^p(a,b) \leq \frac{4}{3}A(a^n,b^n)$
where $p = n-2$, $p \neq -1, 0$.

Proof. Define $f : (0,\infty) \to \mathbb{R}$ by $f(x) = x$, clearly $f \in \mathcal{H}SX(I)$ and so by Example 1, $f \in \mathcal{H}SX(\phi_{h-s}, I)$ since $h(t) \leq t^{1-\frac{1}{s}}$. The remaining part of the proof then follows from Theorem 2. The proofs of (ii) and (iii) similarly follow by setting $f(x) = x^2$ and $f(x) = x^n$ respectively. QED

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