A spaceability result in the context of hypergroups

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Abstract. In this paper, by an elementary constructive technique, it is shown that \( L^r(Z^+) - \bigcup_{q<r} L^q(Z^+) \) is non-empty, where \( Z^+ \) is the dual of a compact countable hypergroup introduced by Dunkl and Ramirez. Also, we prove that for each \( r > 1 \), \( L^r(Z^+) - \bigcup_{q<r} L^q(Z^+) \) is spaceable.

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1 Introduction and Notations

Suppose that \( X \) is a topological vector space. A subset \( S \subseteq X \) is called spaceable if \( S \cup \{0\} \) is large enough to contain a closed infinite dimensional subspace of \( X \). This concept was first introduced in [6] and then the term spaceable was used in [1]. There have been several further works on this notion (for example see [4], [7] and [9]). It is well-known that for each \( 0 < p \), there are sequences in \( l_p \) not belonging to \( \bigcup_{q<p} l_q \). Usually, this fact is proved by non-constructive techniques, although a constructive proof, depending on the Principle of Uniform Boundedness can be found in [8]. Actually, in [3] it is proved that for every \( p > 0 \), the set \( l_p - \bigcup_{q<p} l_q \) is even spaceable.

Let \( Z^+ \) be the set of non-negative integers equipped with discrete topology, \( M(Z^+) \) be the space of all Radon measures on \( Z^+ \), and \( p_0 \) be a fixed prime number. For any \( k \in Z^+ \) and distinct non-zero \( m, n \in Z^+ \) we put

\[
\delta_k \ast \delta_0 = \delta_0 \ast \delta_k := \delta_k, \\
\delta_n \ast \delta_n := \frac{1}{p_0^{n-1}(p_0-1)} \delta_0 + \sum_{k=1}^{n-1} \frac{p_0^{k-1}}{p_0-1} \delta_k + \frac{p_0 - 2}{p_0 - 1} \delta_n,
\]

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and $\delta_m \ast \delta_n := \delta_{\max\{m,n\}}$. Then, by [5], $\mathbb{Z}_+$ equipped with the convolution
\[ * : M(\mathbb{Z}_+) \times M(\mathbb{Z}_+) \to M(\mathbb{Z}_+) \]
defined by
\[ \mu \ast \nu := \int_{\mathbb{Z}_+} \int_{\mathbb{Z}_+} \delta_m \ast \delta_n \, d\mu(m) \, d\nu(n), \quad (\mu, \nu \in M(\mathbb{Z}_+)), \]
and the identity mapping on $\mathbb{Z}_+$ as involution, is a commutative discrete hypergroup. Also, the measure $m$ defined by
\[ m(\{k\}) := \begin{cases} 1, & \text{if } k = 0, \\ (p_0 - 1)p_0^{k-1}, & \text{if } k \geq 1, \end{cases} \]
is a Haar measure for $\mathbb{Z}_+$. For more details about locally compact hypergroups see [2].

In the sequel, we consider $\mathbb{Z}_+$ with above structure, and for each $s > 0$ we denote $L^s(\mathbb{Z}_+) := L^s(\mathbb{Z}_+, m)$.

In this paper, by an elementary technique, we give an algorithm that generates lots of sequences in $L^p(\mathbb{Z}_+)$ but not in $L^q(\mathbb{Z}_+)$ for every $q < p$. This argument can be regarded as a variant of the technique that was used to prove the Banach-Steinhaus Theorem in [8]. Also, we prove that for each $r > 1$, the set $L^r(\mathbb{Z}_+) - \bigcup_{q<r} L^q(\mathbb{Z}_+)$ is spaceable in $L^r(\mathbb{Z}_+)$.

## 2 Main Results

This is well-known that:

**Lemma 1.** There is a sequence $(a_n)_{n=1}^{\infty}$ of real numbers such that $a_n > 1$ for all $n$, $\lim_{n \to \infty} a_n = 1$, and $\sum_{n=1}^{\infty} \frac{1}{n a_n}$ converges.

**Lemma 2.** If $0 < q < r < \infty$, then $L^q(\mathbb{Z}_+) \subseteq L^r(\mathbb{Z}_+)$.

**Proof.** Let $f := (x_n)_{n=0}^{\infty} \in L^q(\mathbb{Z}_+)$. Since
\[ \|f\|_q^q = |x_0|^q + \sum_{n=1}^{\infty} |x_n|^q (p_0 - 1)p_0^{n-1} < \infty, \quad (2.1) \]
there is a positive number $N$ such that for each $n \geq N$,
\[ |x_n| \left((p_0 - 1)p_0^{n-1}\right)^{\frac{1}{q}} < 1. \]

But for each $n \in \mathbb{N}$,
\[ \left((p_0 - 1)p_0^{n-1}\right)^{\frac{1}{q}} \geq 1. \]
Hence, for all $n \geq N$, $|x_n| < 1$, and then $|x_n|^r < |x_n|^q$. Therefore, for each $n \geq N$ we have

$$|x_n|^r (p_0 - 1)p_0^{n-1} < |x_n|^q (p_0 - 1)p_0^{n-1}.$$ 

By (2.1) we get $\sum_{n=1}^{\infty} |x_n|^r (p_0 - 1)p_0^{n-1} < \infty$, i.e. $f \in L^r(\mathbb{Z}_+)$.  \[QED\]

**Lemma 3.** Let $q < r$, $p_0$ be a prime number, and $(a_n)_{n=1}^{\infty}$ be a sequence as in Lemma 1. If the sequence $x = (x_n)_{n=0}^{\infty}$ is defined by

$$x_n := \begin{cases} 1 & \text{if } n = 0, \\ \frac{1}{(p_0^{n-1}(p_0 - 1)n^{a_n})^r} & \text{if } n \geq 1, \end{cases}$$

then $x \in L^r(\mathbb{Z}_+) - L^q(\mathbb{Z}_+)$. 

**Proof.** From

$$\|x\|^r_\mathcal{F} = \sum_{n=0}^{\infty} |x_n|^r \cdot m(\{n\})$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{p_0^{n-1}(p_0 - 1)n^{a_n}(p_0 - 1)p_0^{n-1}}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n^{a_n}} < \infty$$

we conclude that $x \in L^r(\mathbb{Z}_+)$. Let $0 < q < r$. Since $\frac{r}{a_n} < r$ for every $n$ and $\lim_{n \to \infty} \frac{r}{a_n} = r$, there is some $t_0 \in \mathbb{N}$ such that $q < \frac{r}{a_0}$. Since

$$\lim_{n \to \infty} \frac{a_n}{a_{t_0}} = \frac{1}{a_{t_0}} < 1,$$

we can choose $s \in \left(\frac{1}{a_{t_0}}, 1\right)$, and there is $N_0 \in \mathbb{N}$ such that $\frac{a_n}{a_{t_0}} < s < 1$ for all $n \geq N_0$. Hence,

$$\frac{1}{n^s} < \frac{1}{n^{a_n}}$$
for all $n \geq N_0$. We have

$$\|x\|_{\ell_{10}^r}^r = \sum_{n=0}^{\infty} |x_n|^{r_{10}/n_{10}} m(\{n\})$$

$$= 1 + \sum_{n=1}^{\infty} \left( \frac{1}{(p_0^{n-1})(p_0 - 1)n^{a_n}} \right)^{r_{10}/n_{10}} (p_0 - 1)p_0^{n-1}$$

$$= 1 + \sum_{n=1}^{\infty} ((p_0 - 1)p_0^{-1})^{1 - \frac{1}{n_{10}}} \frac{1}{n^{a_{10}}}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^q} = \infty$ and

$$((p_0 - 1)p_0^{-1})^{1 - \frac{1}{n_{10}}} \geq 1,$$

we have

$$\sum_{n=N_0}^{\infty} \frac{1}{n^q} \leq \sum_{n=N_0}^{\infty} \frac{1}{n^{a_{10}}} \leq 1 + \sum_{n=1}^{\infty} ((p_0 - 1)p_0^{-1})^{1 - \frac{1}{n_{10}}} \frac{1}{n^{a_{10}}} \leq \|x\|_{\ell_{10}^r},$$

and so, $x \notin L^r_{n_{10}}(\mathbb{Z}_+)$. Therefore, $x \notin L^q(\mathbb{Z}_+)$ because $q < r_{10}$ and $L^q(\mathbb{Z}_+) \subseteq L^r_{n_{10}}(\mathbb{Z}_+)$.

**Theorem 1.** If $1 \leq q < r < \infty$, then $L^q(\mathbb{Z}_+) \text{ is not closed in } L^r(\mathbb{Z}_+).$

**Proof.** First Proof: Let $C_c(\mathbb{Z}_+)$ be the space of all functions from $\mathbb{Z}_+$ into $\mathbb{C}$ with finite support. We have $C_c(\mathbb{Z}_+) \subseteq L^q(\mathbb{Z}_+) \subseteq L^r(\mathbb{Z}_+), \text{ and } C_c(\mathbb{Z}_+) \text{ is dense in } L^q(\mathbb{Z}_+) \text{ and in } L^r(\mathbb{Z}_+).$ So, if $L^q(\mathbb{Z}_+) \text{ is closed in } L^r(\mathbb{Z}_+)$, then we have $L^q(\mathbb{Z}_+) \subseteq L^r(\mathbb{Z}_+)$, a contradiction.

Second Proof: Let $T : L^q(\mathbb{Z}_+) \rightarrow L^r(\mathbb{Z}_+) \text{ be the inclusion mapping. We claim that } T \text{ is continuous.}$ For this, suppose that $(f_n)_{n=0}^{\infty} \text{ is a sequence in } L^q(\mathbb{Z}_+) \text{ with } f_n := (x_{n,i})_{i=0}^{\infty}$, and $f_n \rightarrow f \text{ in } L^q(\mathbb{Z}_+), \text{ where } f := (x_i)_{i=0}^{\infty} \in L^q(\mathbb{Z}_+)$. For each $0 < \varepsilon < 1$, there is $N \in \mathbb{N}$ such that for all $n \geq N,

$$\|f_n - f\|_q = |x_{n,0} - x_0|^q + \sum_{i=1}^{\infty} |x_{n,i} - x_i|^q(p_0 - 1)p_0^{i-1} < \varepsilon < 1.$$

Hence, $|x_{n,i} - x_i|^q < 1$ for all $n \geq N$ and $i \in \mathbb{Z}_+$. Since $q < r$, we get

$$|x_{n,i} - x_i|^r \leq |x_{n,i} - x_i|^q,$$
where \( n \geq N \) and \( i \in \mathbb{Z}_+ \), and so \( \| f_n - f \|_r \leq \| f_n - f \|_q \) for all \( n \geq N \). Therefore, \( f_n \to f \) with respect to the norm \( \| \cdot \|_r \), and \( T \) is continuous.

Now, we show that \( \| \cdot \|_q \) and \( \| \cdot \|_r \) are not equivalent. In contrast, suppose that there exists \( M > 0 \) such that for each \( f \in L^q(\mathbb{Z}_+) \), \( \| f \|_q \leq M \| f \|_r \). For any \( n \in \mathbb{N} \), let \( \chi_n \) be the characteristic function of the set \( \{ n \} \). Then for all \( n \in \mathbb{N} \), we have

\[
((p_0 - 1)p_0^{n-1})^\frac{1}{q} = \| \chi_n \|_q \leq M \| \chi_n \|_r = M ((p_0 - 1)p_0^{n-1})^\frac{1}{r},
\]

and so,

\[
((p_0 - 1)p_0^{n-1})^\frac{1}{q} - \frac{1}{r} \leq M,
\]

which is a contradiction, since \( \frac{1}{q} - \frac{1}{r} > 0 \). Therefore, the norms \( \| \cdot \|_q \) and \( \| \cdot \|_r \) are not equivalent. Finally, suppose that \( L^q(\mathbb{Z}_+) \) is closed in \( L^r(\mathbb{Z}_+) \). Since \( T \) is continuous, the induced mapping

\[
\tilde{T} : L^q(\mathbb{Z}_+) \to T(L^q(\mathbb{Z}_+)) = (L^q(\mathbb{Z}_+), \| \cdot \|_r), \quad \tilde{T}(f) := T(f) = f,
\]

is a bijective continuous linear operator between Banach spaces. So, by the Open Mapping Theorem, \( \tilde{T} \) is an isomorphism, and in particular the norms \( \| \cdot \|_q \) and \( \| \cdot \|_r \) are equivalent on \( L^q(\mathbb{Z}_+) \), a contradiction. Therefore, \( L^q(\mathbb{Z}_+) \) is not closed in \( L^r(\mathbb{Z}_+) \). \( \square \)

By [9, Theorem 2.4], we can conclude:

**Corollary 1.** If \( 0 < q < r < \infty \), then \( L^r(\mathbb{Z}_+) - L^q(\mathbb{Z}_+) \) is spaceable.

Here, we recall the following result (see [9, Theorem 3.3]).

**Theorem 2.** Suppose that \( X \) is a Frechet space, and for each \( n = 1, 2, \ldots \), \( Z_n \) is a Banach space. Let for each \( n \in \mathbb{N} \), \( T_n : Z_n \to X \) be bounded linear operators. If \( Y := \text{span}(\bigcup_{n=1}^\infty T_n(Z_n)) \) is not closed in \( X \), then \( X - Y \) is spaceable.

**Theorem 3.** If \( r > 1 \), then

\[
L^r(\mathbb{Z}_+) - \bigcup_{q<r} L^q(\mathbb{Z}_+)
\]

is spaceable.

**Proof.** The sequence \( x \in L^r(\mathbb{Z}_+) \) constructed in Lemma 3 does not belong to \( L^q(\mathbb{Z}_+) \) for all \( q < r \), i.e. \( L^r(\mathbb{Z}_+) - \bigcup_{q<r} L^q(\mathbb{Z}_+) \) is non-empty.

Let \( q < r \). Since \( C_c(\mathbb{Z}_+) \) is dense in \( L^r(\mathbb{Z}_+) \), and

\[
C_c(\mathbb{Z}_+) \subseteq L^q(\mathbb{Z}_+) \subseteq L^r(\mathbb{Z}_+),
\]

Proof. The sequence \( x \in L^r(\mathbb{Z}_+) \) constructed in Lemma 3 does not belong to \( L^q(\mathbb{Z}_+) \) for all \( q < r \), i.e. \( L^r(\mathbb{Z}_+) - \bigcup_{q<r} L^q(\mathbb{Z}_+) \) is non-empty.

Let \( q < r \). Since \( C_c(\mathbb{Z}_+) \) is dense in \( L^r(\mathbb{Z}_+) \), and

\[
C_c(\mathbb{Z}_+) \subseteq L^q(\mathbb{Z}_+) \subseteq L^r(\mathbb{Z}_+),
\]
$L^q(Z_+)$ is dense in $L^r(Z_+)$, and so $\bigcup_{q<r} L^q(Z_+)$ is not closed in $L^r(Z_+)$. By Lemma 2, the set $\bigcup_{q<r} L^q(Z_+)$ is actually a countable union:

$$\bigcup_{q<r} L^q(Z_+) = \bigcup_{n=1}^{\infty} L^{q_n}(Z_+),$$

(2.2)

where $q_1 < q_2 < \cdots < r$, and moreover, the sequence $(q_n)_n$ converges to $r$. Easily, one can see that $\text{span} \left( \bigcup_{n=1}^{\infty} L^{q_n}(Z_+) \right) = \bigcup_{n=1}^{\infty} L^{q_n}(Z_+)$. So, applying Theorem 2 (put $Z_n := (L^{q_n}(Z_+), \| - \|_{q_n})$ and $T_n : L^{q_n}(Z_+) \rightarrow L^r(Z_+) \to \bigcup_{q<r} L^q(Z_+)$ is spaceable.

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**References**


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