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Auslander–Reiten sequences and intuition

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Abstract. The aim of this note is to give a visual presentation of irreducible maps and Auslander–Reiten sequences belonging to finite Auslander–Reiten quivers.

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MSC 2000 classification: Primary 16G20, 16G70

1 Introduction and conventions

Auslander–Reiten quivers and sequences seem to have so many aspects that even the best definitions and theorems cannot describe them completely. Their beauty is somehow independent of the knowledge of the whole theory behind them. Perhaps this is the reason why the word "intuition" shows up at the end of Gabriel's paper [G, pages 70–71]. Indeed these are his final words: "Since then, various specialists like Bautista, Brenner, Butler, Riedtmann... have hoarded a few hundred examples in their dossiers, thus getting an intuition which no theoretical argument can replace."

This paper is organized as follows. In section 1 we collect some notation and conventions. In section 2 we describe the Auslander–Reiten quivers of the algebras given by a quiver Q of the form

$$\underset{1}{\bullet} \xrightarrow{\;a\;} \underset{2}{\bullet} \circlearrowleft b \qquad \text{with relations} \quad b^2a = 0 \quad \text{and} \quad b^m = 0$$

for some $1 < m < 6$. In section 3 we assume $m = 5$ and we describe one of the most complicated Auslander–Reiten sequences. For the precise definitions of Auslander–Reiten sequences and Auslander - Reiten quivers we refer to [AReS] or [R2]. Here we only recall that Auslander–Reiten sequences are short exact sequences which do not split, starting and ending at indecomposable modules. The two non–zero maps of an Auslander–Reiten sequence are diagonal maps of irreducible maps, which are morphisms with only obvious factorizations. Moreover the Auslander–Reiten quiver of an algebra is an oriented graph, whose vertices correspond to indecomposable modules and whose arrows correspond to irreducible maps. In the sequel we denote by K an algebraically closed field. Finally we recall the strategy, due to Ringel, used to describe the above indecomposable modules in a very combinatorial and efficient way. First of all, as usual, any indecomposable representation of Q is of the form $M = (V(1), V(2); a, b)$, where $V(1)$ and $V(2)$ are finite dimensional K–vector spaces, while $a : V(1) \to V(2)$ and $b : V(2) \to V(2)$ are K–linear maps satisfying $b^2a = 0$ and $b^m = 0$. If $V(2)$ is different from 0 , then the map $a : V(1) \to V(2)$ is injective. Hence we may view M as a vector space $V(2)$ equipped with an endomorphism b with $b^m = 0$ and a subspace $a(V(1))$ contained in $\ker b^2$. Next, small squares denote the elements of a fixed basis of the underlying vector space $V(2)$, corresponding to the vertex 2, while small black squares denote elements of the fixed basis belonging to the image of the map a. Finally a segment connecting small squares, corresponding to the vectors $v(1), \dots, v(n)$, indicates that the vector $v(1) + \cdots + v(n)$ belongs to the image of the map a.

I wish to thank Claus Michael Ringel who suggested me the problem of determining these Auslander–Reiten quivers, and gave me useful hints to compute them.

2 Some examples of finite Auslander–Reiten quivers.

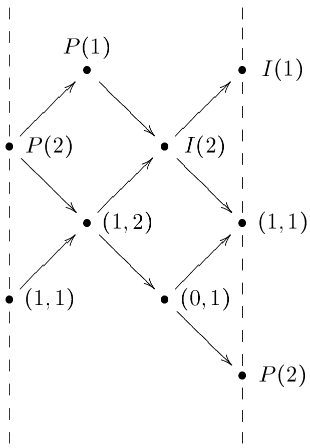
Throughout this section, let Q be a quiver of the form

$$\underset{1}{\bullet} \xrightarrow{\ a\ } \underset{2}{\bullet} \circlearrowleft b \qquad \text{with relations} \quad b^2a = 0 \quad \text{and} \quad b^m = 0$$

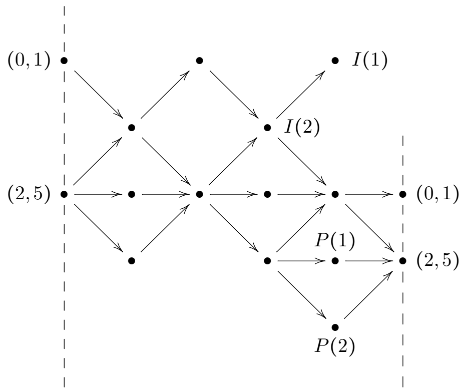
with $m > 1$. Next, let A denote the K–algebra given by Q. Then a direct calculation of the whole Auslander–Reiten quiver of A, by means of the dual of the transpose [AReS] $\tau(M)$ of any indecomposable non projective module M, gives the following result.

Fact. If $m = 2, 3, 4, 5$, then the number of isomorphism classes of the indecomposable left A–modules is equal to 7, 14, 28 and 66 respectively.

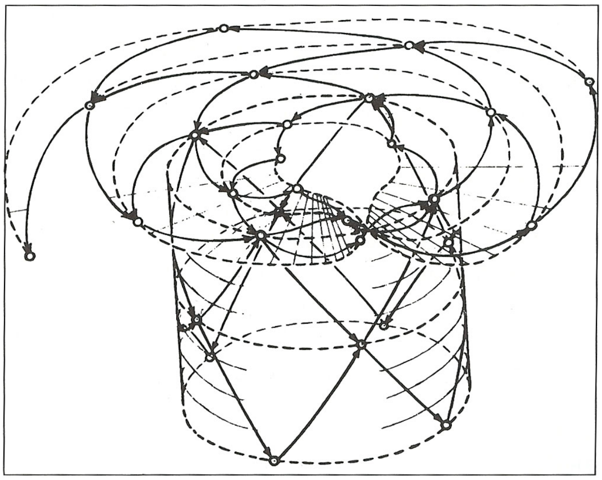
If $m = 2$, then there are 3 stable indecomposable modules. On the other hand, the 4 indecomposable modules which are either projective or injective are exactly the unstable modules, as indicated in the following Auslander–Reiten quiver. In the picture we have to identify the dotted lines and we replace the 3 stable indecomposable modules M by their dimension vectors [R2], that is by the pairs $(\dim_K V(1), \dim_K V(2))$.



If $m = 3$, then all the 14 indecomposable modules are unstable and the two τ–orbits have 6 and 8 elements respectively. In this case the Auslander–Reiten quiver has the following shape, where again we have to identify the dotted lines.



If $m = 4$, then 20 indecomposable modules are stable and 8 are unstable. For an elegant and topological form of the Auslander–Reiten quiver, we refer to Ringel's paper [R1, page 93] . We take from Ringel's home page [R3, Abbildung 3] the following picture of this Auslander–Reiten quiver with 28 vertices.



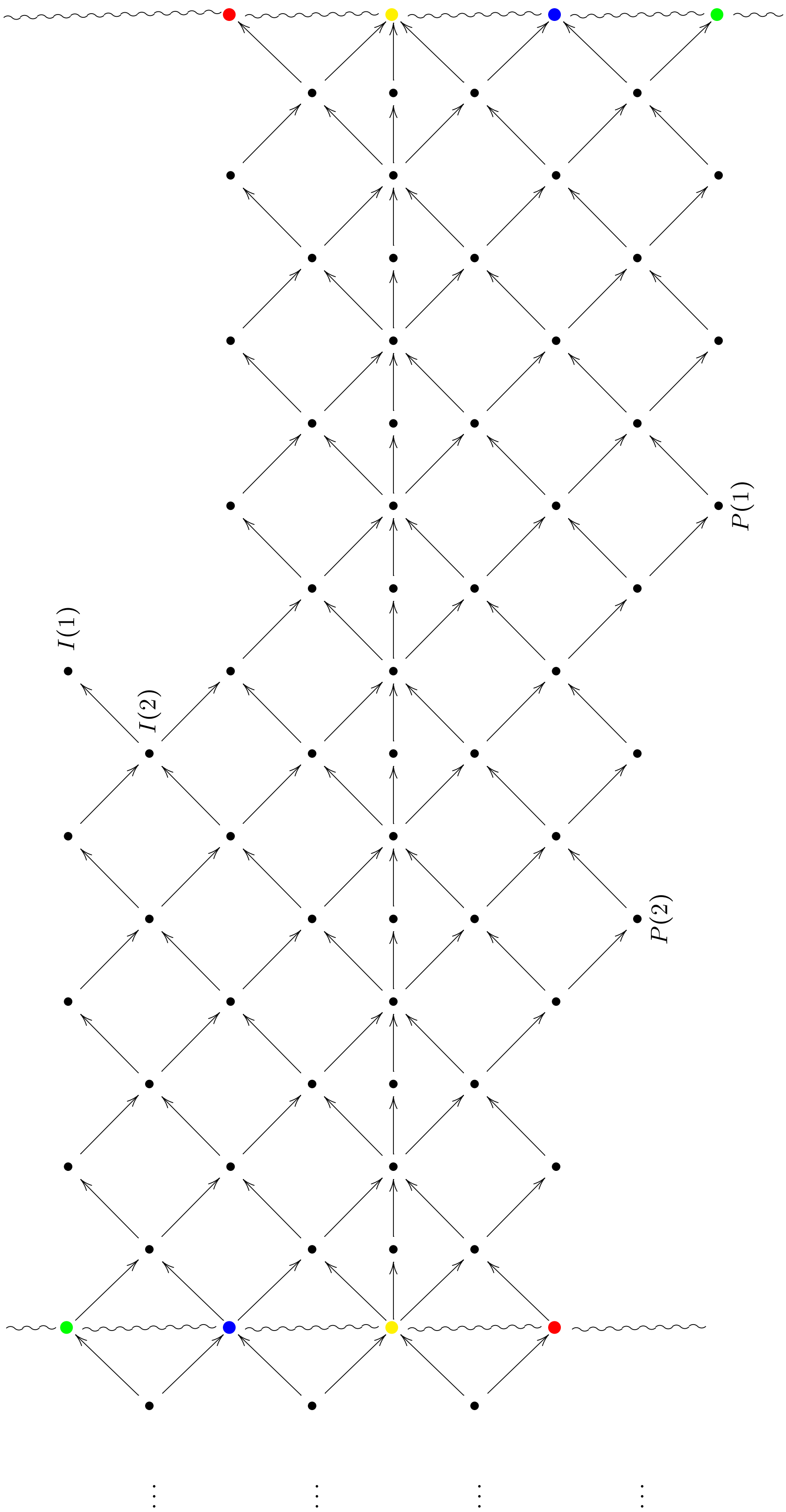
We refer to Section 3.1 of [D] for a naïve description of the same Auslander–Reiten quiver, admitting

- four stable τ–orbits with 4, 4, 4 and 8 elements;
- two unstable τ–orbits with 3 and 5 elements.

If $m = 5$, then there exist

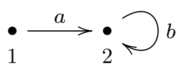
- 48 stable modules, belonging to four τ–orbits with 8, 8, 16 and 16 elements;
- 18 unstable modules, belonging to two τ–orbits with 8 and 10 elements.

The following picture illustrates the shape of the Auslander–Reiten quiver, and we identify the two verical waves lines.


I(1)
I(2)
P(1)
P(2)

3 An Auslander–Reiten sequence with three terms in the middle.

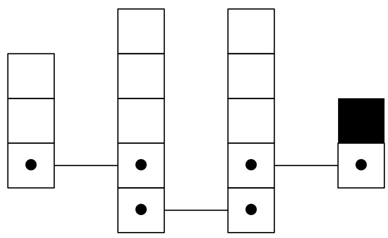
Let A be the K–algebra given by the quiver



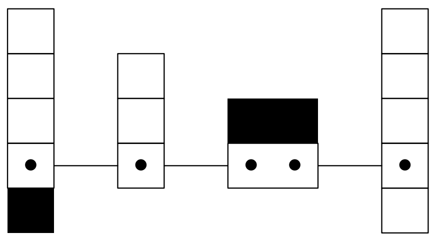
with relations $b^2a = 0$ and $b^5 = 0$.

Next let $M = (4, 15)$ denote the indecomposable non projective A–module with the biggest dimension vector described by one of the following pictures.

(4,15) A



(4,15) B

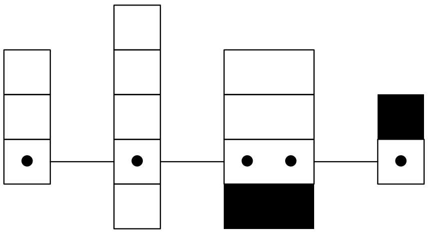


Next, let (#) denote the Auslander–Reiten sequence of A–modules ending at M of the form

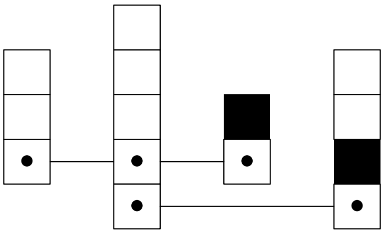
$$0 \longrightarrow (4, 14) \longrightarrow X \longrightarrow (4, 15) \longrightarrow 0. \qquad (\#)$$

In this case the indecomposable module $(4, 14)$ is described by one of the following pictures, and corresponds to the yellow vertex of the Auslander-Reiten quiver.

(4,14) A

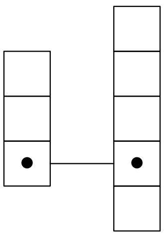


(4,14) B

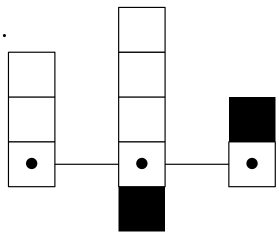


Finally, X is the direct sum of three indecomposable modules of dimension vector $(1, 8)$, $(3, 10)$ and $(4, 11)$ illustrated by the following pictures.

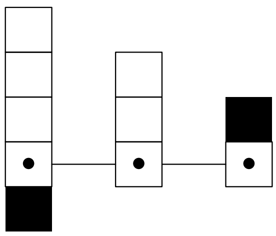
(1,8)



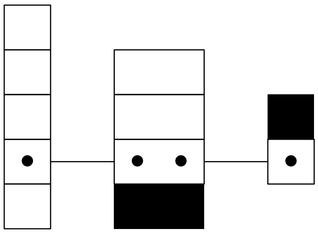
$(\mathbf{3}, \mathbf{10})\mathbf{A}$.



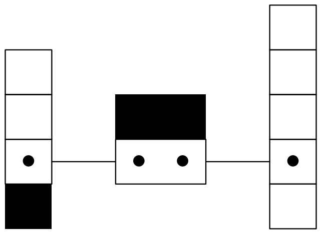
$(\mathbf{3}, \mathbf{10})\mathbf{B}$.



(4, 11)A.

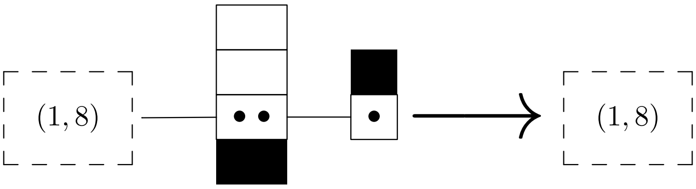


(4, 11)B.

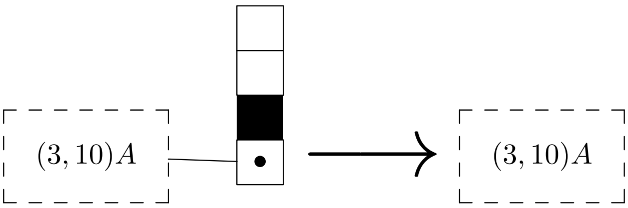


With respect to suitable bases all the irreducible maps in (#) look like cancellations, additions or diagonal embeddings, as the following pictures show. More precisely, the three irreducible epimorphisms are

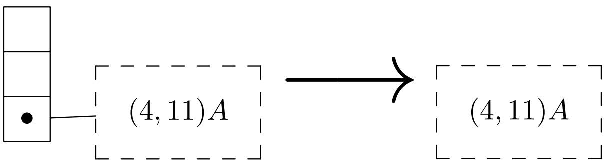
- the right cancellation $(4, 14)A \longrightarrow (1, 8)$ described by



- the right cancellation $(4, 14)B \longrightarrow (3, 10)A$ described by

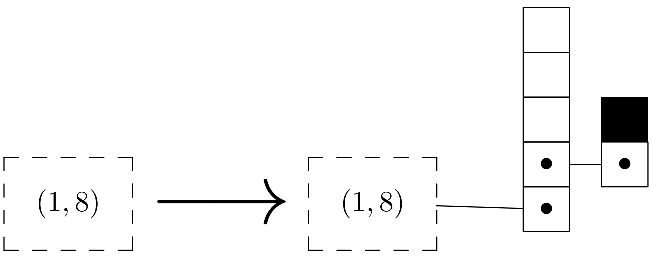


- the left cancellation $(4, 14)A \longrightarrow (4, 11)A$ described by

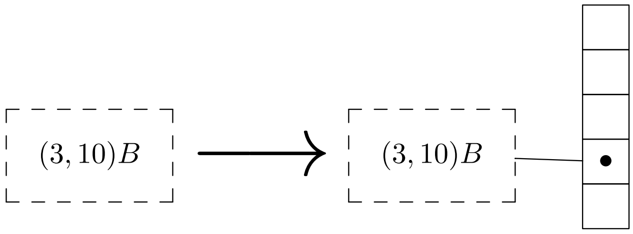


The three irreducible monomorphisms are

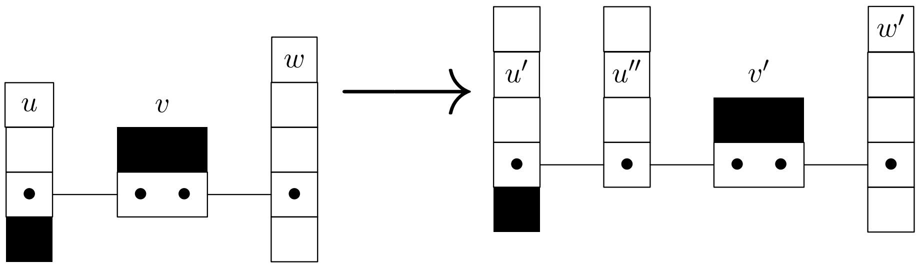
- the right addition $(1, 8) \longrightarrow (4, 15)A$ described by



- the right addition $(3,10)B \longrightarrow (4,15)B$ described by



- the left diagonal embedding $(4,11)B \longrightarrow (4,15)B$ described by



which satisfies the following conditions:
$u \longrightarrow u' + u''$, $\quad v \longrightarrow v'$, $\quad w \longrightarrow w'$.

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