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Multivalued functions in digital topology

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Abstract. We study several types of multivalued functions in digital topology.

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1 Introduction

A common method of composing a movie or a video is via a set of “frames” or images that are projected sequentially. If a frame is an $m \times n$ grid of pixels, then a transition between projections of consecutive frames requires reading and then outputting to the screen each of the mn pixels of the incoming frame. Since many frames must be displayed each second (in current technology, 30 per second is common), a video composed in this fashion uses a lot of memory, and requires that a lot of data be processed rapidly.

Often, consecutive frames have a great deal of resemblance. In particular, there are many pairs (i, j) such that the pixel in row i and column j is unchanged between one frame and its successor. If, further, one can efficiently compute for all changing pixels how they change between successive frames, then it is not necessary to use as much storage for the video, as the pixels not changed between successive frames need not be subjected to the i/o described above; and a larger number of frames or their equivalent can be processed per second. E.g., this approach can be taken in computer graphics when the changes in the viewer's screen are due to the movements or transformations of sprites. Thus, the world of applications motivates us to understand the properties of structured single-valued and multivalued functions between digital images.

Continuous (single-valued and multivalued) functions can often handle changes between successive frames that seem modeled on continuous Euclidean changes. Other changes, such as the sudden breaking of an object, may be discontinuous.

In this paper, we develop tools for modeling changes in digital images. We are particularly concerned with properties of multivalued functions between digital images that are characterized by any of the following.

- Continuity [5, 6]
- Weak continuity [11]
- Strong continuity [11]
- Connectivity preservation [9]

2 Preliminaries

Much of this section is quoted or paraphrased from [4] and other papers cited.

2.1 Basic notions of digital topology

We will assume familiarity with the topological theory of digital images. See, e.g., [1] for the standard definitions. All digital images X are assumed to carry their own adjacency relations (which may differ from one image to another). When we wish to emphasize the particular adjacency relation we write the image as (X, κ), where κ represents the adjacency relation.

Among the commonly used adjacencies are the c_u-adjacencies. Let $x, y \in \mathbb{Z}^n$, $x \neq y$, where we consider these points as n-tuples of integers:

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n).$$

Let u be an integer, $1 \leq u \leq n$. We say x and y are c_u*-adjacent* if

- There are at most u indices i for which $|x_i - y_i| = 1$.
- For all indices j such that $|x_j - y_j| \neq 1$ we have $x_j = y_j$.

We often label a c_u-adjacency by the number of points adjacent to a given point in $\mathbb{Z}^n$ using this adjacency. E.g.,

- In $\mathbb{Z}^1$, c_1-adjacency is 2-adjacency.
- In $\mathbb{Z}^2$, c_1-adjacency is 4-adjacency and c_2-adjacency is 8-adjacency.
- In $\mathbb{Z}^3$, c_1-adjacency is 6-adjacency, c_2-adjacency is 18-adjacency, and c_3-adjacency is 26-adjacency.
- In $\mathbb{Z}^n$, c_1-adjacency is $2n$-adjacency and c_n-adjacency is $(3^n - 1)$-adjacency.

For κ-adjacent x, y, we write $x \leftrightarrow_\kappa y$ or $x \leftrightarrow y$ when κ is understood. We write $x \leftrightarrows_\kappa y$ or $x \leftrightarrows y$ to mean that either $x \leftrightarrow_\kappa y$ or $x = y$.

A subset Y of a digital image (X, κ) is *κ-connected* [10], or *connected* when κ is understood, if for every pair of points $a, b \in Y$ there exists a sequence $\{y_i\}_{i=0}^m \subset Y$ such that $a = y_0$, $b = y_m$, and $y_i \leftrightarrow_\kappa y_{i+1}$ for $0 \leq i < m$. The following generalizes a definition of [10].

Definition 2.1. [2] Let (X, κ) and (Y, λ) be digital images. A single-valued function $f : X \to Y$ is (κ, λ)-continuous if for every κ-connected $A \subset X$ we have that $f(A)$ is a λ-connected subset of Y. □

When the adjacency relations are understood, we will simply say that f is *continuous.* Continuity can be reformulated in terms of adjacency of points:

Theorem 2.1. [10, 2] *A single-valued function $f : X \to Y$ is continuous if and only if $x \leftrightarrow x'$ in X implies $f(x) \leftrightarrows f(x')$.* □

For two subsets $A, B \subset X$, we will say that A and B are *adjacent* when there exist points $a \in A$ and $b \in B$ such that $a \leftrightarrows b$. Thus sets with nonempty intersection are automatically adjacent, while disjoint sets may or may not be adjacent.

2.2 Multivalued functions

A *multivalued function* f from X to Y assigns a subset of Y to each point of x. We will write $f : X \multimap Y$. For $A \subset X$ and a multivalued function $f : X \multimap Y$, let $f(A) = \bigcup_{x \in A} f(x)$.

Definition 2.2. [9] A multivalued function $f : X \multimap Y$ is *connectivity preserving* if $f(A) \subset Y$ is connected whenever $A \subset X$ is connected. □

As with Definition 2.1, we can reformulate connectivity preservation in terms of adjacencies.

Theorem 2.2. [4] A multivalued function $f : X \multimap Y$ is *connectivity preserving* if and only if the following are satisfied:

- For every $x \in X$, $f(x)$ is a connected subset of Y.
- For any adjacent points $x, x' \in X$, the sets $f(x)$ and $f(x')$ are adjacent. □

The papers [5, 6] define continuity for multivalued functions between digital images based on subdivisions. (These papers make an error with respect to compositions, which is corrected in [7].) We have the following.

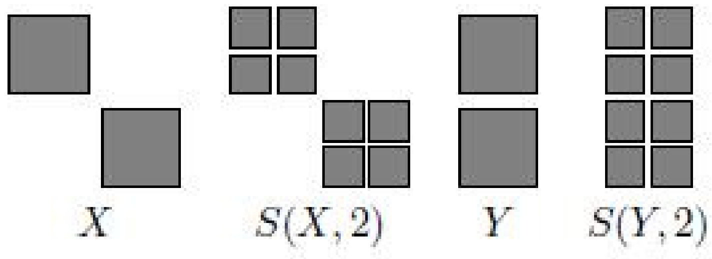

Figure 1. [4] Two images X and Y with their second subdivisions.

Definition 2.3. [5, 6] For any positive integer r, the *r-th subdivision* of $\mathbb{Z}^n$ is

$$\mathbb{Z}_r^n = \{(z_1/r, \ldots, z_n/r) \mid z_i \in \mathbb{Z}\}.$$

An adjacency relation κ on $\mathbb{Z}^n$ naturally induces an adjacency relation (which we also call κ) on $\mathbb{Z}_r^n$ as follows: $(z_1/r, \ldots, z_n/r) \leftrightarrow_\kappa (z_1'/r, \ldots, z_n'/r)$ in $\mathbb{Z}_r^n$ if and only if $(z_1, \ldots, z_n) \leftrightarrow_\kappa (z_1', \ldots, z_n')$ in $\mathbb{Z}^n$.

Given a digital image $(X, \kappa) \subset (\mathbb{Z}^n, \kappa)$, the *$r$-th subdivision* of X is

$$S(X, r) = \{(x_1, \ldots, x_n) \in \mathbb{Z}_r^n \mid (\lfloor x_1 \rfloor, \ldots, \lfloor x_n \rfloor) \in X\}.$$

Let $E_r : S(X, r) \to X$ be the natural map sending $(x_1, \ldots, x_n) \in S(X, r)$ to $(\lfloor x_1 \rfloor, \ldots, \lfloor x_n \rfloor)$.

For a digital image $(X, \kappa) \subset (\mathbb{Z}^n, \kappa)$, a function $f : S(X, r) \to Y$ *induces a multivalued function* $F : X \multimap Y$ as follows:

$$F(x) = \bigcup_{x' \in E_r^{-1}(x)} \{f(x')\}.$$

A multivalued function $F : X \multimap Y$ is called *continuous* when there is some r such that F is induced by some single valued continuous function $f : S(X, r) \to Y$. $\square$

Note that in contrast with the definition of continuity, the definition of connectivity preservation makes no reference to X as being embedded inside of any particular integer lattice $\mathbb{Z}^n$.

Example 2.3. [4] An example of two digital images and their subdivisions is given in Figure 1. Note that the subdivision construction (and thus the notion of continuity) depends on the particular embedding of X as a subset of $\mathbb{Z}^n$. In particular we may have $X, Y \subset \mathbb{Z}^n$ with X isomorphic to Y but $S(X, r)$ not isomorphic to $S(Y, r)$. This is the case for the two images in Figure 1, when we use 8-adjacency for all images: X and Y in the figure are isomorphic, each being a set of two adjacent points, but $S(X, 2)$ and $S(Y, 2)$ are not isomorphic since $S(X, 2)$ can be disconnected by removing a single point, while this is impossible in $S(Y, 2)$. $\square$

Lemma 2.4. [3] Let $F : (X, c_u) \multimap (Y, c_v)$ be a continuous multivalued function generated by the continuous single-valued function $f : (S(X, r), c_u) \to (Y, c_v)$. Then for all $n \in \mathbb{N}$ there is a continuous single-valued function $f_n : (S(X, nr), c_u) \to (Y, c_v)$ that generates F. $\square$

Proposition 2.5. [5, 6] Let $F : X \multimap Y$ be a continuous multivalued function between digital images. Then

- for all $x \in X$, $F(x)$ is connected; and
- F is connectivity preserving. $\square$

Proposition 2.6. [4] Let X and Y be digital images. Suppose Y is connected. Then the multivalued function $f : X \multimap Y$ defined by $f(x) = Y$ for all $x \in X$ is connectivity preserving. $\square$

Proposition 2.7. [4] Let $F : (X, \kappa) \multimap (Y, \lambda)$ be a multivalued surjection between digital images (X, κ) and (Y, κ). If X is finite and Y is infinite, then F is not continuous. $\square$

Corollary 2.8. [4] Let $F : X \multimap Y$ be the multivalued function between digital images defined by $F(x) = Y$ for all $x \in X$. If X is finite and Y is infinite and connected, then F is connectivity preserving but not continuous. $\square$

Other examples of connectivity preserving but not continuous multivalued functions on finite spaces are given in [4].

Other terminology we use includes the following. Given a digital image $(X, \kappa) \subset \mathbb{Z}^n$ and $x \in X$, the set of points adjacent to $x \in \mathbb{Z}^n$ is

$$N_\kappa(x) = \{y \in \mathbb{Z}^n \mid y \leftrightarrow_\kappa x\}.$$

2.3 Weak and strong multivalued continuity

Other notions of continuity have been given for multivalued functions between graphs (equivalently, between digital images). We have the following.

Definition 2.4. [11] Let $F : X \multimap Y$ be a multivalued function between digital images.

- F has *weak continuity*, or is *weakly continuous*, if for each pair of adjacent $x, y \in X$, $f(x)$ and $f(y)$ are adjacent subsets of Y.
- F has *strong continuity*, or is *strongly continuous*, if for each pair of adjacent $x, y \in X$, every point of $f(x)$ is adjacent or equal to some point of $f(y)$ and every point of $f(y)$ is adjacent or equal to some point of $f(x)$. □

Clearly, strong continuity implies weak continuity. Example 2.11 below shows that the converse assertion is false.

Theorem 2.9. [4] A multivalued function $F : X \multimap Y$ is connectivity preserving if and only if the following are satisfied:

- F has weak continuity.
- For every $x \in X$, $F(x)$ is a connected subset of Y. □

Example 2.10. [4] If $F : [0,1]_{\mathbb{Z}} \multimap [0,2]_{\mathbb{Z}}$ is defined by $F(0) = \{0,2\}$, $F(1) = \{1\}$, then F has both weak and strong continuity. Thus a multivalued function between digital images that has weak or strong continuity need not have connected point-images. By Theorem 2.2 and Proposition 2.5 it follows that neither having weak continuity nor having strong continuity implies that a multivalued function is connectivity preserving or continuous. □

Example 2.11. [4] Let $F : [0,1]_{\mathbb{Z}} \multimap [0,2]_{\mathbb{Z}}$ be defined by $F(0) = \{0,1\}$, $F(1) = \{2\}$. Then F is continuous and has weak continuity but does not have strong continuity. □

3 Continuous and other structured multivalued functions

We say a digital image (X, κ) has *subdivisions preserving adjacency* if whenever $x \leftrightarrow_{\kappa} x'$ in X, there exist $x_0 \in S(\{x\}, r)$ and $x'_0 \in S(\{x'\}, r)$ such that $x_0 \leftrightarrow_{\kappa} x'_0$. Images with any c_u-adjacency or with the generalized normal product adjacency [3] $NP_u(c_{u_1}, \ldots, c_{u_v})$ on $\Pi_{i=1}^{v} X_i$, where c_{u_i} is the adjacency of X_i, have subdivisions preserving adjacency.

Theorem 3.1. [5, 6, 4] Let (X, κ) and (Y, λ) have subdivisions preserving adjacency. If $F : (X, \kappa) \multimap (Y, \lambda)$ is a continuous multivalued function, then F is connectivity preserving. □

Theorem 3.2. *Let (X, κ) and (Y, λ) have subdivisions preserving adjacency. Let $F : (X, \kappa) \multimap (Y, \lambda)$ be continuous. Then F is (κ, λ)-weakly continuous.*

Proof. Let $f : S(X, r) \to Y$ be a (κ, λ)-continuous function that generates F. Let $x \leftrightarrow_\kappa x'$ in X. Then there exist $x_0 \in S(\{x\}, r)$ and $x'_0 \in S(\{x'\}, r)$ such that $x_0 \leftrightarrow_\kappa x'_0$. Then we have $f(x_0) \in F(x)$, $f(x'_0) \in F(x')$, and $f(x_0) \leftrightarrows_\lambda f(x'_0)$. Thus, $F(X)$ and $F(x')$ are λ-adjacent sets, so F is weakly continuous. QED

The following example shows that neither weak continuity nor strong continuity implies continuity.

Example 3.3. Let $F : [0, 1]_\mathbb{Z} \multimap [0, 2]_\mathbb{Z}$ be defined by $F(0) = \{1\}$, $F(1) = \{0, 2\}$. Then F is (c_1, c_1)-weakly continuous and (c_1, c_1)-strongly continuous but is not (c_1, c_1)-continuous.

Proof. It is easily seen that F is weakly and strongly continuous. Since $F(1)$ is not c_1-connected, by Theorem 3.1 we can conclude that F is not continuous. QED

4 Composition

Suppose $f : (X, \kappa) \multimap (Y, \lambda)$ and $g : (Y, \lambda) \multimap (W, \mu)$ are multivalued functions between digital images. What properties of f and g are preserved by their composition? The following are known concerning the multivalued function $g \circ f : X \multimap W$.

- [7]

  $$\text{If } f \text{ and } g \text{ are both continuous, } g \circ f \text{ need not be continuous.} \qquad (4.1)$$

  There are additional hypotheses explored in [7] under which a composition $g \circ f$ of continuous multivalued functions is continuous.

- [4] If f and g are both connectivity preserving, then $g \circ f$ is connectivity preserving.

We have the following.

Theorem 4.1. *Let $f : (X, \kappa) \multimap (Y, \lambda)$ and $g : (Y, \lambda) \multimap (W, \mu)$ be multivalued functions between digital images.*

- *If f and g are both weakly continuous, then $g \circ f$ is weakly continuous.*

- *If f and g are both strongly continuous, then $g \circ f$ is strongly continuous.*

Proof. Suppose f and g are both weakly continuous. Let $x \leftrightarrow_\kappa x'$ in X. Then there exist $y \in f(x)$ and $y' \in f(x')$ such that $y \leftrightarroweq_\lambda y'$. Therefore, there exist $w \in g(y) \subset g \circ f(x)$ and $w' \in g(y') \subset g \circ f(x')$ such that $w \leftrightarroweq_\mu w'$. Thus, $g \circ f$ is weakly continuous.

Suppose f and g are both strongly continuous. Let $x \leftrightarrow_\kappa x'$ in X. Then for each $y \in f(x)$ there exists $y' \in f(x')$ such that $y \leftrightarroweq_\lambda y'$. Then for each $w \in g(y) \subset g \circ f(x)$ there exists $w' \in g(y') \subset g \circ f(x')$ such that $w \leftrightarroweq_\mu w'$. Since y was taken as an arbitrary member of $f(x)$, it follows that for each $w \in g \circ f(x)$ there exists $w' \in g \circ f(x')$ such that $w \leftrightarroweq_\mu w'$. Similarly, for each $w' \in g \circ f(x')$ there exists $w \in g \circ f(x)$ such that $w' \leftrightarroweq_\mu w$. Thus, $g \circ f$ is strongly continuous. QED

5 Retractions and extensions

Retractions and extensions are studied for single-valued continuous functions between digital images in [1], for continuous multivalued functions in [5, 6], and for connectivity preserving multivalued functions between digital images in [4]. In this section, we obtain more results for retractions and extensions among multivalued functions.

Since we wish to study multivalued functions that have properties of retractions and that are any of continuous, weakly continuous, strongly continuous, or connectivity preserving, we will call a multivalued function $F : X \multimap A$ a retraction if $A \subset X$ and for all $a \in A$, $F(a) = \{a\}$.

Theorem 5.1. *Let $A \subset (X, \kappa)$. Then there is a multivalued continuous retraction $R : X \multimap A$ if and only if for each continuous single-valued function $f : A \to Y$, there is a continuous multivalued function $F : X \multimap Y$ that extends f.*

Proof. Suppose there is a multivalued continuous retraction $R : X \multimap A$. Then there is a continuous single-valued function $R' : S(X, r) \to A$ that generates R. Given a continuous single-valued function $f : A \to Y$, $f \circ R' : S(X, r) \to Y$ is a continuous extension of f and therefore generates a continuous multivalued extension of f from X to Y.

Suppose given any continuous single-valued function $f : A \to Y$ for any Y, there is a continuous multivalued function $F : X \multimap Y$ that extends f. Then, in particular, $1_A : A \to A$ extends to a continuous multivalued retraction $R : X \multimap A$. QED

The composition of continuous multivalued functions between digital images need not be continuous [7]. However, we have the following (note for $X \subset Z^m$, c_m-adjacency is $(3^m - 1)$-adjacency [8]).

Theorem 5.2. [7] Let $X \subset (Z^m, 3^m - 1)$, $Y \subset (\mathbb{Z}^n, c_u)$ where $1 \leq u \leq n$, $W \subset (\mathbb{Z}^p, c_v)$ where $1 \leq v \leq p$. Suppose $F : X \multimap Y$ and $G : Y \multimap W$ are continuous multivalued functions. Then $G \circ F : (X, 3^m - 1) \multimap (W, c_v)$ is a continuous multivalued function. $\square$

This enables us to examine cases for which we can obtain a stronger result than Theorem 5.1.

Theorem 5.3. *Let $A \subset X \subset (Z^m, 3^m - 1)$. Then there is a multivalued continuous retraction $R : X \multimap A$ if and only if for each continuous multivalued function $F : (A, 3^m - 1) \multimap (Y, c_u)$, there is a continuous multivalued function $F' : X \multimap Y$ that extends F.*

Proof. This assertion follows from an argument like that of the proof of Theorem 5.1, using for one of the implications that by Theorem 5.2, $F \circ R$ is a continuous multivalued function. QED

We say a multivalued function $f : X \multimap Y$ is a *surjection* if for every $y \in Y$ there exists $x \in X$ such that $y \in f(x)$.

Proposition 5.4. *Let (X, κ) and (Y, λ) be nonempty digital images. Then there is a (κ, λ)-strongly continuous multivalued function $F : X \multimap Y$ that is a surjection. Further, if Y is connected then there exists such a multivalued function F that is also connectivity preserving.*

Proof. For all $x \in X$, let $F(x) = Y$. Let $x \leftrightarrow_\kappa x'$ in X. Then for any $y \in F(x)$ we also have $y \in F(x')$, and for any $y' \in F(x')$ we also have $y' \in F(x)$, so F is strongly continuous.

Clearly, if Y is connected then F is connectivity preserving. QED

Corollary 5.5. *Let (X, κ) and (Y, λ) be nonempty digital images such that the number of κ-components of X is greater than or equal to the number of λ-components of Y. Then there is a (κ, λ)-strongly continuous multivalued function $F : X \multimap Y$ that is a connectivity preserving surjection.*

Proof. Let $\{X_u\}_{u \in U}$ be the set of distinct κ-components of X. Let $\{Y_v\}_{v \in V}$ be the set of distinct λ-components of Y. Since $|U| \geq |V|$, there is a surjection $s : U \to V$. By Proposition 5.4, there is a (κ, λ)-strongly continuous, connectivity preserving surjective multivalued function $F_u : X_u \multimap Y_{s(u)}$ for every $u \in U$. Then the multivalued function $F : X \multimap Y$ defined by $F(x) = F_u(x)$ for $x \in X_u$ is easily seen to be a strongly continuous connectivity preserving surjection. QED

Lemma 5.6. *Let $A \subset (X, \kappa)$, $A \neq \emptyset$, where X is connected. For all $x \in X$, let $l^\kappa_X(x, A)$ be the length of a shortest κ-path in X from x to any point of A. For all $x, x' \in X$, if $x \leftrightarrow_\kappa x'$ then $|l^\kappa_X(x, A) - l^\kappa_X(x', A)| \leq 1$.*

Proof. Note the assumption that X is connected guarantees that for all $x \in X$, $l^\kappa_X(x, A)$ is finite.

Let $\{x_i\}_{i=0}^m$ be a path in X from $x = x_0$ to $x_m \in A$, where $m = l^\kappa_X(x, A)$. Then $\{x'\} \cup \{x_i\}_{i=0}^m$ is a path of length $m + 1$ from x' to $x_m \in A$. Therefore, $l^\kappa_X(x', A) \leq m + 1 = l^\kappa_X(x, A) + 1$. Similarly, $l^\kappa_X(x, A) \leq l^\kappa_X(x', A) + 1$. The assertion follows. QED

In the following, for $x \in X$, $A \subset X$, let $L^\kappa_A(x, X)$ be the set such that $a \in L^\kappa_A(x, X)$ implies $a \in A$ and there is a path in X from x to a of length in $\{l^\kappa_X(x, A), l^\kappa_X(x, A) + 1\}$.

Theorem 5.7. *Let (X, κ) be connected. Suppose $\emptyset \neq A \subset X$. Then the multi-valued function $f : X \multimap A$ defined by*

$$f(x) = \begin{cases} \{x\} & \text{if } x \in A; \\ L^\kappa_A(x, X) & \text{if } x \in X \setminus A, \end{cases}$$

is a weakly continuous retraction.

Proof. Let $x \leftrightarrow_\kappa x'$ in X. We must consider the following cases.

- $x, x' \in A$. Then $f(x) = \{x\}$ and $f(x') = \{x'\}$ are clearly adjacent sets.
- $x \in A$, $x' \in X \setminus A$. Then $l^\kappa_X(x', A) = 1$ and $x \in f(x) \cap f(x')$, so $f(x)$ and $f(x')$ are κ-adjacent sets.
- $x' \in A$, $x \in X \setminus A$. This is similar to the previous case.
- $x, x' \in X \setminus A$. Without loss of generality,

$$l^\kappa_X(x, A) \leq l^\kappa_X(x', A). \tag{5.1}$$

  Let $a \in A$ be such that there is a κ-path $P = \{x_i\}_{i=0}^m$ in X from $x' = x_0$ to $x_m = a$, such that $m = l^\kappa_X(x', A)$. By Lemma 5.6 and statement (5.1), $\{x\} \cup P$ is a path in X from x to a of length at most $L^\kappa_X(x, A) + 1$. Therefore, $a \in f(x) \cap f(x')$.

In all cases, $f(x)$ and $f(x')$ are adjacent sets, so f has weak continuity.

Clearly, f is a retraction. QED

The function constructed for Theorem 5.7 does not generally have strong continuity, as shown in the following example.

Example 5.8. Let $X = [0,4]_{\mathbb{Z}}$. Let $A = \{0,4\}$. Then the multivalued function $f : (X, c_1) \multimap (A, c_1)$ of Theorem 5.7 does not have strong continuity.

Proof. This follows from the observations that $f(1) = \{0\}$, $f(2) = \{0,4\}$, $1 \leftrightarrow_{c_1} 2$, and $4 \in f(2)$ is not c_1-adjacent to any member of $f(1)$. QED

For $A \subset X$, let $Bd^{\kappa}_X(A) = \{a \in A \mid N^{\kappa}_X(a) \setminus A \neq \emptyset\}$.

As an alternative to the multivalued function f of Theorem 5.7, we can consider the following.

Theorem 5.9. *Suppose $\emptyset \neq A \subset (X, \kappa)$. Then the multivalued function $g : X \multimap A$ defined by*

$$g(x) = \begin{cases} \{x\} & \text{if } x \in A; \\ Bd^{\kappa}_X(A) & \text{if } x \in X \setminus A, \end{cases}$$

is a weakly continuous retraction. Further, if $Bd^{\kappa}_X(A)$ is connected, then g is connectivity preserving.

Proof. Clearly g is a retraction. To show g is weakly continuous, suppose $x \leftrightarrow_{\kappa} x'$ in X. We consider the following cases.

- $x, x' \in A$. Then $x \in g(x)$ and $x' \in g(x')$, so $g(x)$ and $g(x')$ are κ-adjacent sets.
- $x \in A$, $x' \in X \setminus A$. Then $x \in Bd^{\kappa}_X(A)$, so $x \in g(x) \cap g(x')$. Thus, $g(x)$ and $g(x')$ are κ-adjacent sets.
- $x' \in A$, $x \in X \setminus A$. This is similar to the previous case.
- $x, x' \in X \setminus A$. Then $g(x) = g(x')$.

Thus, in all cases, $g(x)$ and $g(x')$ are adjacent sets. Therefore, g is weakly continuous.

Suppose $Bd^{\kappa}_X(A)$ is connected. Then for every $x \in X$, $g(x)$ is connected. It follows from Theorem 2.9 that g is connectivity preserving. QED

Example 5.10. Let $X = [0,4]_{\mathbb{Z}} \subset (\mathbb{Z}, c_1)$. Let $A = [1,3]_{\mathbb{Z}} \subset (\mathbb{Z}, c_1)$. Then the multivalued function g of Theorem 5.9 is not strongly continuous.

Proof. We have $g(1) = \{1\}$ and $g(0) = \{1,3\}$. Thus, $0 \leftrightarrow_{c_1} 1$ but $3 \in g(0)$ has no c_1-neighbor in $g(1)$. QED

Theorem 5.11. [1] X_0 is a (single-valued) retract of (X, κ) if and only if for every digital image (Y, λ) and every (κ, λ)-continuous single-valued function $f : X_0 \to Y$, there exists a (κ, λ)-continuous extension $f' : X \to Y$. $\square$

The proof of this assertion depends on the fact that the composition of continuous single-valued functions is continuous. Since statement (4.1) implies that a similar argument does not work for multivalued functions, it is not known whether an analog of Theorem 5.11 for multivalued functions is correct.

Theorem 5.12. *Let $X_0 \subset (X, \kappa)$ and let $F : X_0 \multimap (Y, \lambda)$ be a weakly continuous multivalued function. Then there is an extension $F' : X \multimap Y$ that is (κ, λ)-weakly continuous. Further, if F is connectivity preserving and $F(X_0)$ is λ-connected, then F' can be taken to be connectivity preserving.*

Proof. Let F' be defined by

$$F'(x) = \begin{cases} F(x) & \text{if } x \in X_0; \\ F(X_0) & \text{if } x \in X \setminus X_0. \end{cases}$$

Let $x \leftrightarrow_\kappa x'$. If $\{x, x'\} \subset X_0$, then $F'(x) = F(x)$ and $F(x') = F'(x')$ are λ-adjacent subsets of Y. If $\{x, x'\} \subset X \setminus X_0$, then $F'(x) = F(X_0) = F'(x')$. Otherwise, without loss of generality we have $x \in X_0$ and $x' \in X \setminus X_0$. Then $F'(x) = F(x) \subset F(X_0) = F'(x')$. Thus, in all cases, $F'(x)$ and $F'(x')$ are λ-adjacent subsets of Y. Therefore, F' is weakly continuous.

Suppose F is connectivity preserving and $F(X_0)$ is λ-connected. Let A be a κ-connected subset of X. If $A \subset X_0$, then $F'(A) = F(A)$ is connected. Otherwise, $F'(A) = F(X_0)$ is λ-connected. Thus, F' is connectivity preserving. QED

Corollary 5.13. *Let $X_0 \subset (X, \kappa)$. Then X_0 is a weakly continuous multivalued retract of X.*

Proof. Since the identity function on X_0 is a weakly continuous multivalued function, the assertion follows from Theorem 5.12. QED

Theorem 5.14. *Let $\emptyset \neq X_0 \subset X$ and let $F : X_0 \multimap Y$ be a (κ, λ)-connectivity preserving multivalued function. If*

- *Y is connected, or*
- *$F(X_0)$ is connected, or*
- *$F(Bd_X^\kappa(X_0))$ is connected,*

then there is a (κ, λ)-connectivity preserving extension $F' : X \multimap Y$ of F.

Proof. Suppose Y is connected. Let $F' : X \multimap Y$ be defined by

$$F'(x) = \begin{cases} F(x) & \text{if } x \in X_0; \\ Y & \text{if } x \in X \setminus X_0. \end{cases}$$

Let A be a κ-connected subset of X. If $A \subset X_0$, then $F'(A) = F(A)$ is connected. Otherwise, $F'(A) = Y$ is connected. Thus, F' is connectivity preserving.

Suppose $F(X_0)$ is connected. Let $F' : X \multimap Y$ be defined by

$$F'(x) = \begin{cases} F(x) & \text{if } x \in X_0; \\ F(X_0) & \text{if } x \in X \setminus X_0. \end{cases}$$

Let A be a κ-connected subset of X. If $A \subset X_0$, then $F'(A) = F(A)$ is connected. Otherwise, $F'(A) = F(X_0)$ is connected. Thus, F' is connectivity preserving.

Suppose $Bd_X^\kappa(X_0)$ is connected. Let $F' : X \multimap Y$ be defined by

$$F'(x) = \begin{cases} F(x) & \text{if } x \in X_0; \\ F(Bd_X^\kappa(X_0)) & \text{if } x \in X \setminus X_0. \end{cases}$$

Let A be a κ-connected subset of X.

- If $A \subset X_0$, then $F'(A) = F(A)$ is connected.
- If $A \subset X \setminus X_0$, then $F'(A) = F(Bd_X^\kappa(X_0))$ is connected.
- Otherwise, there exists $a_0 \in A \setminus X_0$ and, for all $a \in A$, a κ-path P_a in A from a to a_0. Then
$$F'(A) = \bigcup_{a \in A} F'(P_a). \tag{5.2}$$
Since $F'(a_0) \subset F'(P_a)$ for all $a \in A$, if we can show each $F'(P_a)$ is λ-connected then it will follow from equation (5.2) that $F'(A)$ is λ-connected.
Suppose $P_a = \{x_i\}_{i=1}^m$ where $x_1 = a$, $x_m = a_0$, and $x_{i+1} \leftrightarrow_\kappa x_i$ for $1 \leq i < m$. For each maximal segment $\{x_i\}_{i=u}^v$ of P_a contained in $A \cap X_0$, we have $x_v \in Bd_X^\kappa(X_0)$, so
$$F'(x_v) = F(x_v) \subset F(Bd_X^\kappa(X_0)) = F'(x_{v+1}).$$
Thus, $F(\{x_v, x_{v+1}\})$ is λ-connected. Similarly, if $u > 1$ then $F(\{x_{u-1}, x_u\})$ is λ-connected. It follows that $F(P_a)$ is connected.

Thus, F' is connectivity preserving. QED

Corollary 5.15. *Let $\emptyset \neq X_0 \subset (X, \kappa)$. If X_0 is connected or if $Bd_X^\kappa(X_0)$ is connected, then X_0 is a connectivity preserving multivalued retract of X.*

Proof. Since the identity function on X_0 is a connectivity preserving multivalued function, the assertion follows from Theorem 5.14. QED

An alternate proof of the first assertion of Corollary 5.15 is given for Theorem 7.2 of [4].

6 Wedges

Let (W, κ) be a digital image such that $W = X \cup X'$, where $X \cap X' = \{x_0\}$ for some $x_0 \in W$. We say W is the *wedge* of X and X', written $W = X \wedge X'$ or $(W, \kappa) = (X, \kappa) \wedge (X', \kappa)$.

Let $X \cap X' = \{x_0\}$, $Y \cap Y' = \{y_0\}$, $F : X \multimap Y$ and $F' : X' \multimap Y'$ be multivalued functions such that $F(x_0) = \{y_0\} = F'(x_0)$. Define $F \wedge F' : X \wedge X' \multimap Y \wedge Y'$ by

$$(F \wedge F')(a) = \begin{cases} F(a) & \text{if } a \in X \setminus \{x_0\}; \\ F'(a) & \text{if } a \in X' \setminus \{x_0\}; \\ \{y_0\} & \text{if } a = x_0. \end{cases}$$

Lemma 6.1. *Let $(W, \kappa) = (X, \kappa) \wedge (X', \kappa)$, where $\{x_0\} = X \cap X'$. Let $A \subset W$. Then A is κ-connected if and only if each of $A \cap X$ and $A \cap X'$ is κ-connected.*

Proof. Suppose A is κ-connected. Let $a_0, a_1 \in A \cap X$. Then there is a κ-path $P = \{x_i\}_{i=1}^n$ in A from a_0 to a_1. If P is not a subset of $A \cap X$ then there are smallest and largest indices u, v such that $\{x_u, x_v\} \subset (A \cap X') \setminus \{x_0\}$. Then $P_0 = \{x_i\}_{i=1}^{u-1}$ is a path in $A \cap X$ from a_0 to x_0, and $P_1 = \{x_i\}_{i=v+1}^n$ is a path in $A \cap X$ from x_0 to a_1. Therefore, $P_0 \cup P_1$ is a path in $A \cap X$ from a_0 to a_1. Since a_0 and a_1 were arbitrarily chosen, $A \cap X$ is connected. Similarly, $A \cap X'$ is connected.

Suppose each of $A \cap X$ and $A \cap X'$ is κ-connected. Let $a_0, a_1 \in A$. If $a_0 \in A \cap X$ then there is a path P in $A \cap X$ from a_0 to x_0. Similarly, if $a_0 \in A \cap X'$ then there is a path P in $A \cap X'$ from a_0 to x_0. In either case, there is a path P in A from a_0 to x_0. Similarly, there is a path P' in A from x_0 to a_1. Therefore, $P \cup P'$ is a path in A from a_0 to a_1. Since a_0 and a_1 were arbitrarily chosen, it follows that A is connected. QED

Theorem 6.2. *Let $X \cap X' = \{x_0\}$, $Y \cap Y' = \{y_0\}$. Let $F : X \multimap Y$ and $F' : X' \multimap Y'$ be multivalued functions such that $\{y_0\} = F(x_0) = F'(x_0)$.*

(1) If F and F' are both (κ, λ)-continuous, then $F \wedge F' : X \wedge X' \multimap Y \wedge Y'$ is (κ, λ)-continuous.

(2) If F and F' are both (κ, λ)-connectivity preserving, then $F \wedge F' : X \wedge X' \multimap Y \wedge Y'$ is (κ, λ)-connectivity preserving.

(3) If F and F' are both (κ, λ)-weakly continuous, then $F \wedge F' : X \wedge X' \multimap Y \wedge Y'$ is (κ, λ)-weakly continuous.

(4) If F and F' are both (κ, λ)-strongly continuous, then $F \wedge F' : X \wedge X' \multimap Y \wedge Y'$ is (κ, λ)-strongly continuous.

Proof. We argue as follows.

(1) Suppose F and F' are both (κ, λ)-continuous. From Lemma 2.4, we conclude that for some $r \in \mathbb{N}$, there are continuous functions $f : S(X, r) \to Y$ and $f' : S(X', r) \to Y'$ that generate F and F', respectively. Then the single-valued function $\hat{f} : S(X, r) \cup S(X', r) = S(X \wedge X', r) \to Y \wedge Y'$ defined by

$$\hat{f}(a) = \begin{cases} f(a) & \text{if } a \in S(X, r); \\ f'(a) & \text{if } a \in S(X', r) \end{cases}$$

is well defined since $F(x_0) = \{y_0\} = F'(x_0)$, and is continuous and generates $F \wedge F'$. Thus, $F \wedge F'$ is continuous.

(2) Suppose F and F' are both (κ, λ)-continuity preserving. Let A be a connected subset of $X \wedge X'$. If $A \subset X$ or if $A \subset X'$ then $(F \wedge F')(A)$ is either $F(A)$ or $F'(A)$, hence is connected. Otherwise, by Lemma 6.1, each of $A \cap X$ and $A \cap X'$ is connected. Therefore, $F(A \cap X)$ and $F'(A \cap X')$ are connected, and $F(A \cap X) \cap F'(A \cap X') = \{y_0\}$. Thus, $(F \wedge F')(A) = F(A \cap X) \cup F'(A \cap X')$ is connected. Hence $F \wedge F'$ is connectivity preserving.

(3) Suppose F and F' are both (κ, λ)-weakly continuous. Let $a_0 \leftrightarrow_\kappa a_1$ in $X \wedge X'$. Then either $\{a_0, a_1\} \subset X$ or $\{a_0, a_1\} \subset X'$. Therefore either $(F \wedge F')(\{a_0, a_1\}) = F(\{a_0, a_1\})$ or $(F \wedge F')(\{a_0, a_1\}) = F'(\{a_0, a_1\})$. In either case, $(F \wedge F')(a_0)$ and $(F \wedge F')(a_1\})$ are adjacent sets. Thus, $(F \wedge F')$ is weakly continuous.

(4) Suppose F and F' are both (κ, λ)-strongly continuous. Let $a_0 \leftrightarrow_\kappa a_1$ in $X \wedge X'$. As above for our argument concerning weak continuity, either $(F \wedge F')(\{a_0, a_1\}) = F(\{a_0, a_1\})$ or $(F \wedge F')(\{a_0, a_1\}) = F'(\{a_0, a_1\})$. In either case, one sees from Definition 2.4 that $F \wedge F'$ is strongly continuous.

QED

7 Further remarks

We have studied properties of structured multivalued functions between digital images. In section 3, we studied relations between continuous multivalued functions and other structured types of multivalued functions. In section 4, we studied properties of multivalued functions that are preserved by composition. In section 5, we studied retractions and extensions of structured multivalued functions. In section 6, we studied properties of multivalued functions that are preserved by the wedge operation.

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