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Certain generating funtion of generalized Apostol type Legendre-based polynomials

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Abstract. In this paper, we aim to introduce a generating function for generalized Apostol type Legendre-Based polynomials which extends some known results. We also deduce some properties of the generalized Apostol-Bernoulli polynomials, the generalized Apostol-Euler polynomials and the generalized Apostol-Genocchi polynomials of higher order. By making use of the generating function method and some functional equations mentioned in the paper, we conduct a further investigation in order to obtain some implicit summation formulae and general symmetry identities for the generalized Apostol type Legendre-Based polynomials.

Keywords: Hermite polynomials and their two variable extensions, generalized Apostol Bernoulli numbers and polynomials, generalized Apostol Euler numbers and polynomials, generalized Apostol Genocchi numbers and polynomials, Legendre polynomials and their two variable extensions, 0th order Tricomi function, generalized Apostol type Legendre-Based polynomials, summation formulae, symmetric identities.

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1 Introduction

Generalized and multivariable forms of the special functions of mathematical physics have witnessed a significant evolution during the recent years. In particular, the special polynomials of more than one variable provided new means of analysis for the solution of large classes of partial differential equations often encountered in physical problems. Most of the special function of mathematical physics and their generalization have been suggested by physical problems.

To give an example, we recall that the 2-variable Hermite Kampe' de Fe'riet polynomials $H_n(x,y)$ [2] defined by the generating function

$$\exp(xt+yt^2)=\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!} \tag{1.1}$$

are the solution of heat equation

$$\frac{\partial}{\partial y}H_n(x,y)=\frac{\partial^2}{\partial x^2}H_n(x,y), \qquad H_n(x,0)=x^n \tag{1.2}$$

The higher order Hermite polynomials, sometimes called the Kampe' de Fe'riet polynomials of order m or the Gould-Hopper polynomials $H_n^{(m)}(x,y)$ defined by the generating function ([11], p.58 (6.3))

$$\exp(xt+yt^m)=\sum_{n=0}^{\infty}H_n^{(m)}(x,y)\frac{t^n}{n!} \tag{1.3}$$

are the solution of the generalized heat equation [7]

$$\frac{\partial}{\partial y}f(x,y)=\frac{\partial^m}{\partial x^m}f(x,y), \qquad f(x,0)=x^n \tag{1.4}$$

Also, we note that

$$H_n^{(2)}(x,y)=H_n(x,y), \tag{1.5}$$

$$H_n(2x,-1)=H_n(x), \tag{1.6}$$

where $H_n(x)$ are the classical Hermite polynomials [1].

Next, we recall that the 2-variable Legendre polynomials $S_n(x,y)$ and $R_n(x,y)$ are given by Dattoli et al. [8]

$$S_n(x,y)=n!\sum_{k=0}^{[\frac{n}{2}]}\frac{x^k y^{n-2k}}{[(n-2k)!(k!)^2]} \tag{1.7}$$

and

$$R_n(x,y)=(n!)^2\sum_{k=0}^{\infty}\frac{(-1)^{n-k}x^{n-k}y^k}{[(n-2k)!]^2(k!)^2} \tag{1.8}$$

respectively, and are related with the ordinary Legendre polynomials $P_n(x)$ [27] as

$$P_n(x) = S_n\left(-\frac{1-x^2}{4}, x\right) = R_n\left(\frac{1-x}{2}, \frac{1+x}{2}\right).$$

From equation (7) and (8), we have

$$S_n(x,0) = n!\frac{x^{[\frac{n}{2}]}}{[(\frac{n}{2})!]^2}, \qquad S_n(0,y) = y^n, \tag{1.9}$$

$$R_n(x,0) = (-x)^n, \qquad R_n(0,y) = y^n. \tag{1.10}$$

The generating functions for two variable Legendre polynomials $S_n(x,y)$ and $R_n(x,y)$ are given by [8]

$$e^{yt}C_0(-xt^2) = \sum_{n=0}^{\infty} S_n(x,y)\frac{t^n}{n!} \tag{1.11}$$

$$C_0(xt)C_0(-yt) = \sum_{n=0}^{\infty} R_n(x,y)\frac{t^n}{(n!)^2} \tag{1.12}$$

where $C_0(x)$ is the 0-th order Tricomi function [27]

$$C_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{(r!)^2} \tag{1.13}$$

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order $\alpha \in C$, the generalized Euler polynomials $E_n^{(\alpha)}(x)$ of order $\alpha \in C$ and generalized Genocchi polynomials $G_n^{(\alpha)}(x)$ of order $\alpha \in C$, are defined respectively by the following generating functions (see [10], vol. 3, p. 253 et seq.), ([16], section 2.8) and [18]):

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \qquad (|t| < 2\pi; 1^{\alpha} := 1) \tag{1.14}$$

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!}, \qquad (|t| < \pi; 1^{\alpha} := 1) \tag{1.15}$$

$$\left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\frac{t^n}{n!}, \qquad (|t| < \pi; 1^{\alpha} := 1) \tag{1.16}$$

The literature contains a large number of interesting properties and relationship involving these polynomials (see [3],[4],[5],[10],[12],[24]). Luo and Srivastava

([20],[22]) introduced the generalized Apostol Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ of order α, Luo [17] investigated Apostol Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ of order α and the generalized Apostol Genocchi polynomials $G_n^{(\alpha)}(x;\lambda)$ of order α (see also [18],[19],[21]).

The generalized Apostol Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in C$, the generalized Apostol Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in C$ and generalized Apostol Genocchi polynomials $G_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in C$, are defined respectively by the following generating functions:

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, \qquad (|t + \ln\,\lambda| < 2\pi; 1^{\alpha} := 1) \tag{1.17}$$

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, \qquad |t + \ln\,\lambda| < \pi; 1^{\alpha} := 1) \tag{1.18}$$

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, \qquad (|t + \ln\,\lambda| < \pi; 1^{\alpha} := 1) \tag{1.19}$$

where, if we take $x = 0$ in the above, we have

$$B_n^{(\alpha)}(0;\lambda) := B_n^{(\alpha)}(\lambda),\ E_n^{(\alpha)}(0;\lambda) := E_n^{(\alpha)}(\lambda),\ G_n^{(\alpha)}(0;\lambda) = G_n^{(\alpha)}(\lambda) \tag{1.20}$$

calling Apostol-Bernoulli number of order α, Apostol-Euler number of order α and Apostol-Genocchi number of order α, respectivily. Also

$$B_n^{(\alpha)}(x) := B_n^{(\alpha)}(x;1),\ E_n^{(\alpha)}(x) := E_n^{(\alpha)}(x;1),\ G_n^{(\alpha)}(x) = G_n^{(\alpha)}(x;1). \tag{1.21}$$

Srivastava et al. [29],[30] have investigated the new class of generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda;a,b,e)$ of order α, Apostol-Euler polynomials $E_n^{(\alpha)}(x;\lambda;a,b,e)$ of order α and Apostol-Genocchi polynomials $G_n^{(\alpha)}(x;\lambda;a,b,e)$ of order α, are defined respectively by the following generating functions:

$$\left(\frac{t}{\lambda b^t - a^t}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda;a,b,e)\frac{t^n}{n!}, \qquad (|t\,\ln\left(\frac{\mathrm{a}}{\mathrm{b}}\right) + \ln\,\lambda| < 2\pi; 1^{\alpha} := 1) \tag{1.22}$$

$$\left(\frac{2}{\lambda b^t + a^t}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda;a,b,e)\frac{t^n}{n!}, \qquad \left(\left|t\ \ln\left(\frac{\mathrm{a}}{\mathrm{b}}\right) + \ln\ \lambda\right| < \pi; 1^{\alpha} := 1\right) \tag{1.23}$$

$$\left(\frac{2t}{\lambda b^t + a^t}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda;a,b,e)\frac{t^n}{n!}, \qquad \left(\left|t\ \ln\left(\frac{\mathrm{a}}{\mathrm{b}}\right) + \ln\ \lambda\right| < \pi; 1^{\alpha} := 1\right) \tag{1.24}$$

If we take $a = 1,\ b = e$ in (22), (23) and (24) respectively, we have (17), (18) and (19). Obviously when we set $\lambda = 1,\ \alpha = 1,\ b = e$ in (22), (23) and (24), we have classical Bernoulli polynomials $B_n(x)$, classical Euler polynomials $E_n(x)$ and classical Genocchi polynomials $G_n(x)$.

Recently, Luo et al. [23] introduced a generalized Apostol type polynomials $F_n^{(\alpha)}(x;\lambda;\mu,\nu)\quad (\alpha \in N_0, \mu, \nu \in C)$ of order α, are defined by means of the following generating function:

$$\left(\frac{2^{\mu}t^{\nu}}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} F_n^{(\alpha)}(x;\lambda;\mu,\nu)\frac{t^n}{n!}, \qquad |t| < |\log(-\lambda))\tag{1.25}$$

where

$$F_n^{(\alpha)}(\lambda;\mu,\nu) = F_n^{(\alpha)}(0;\lambda;\mu,\nu)\tag{1.26}$$

denotes the so called Apostol type number of order α.
So that by comparing equation (17), (18) and (19), we have

$$B_n^{(\alpha)}(x;\lambda) = (-1)^{\alpha} F_n^{(\alpha)}(x;-\lambda;0,1)\tag{1.27}$$

$$E_n^{(\alpha)}(x;\lambda) = F_n^{(\alpha)}(x;\lambda;1,0)\tag{1.28}$$

$$G_n^{(\alpha)}(x;\lambda) = F_n^{(\alpha)}(x;\lambda;1,1)\tag{1.29}$$

The special polynomials of more than one variable provide new means of analysis for the solutions of a wide class of partial differential equations often encountered in physical problems. It happens very often that the solution of a given problem in physics or applied mathematics requires the evaluation of infinite sum, involving special functions. Problem of this type arise, for example, in the computation of the higher-order moments of a distribution or in evaluation of transition matrix elements in quantum mechanics. In [6], Dattoli showed that the summation formulae of special functions, often encountered in applications ranging from electromagnetic process to combinatorics, can be written in terms of Hermite polynomials of more than one variable.

In this paper, we first give definition of the generalized Apostol type Legendre-Based polynomials ${}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu)$ which generalizes the concept stated above and then find their basic properties and relationships with Apostol type Hermite-Based polynomials ${}_HF_n^{(\alpha)}(x, y; \lambda; \mu, \nu)$ of Lu et al. [23]. Some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions. These result extends some known summation and identities of generalized Apostol type Hermite-Bernoulli, Euler and Genocchi polynomials studied by Dattoli et al. [9], Yang [31], Khan et al. [13]-[15], Pathan [25], Pathan and Khan [26], Yang et al. [32] and Zhang et al. [33].

2 Definition and Properties of the Generalized Apostol type Legendre-Based polynomials

In this section, we present further definition and properties for the generalized Apostol type Legendre-Based polynomials ${}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu)$.

Definition 2.1. The generalized Apostol type Legendre-Based polynomials ${}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu)$ $(\alpha \in N_0, \mu, \nu \in C)$ for nonnegative integer n, are defined by

$$\sum_{n=0}^{\infty} {}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu)\frac{t^n}{n!} = \left(\frac{2^\mu t^\nu}{\lambda e^t + 1}\right)^\alpha e^{yt+zt^2} C_0(-xt^2), \quad (|t| < |\log(-\lambda)) \tag{2.1}$$

so that

$${}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu) = \sum_{m=0}^{n} \sum_{k=0}^{[\frac{m}{2}]} \frac{F_{n-m}^{(\alpha)}(\lambda; \mu, \nu)\ S_{m-2k}(x, y) z^k n!}{(m-2k)!k!(n-m)!} \tag{2.2}$$

For $\alpha = 1$, in (30) we obtain the following generating function

$$\sum_{n=0}^{\infty} {}_SF_n(x, y, z; \lambda; \mu, \nu)\frac{t^n}{n!} = \left(\frac{2^\mu t^\nu}{\lambda e^t + 1}\right) e^{yt+zt^2} C_0(-xt^2), \quad (|t| < |\log(-\lambda)) \tag{2.3}$$

For $x = 0$ in (30), the result reduces to Hermite-Based generalized Apostol type polynomials of Lu et al. [23] is defined as

$$\sum_{n=0}^{\infty} {}_H F_n^{(\alpha)}(y, z; \lambda; \mu, \nu)\frac{t^n}{n!} = \left(\frac{2^\mu t^\nu}{\lambda e^t + 1}\right)^\alpha e^{yt+zt^2}, \quad (|t| < |\log(-\lambda)) \tag{2.4}$$

As in the case $y = z = 0$ in (30), it leads to an extension of the generalized Apostol type polynomials denoted by $F_n^{(\alpha)}(x; \lambda; \mu, \nu)$ for a nonnegative integer n defined earlier by (25).

The generalized Apostol type Legendre-Based polynomials ${}_S F_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu, e)$ defined by (30) have the following properties which are stated as theorem below.

Theorem 2.1. For any integral $n \geq 1$, $x, y, z \in R$, $\lambda \in C$ and $\alpha \in N$. The following relation for the generalized Apostol type Legendre-Based polynomials ${}_S F_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu)$ holds true:

$$\begin{aligned}
&{}_S F_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu, e) = {}_S F_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu), \\
&(-1)^\alpha {}_S F_n^{(\alpha)}(x, y, z; -\lambda; 0, 1) = {}_S B_n^{(\alpha)}(x, y, z; \lambda) \\
&{}_S F_n^{(\alpha)}(x, y, z; \lambda; 1, 0) = {}_S E_n^{(\alpha)}(x, y, z; \lambda), \\
&{}_S F_n^{(\alpha)}(x, y, z; \lambda; 1, 1) = {}_S G_n^{(\alpha)}(x, y, z; \lambda)
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
&{}_S F_n^{(\alpha+\beta)}(x, y + z, v + u; \lambda; \mu, \nu) = \\
&\qquad \sum_{k=0}^{n} \binom{n}{k} {}_S F_{n-k}^{(\alpha)}(x, z, v; \lambda; \mu, \nu) {}_H F_k^{(\beta)}(y, u; \lambda; \mu, \nu)
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&{}_S F_n^{(\alpha+\beta)}(x, y + v, z; \lambda; \mu, \nu) = \\
&\qquad \sum_{k=0}^{n} \binom{n}{k} {}_S F_{n-k}^{(\alpha)}(x, y, z; \lambda; \mu, \nu) F_k^{(\beta)}(v; \lambda; \mu, \nu)
\end{aligned} \tag{2.7}$$

Proof. The proof of (34) are obvious. Applying definition (30), we have

$$\sum_{n=0}^{\infty} {}_S F_n^{(\alpha+\beta)}(x, y + z, v + u; \lambda; \mu, \nu)\frac{t^n}{n!}$$

$$= \left(\sum_{n=0}^{\infty} {}_S F_n^{(\alpha)}(x, z, v; \lambda; \mu, \nu)\frac{t^n}{n!}\right) \left(\sum_{k=0}^{\infty} {}_H F_k^{(\beta)}(y, u; \lambda; \mu, \nu)\frac{t^k}{k!}\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} {}_S F_{n-k}^{(\alpha)}(x,z,v;\lambda;\mu,\nu) {}_H F_k^{(\beta)}(x,y,u;\lambda;\mu,\nu) \right) \frac{t^n}{n!}$$

Now equating the coefficient of $\frac{t^n}{n!}$ in the above equation, we get the result (35). Again by definition (30) of Apostol type Legendre-Based polynomials, we have

$$\sum_{n=0}^{\infty} {}_S F_n^{(\alpha+\beta)}(x,y+v,z;\lambda;\mu,\nu) \frac{t^n}{n!} = \left(\frac{2^\mu t^\nu}{\lambda e^t + 1} \right)^{\alpha+\beta} e^{(y+v)t+zt^2} C_0(-xt^2)$$

$$= \left(\left(\frac{2^\mu t^\nu}{\lambda e^t + 1} \right)^{\alpha} e^{yt+zt^2} C_0(-xt^2) \right) \left(\left(\frac{2^\mu t^\nu}{\lambda e^t + 1} \right)^{\beta} e^{vt} \right)$$

which can be written as

$$= \sum_{n=0}^{\infty} {}_S F_n^{(\alpha)}(x,y,z;\lambda;\mu,\nu) \frac{t^n}{n!} \sum_{k=0}^{\infty} F_k^{(\beta)}(v;\lambda;\mu,\nu) \frac{t^k}{k!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} {}_S F_{n-k}^{(\alpha)}(x,y,z;\lambda;\mu,\nu) F_k^{(\beta)}(v;\lambda;\mu,\nu) \right) \frac{t^n}{n!}$$

Now equating the coefficient of the like power of $\frac{t^n}{n!}$ in the above equation, we get the result (36).

3 Implicit Summation Formulae Involving Apostol type Legendre-Based Polynomials

For the derivation of implicit formulae involving generalized Apostol type Legendre-Based polynomials ${}_S F_n^{(\alpha)}(x,y,z;\lambda;\mu,\nu)$ the same consideration as developed for the ordinary Hermite and related polynomials in Khan et al. [14] and Hermite-Bernoulli polynomials in Pathan [25], Pathan and Khan [26] and Khan et al. [13]-[15] holds as well. First we prove the following results involving generalized Apostol type Legendre-Based polynomials ${}_S F_n^{(\alpha)}(x,y,z;\lambda;\mu,\nu)$.

Theorem 3.1. For any integral $n \geq 1$, $x,y,z \in R$, $\lambda \in C$ and $\alpha \in N$. The following implicit summation formulae for the generalized Apostol type Legendre-Based polynomials ${}_S F_n^{(\alpha)}(x,y,z;\lambda;\mu,\nu)$ holds true:

$${}_S F_{m+n}^{(\alpha)}(x,v,z;\lambda;\mu,\nu) = \sum_{s,k=0}^{m,n} \begin{pmatrix} m \\ s \end{pmatrix} \begin{pmatrix} n \\ k \end{pmatrix} (v-y)^{s+k} {}_S F_{m+n-s-k}^{(\alpha)}(x,v,z;\lambda;\mu,\nu) \quad (3.1)$$

Proof. We replace t by $t+u$ and rewrite the generating function (30) as

$$\left(\frac{2^{\mu}(t+u)^{\nu}}{\lambda e^{t+u}+1}\right)^{\alpha} e^{z(t+u)^2} C_0(-x(t+u)^2)$$
$$= e^{-y(t+u)} \sum_{m,n=0}^{\infty} {}_SF_{m+n}^{(\alpha)}(x,y,z;\lambda;\mu,\nu) \frac{t^n}{n!} \frac{u^m}{m!} \tag{3.2}$$

Replacing y by v in the above equation and equating the resulting equation to the above equation, we get

$$e^{(v-y)(t+u)} \sum_{m,n=0}^{\infty} {}_SF_{m+n}^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\frac{t^n}{n!}\frac{u^m}{m!}$$
$$= \sum_{m,n=0}^{\infty} {}_SF_{m+n}^{(\alpha)}(x,v,z;\lambda;\mu,\nu)\frac{t^n}{n!}\frac{u^m}{m!} \tag{3.3}$$

On expanding exponential function (39) gives

$$\sum_{N=0}^{\infty} \frac{[(v-y)(t+u)]^N}{N!} \sum_{m,n=0}^{\infty} {}_SF_{m+n}^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\frac{t^n}{n!}\frac{u^m}{m!}$$
$$= \sum_{m,n=0}^{\infty} {}_SF_{m+n}^{(\alpha)}(x,v,z;\lambda;\mu,\nu)\frac{t^n}{n!}\frac{u^m}{m!} \tag{3.4}$$

which on using the following formula ([28], p. 52(2))

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(m+n)\frac{x^n}{n!}\frac{y^m}{m!} \tag{3.5}$$

in the left hand side becomes

$$\sum_{k,s=0}^{\infty} \frac{(v-y)^{k+s}\ t^k u^s}{k!s!} \sum_{m,n=0}^{\infty} {}_SF_{m+n}^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\frac{t^n}{n!}\frac{u^m}{m!}$$
$$= \sum_{m,n=0}^{\infty} {}_SF_{m+n}^{(\alpha)}(x,v,z;\lambda;\mu,\nu)\frac{t^n}{n!}\frac{u^m}{m!} \tag{3.6}$$

Now replacing n by $n-k$, s by $n-s$ and using the lemma ([28], p. 100(1)) in the left hand side of (42), we get

$$\sum_{m,n=0}^{\infty} \sum_{k,s=0}^{m,n} \frac{(v-y)^{k+s}}{k!s!} {}_SF_{m+n-k-s}^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\frac{t^n}{(n-k)!}\frac{u^m}{(m-s)!}$$

$$= \sum_{m,n=0}^{\infty} {}_S F_{m+n}^{(\alpha)}(x,v,z;\lambda;\mu,\nu)\frac{t^n}{n!}\frac{u^m}{m!} \tag{3.7}$$

Finally, on equating the coefficient of the like powers of t^n and u^m in the above equation, we get the required result.

Remark 3.1.1. Replacing $\lambda = -\lambda$, $\mu = 0$ and $\nu = 1$ in Theorem (3.1) and then multiplying $(-1)^\alpha$ on both side of the result, we immediately deduce the following corollary.

Corollary 3.1.1. The following implicit summation formula for the generalized Apostol type Legendre-Bernoulli polynomials ${}_S B_n^{(\alpha)}(x,y,z;\lambda)$ holds true:

$${}_S B_{m+n}^{(\alpha)}(x,v,z;\lambda) = \sum_{s,k=0}^{m,n} \binom{m}{s}\binom{n}{k}(v-y)^{s+k}{}_S B_{m+n-s-k}^{(\alpha)}(x,v,z;\lambda) \tag{3.8}$$

Remark 3.1.2. By taking $\mu = 1$ and $\nu = 0$ in Theorem (3.1), we immediately deduce the following corollary.

Corollary 3.1.2. The following implicit summation formula for the generalized Apostol type Legendre-Euler polynomials ${}_S E_n^{(\alpha)}(x,y,z;\lambda)$ holds true:

$${}_S E_{m+n}^{(\alpha)}(x,v,z;\lambda) = \sum_{s,k=0}^{m,n} \binom{m}{s}\binom{n}{k}(v-y)^{s+k}{}_S E_{m+n-s-k}^{(\alpha)}(x,v,z;\lambda) \tag{3.9}$$

Remark 3.1.3. By taking $\mu = 1$ and $\nu = 1$ in Theorem (3.1), we immediately deduce the following corollary.

Corollary 3.1.3. The following implicit summation formula for the generalized Apostol type Legendre-Genocchi polynomials ${}_S G_n^{(\alpha)}(x,y,z;\lambda)$ holds true:

$${}_S G_{m+n}^{(\alpha)}(x,v,z;\lambda) = \sum_{s,k=0}^{m,n} \binom{m}{s}\binom{n}{k}(v-y)^{s+k}{}_S G_{m+n-s-k}^{(\alpha)}(x,v,z;\lambda) \tag{3.10}$$

Theorem 3.2. For any integral $n \geq 1$, $x,y,z \in R$, $\lambda \in C$ and $\alpha \in N$. The following implicit summation formula for the generalized Apostol type Legendre-Based polynomials ${}_S F_n^{(\alpha)}(x,y,z;\lambda;\mu,\nu)$ holds true:

$${}_S F_n^{(\alpha)}(x,y+u,z;\lambda;\mu,\nu) = \sum_{j=0}^{n} \binom{n}{j} u^j {}_S F_{n-j}^{(\alpha)}(x,y,z;\lambda;\mu,\nu) \tag{3.11}$$

Proof. Since

$$\sum_{n=0}^{\infty} {}_SF_n^{(\alpha)}(x,y+u,z;\lambda;\mu,\nu)\frac{t^n}{n!} = \left(\frac{2^\mu t^\nu}{\lambda e^t+1}\right)^\alpha \; e^{(y+u)t+zt^2}C_0(-xt^2)$$

$$\sum_{n=0}^{\infty} {}_SF_n^{(\alpha)}(x,y+u,z;\lambda;\mu,\nu)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} {}_SF_n^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty} u^j\frac{t^j}{j!}\right)$$

Now, replacing n by $n-j$ and comparing the coefficient of t^n, we get the result (47).

Remark 3.2.1. Replacing $\lambda = -\lambda$, $\mu = 0$ and $\nu = 1$ in Theorem (3.2) and then multiplying $(-1)^\alpha$ on both side of the result, we immediately deduce the following corollary.

Corollary 3.2.1. The following implicit summation formula for the generalized Apostol type Legendre-Bernoulli polynomials ${}_SB_n^{(\alpha)}(x,y,z;\lambda)$ holds true:

$$ {}_SB_n^{(\alpha)}(x,y+u,z;\lambda) = \sum_{j=0}^{n}\binom{n}{j} u^j\,{}_SB_{n-j}^{(\alpha)}(x,y,z;\lambda) \tag{3.12}$$

Remark 3.2.2. By taking $\mu = 1$ and $\nu = 0$ in Theorem (3.2), we immediately deduce the following corollary.

Corollary 3.2.2. The following implicit summation formula for the generalized Apostol type Legendre-Euler polynomials ${}_SE_n^{(\alpha)}(x,y,z;\lambda)$ holds true:

$$ {}_SE_n^{(\alpha)}(x,y+u,z;\lambda) = \sum_{j=0}^{n}\binom{n}{j} u^j\,{}_SE_{n-j}^{(\alpha)}(x,y,z;\lambda) \tag{3.13}$$

Remark 3.2.3. By taking $\mu = 1$ and $\nu = 1$ in Theorem (3.2), we immediately deduce the following corollary.

Corollary 3.2.3. The following implicit summation formula for the generalized Apostol type Legendre-Genocchi polynomials ${}_SG_n^{(\alpha)}(x,y,z;\lambda)$ holds true:

$$ {}_SG_n^{(\alpha)}(x,y+u,z;\lambda) = \sum_{j=0}^{n}\binom{n}{j} u^j\,{}_SG_{n-j}^{(\alpha)}(x,y,z;\lambda) \tag{3.14}$$

Theorem 3.3. For any integral $n \geq 1,\ x, y, z \in R,\ \lambda \in C$ and $\alpha \in N$. The following implicit summation formula for the generalized Apostol type Legendre-Based polynomials ${}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu)$ holds true:

$$ {}_SF_n^{(\alpha)}(x, y+u, z+w; \lambda; \mu, \nu) = \sum_{m=0}^{n} \binom{n}{m} {}_SF_{n-m}^{(\alpha)}(x, y, z; \lambda; \mu, \nu) H_m(u, w) \tag{3.15} $$

Proof. By the definition of Apostol type Legendre-Based polynomials and the definition (1), we have

$$ \left(\frac{2^\mu t^\nu}{\lambda e^t + 1}\right)^\alpha e^{(y+u)t+(z+w)t^2} C_0(-xt^2) = \left(\sum_{n=0}^{\infty} {}_SF_n^{(k)}(x, y, z)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} H_m(u, w)\frac{t^m}{m!}\right) \tag{3.16} $$

Now, replacing n by $n-m$ and comparing the coefficient of t^n, we get the result (51).

Remark 3.3.1. Replacing $\lambda = -\lambda,\ \mu = 0$ and $\nu = 1$ in Theorem (3.3) and then multiplying $(-1)^\alpha$ on both side of the result, we immediately deduce the following corollary.

Corollary 3.3.1. The following implicit summation formula for the generalized Apostol type Legendre-Bernoulli polynomials ${}_SB_n^{(\alpha)}(x, y, z; \lambda)$ holds true:

$$ {}_SB_n^{(\alpha)}(x, y+u, z+w; \lambda) = \sum_{m=0}^{n} \binom{n}{m} {}_SB_{n-m}^{(\alpha)}(x, y, z; \lambda) H_m(u, w) \tag{3.17} $$

Remark 3.3.2. By taking $\mu = 1$ and $\nu = 0$ in Theorem (3.3), we immediately deduce the following corollary.

Corollary 3.3.2. The following implicit summation formula for the generalized Apostol type Legendre-Euler polynomials ${}_SE_n^{(\alpha)}(x, y, z; \lambda)$ holds true:

$$ {}_SE_n^{(\alpha)}(x, y+u, z+w; \lambda) = \sum_{m=0}^{n} \binom{n}{m} {}_SE_{n-m}^{(\alpha)}(x, y, z; \lambda) H_m(u, w) \tag{3.18} $$

Remark 3.3.3. By taking $\mu = 1$ and $\nu = 1$ in Theorem (3.3), we immediately deduce the following corollary.

Corollary 3.3.3. The following implicit summation formula for the generalized Apostol type Legendre-Genocchi polynomials ${}_SG_n^{(\alpha)}(x, y, z; \lambda)$ holds true:

$$ {}_SG_n^{(\alpha)}(x, y + u, z + w; \lambda) = \sum_{m=0}^{n} \binom{n}{m} {}_SG_{n-m}^{(\alpha)}(x, y, z; \lambda) H_m(u, w) \tag{3.19} $$

Theorem 3.4. For any integral $n \geq 1$, $x, y, z \in R$, $\lambda \in C$ and $\alpha \in N$. The following implicit summation formula for the generalized Apostol type Legendre-Based polynomials ${}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu)$ holds true:

$$ {}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu) = \sum_{m=0}^{n-2j} \sum_{j=0}^{[\frac{n}{2}]} \frac{F_m^{(\alpha)}(\lambda; \mu, \nu) S_{n-m-2j}(x, y) z^j n!}{m! j! (n - m - 2j)!} \tag{3.20} $$

Proof. Applying the definition (30) to the term $\left(\frac{2^\mu t^\nu}{\lambda e^t + 1}\right)^\alpha$ and expanding the exponential and tricomi function $e^{yt+zt^2} C_0(-xt^2)$ at $t = 0$ yields

$$ \left(\frac{2^\mu t^\nu}{\lambda e^t + 1}\right)^\alpha e^{yt+zt^2} C_0(-xt^2) = $$

$$ \left(\sum_{m=0}^{\infty} F_m^{(\alpha)}(\lambda; \mu, \nu) \frac{t^m}{m!}\right) \left(\sum_{n=0}^{\infty} S_n(x, y) \frac{t^n}{n!}\right) \left(\sum_{j=0}^{\infty} z^j \frac{t^{2j}}{j!}\right) $$

$$ \sum_{n=0}^{\infty} {}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu) \frac{t^n}{n!} = $$

$$ \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{F_m^{(\alpha)}(\lambda; \mu, \nu) S_{n-m}(x, y)}{(n - m)! m!}\right) t^n \left(\sum_{j=0}^{\infty} z^j \frac{t^{2j}}{j!}\right) $$

Now, replacing n by $n - 2j$ and comparing the coefficient of t^n, we get the result (55).

Remark 3.4.1. Replacing $\lambda = -\lambda$, $\mu = 0$ and $\nu = 1$ in Theorem (3.4) and then multiplying $(-1)^\alpha$ on both side of the result, we immediately deduce the following corollary.

Corollary 3.4.1. The following implicit summation formula for the generalized Apostol type Legendre-Bernoulli polynomials ${}_SB_n^{(\alpha)}(x, y, z; \lambda)$ holds true:

$$ {}_SB_n^{(\alpha)}(x, y, z; \lambda) = \sum_{m=0}^{n-2j} \sum_{j=0}^{[\frac{n}{2}]} \frac{B_m^{(\alpha)}(\lambda) S_{n-m-2j}(x, y) z^j n!}{m! j! (n - m - 2j)!} \tag{3.21} $$

Remark 3.4.2. By taking $\mu = 1$ and $\nu = 0$ in Theorem (3.4), we immediately deduce the following corollary.

Corollary 3.4.2. The following implicit summation formula for the generalized Apostol type Legendre-Euler polynomials ${}_SE_n^{(\alpha)}(x, y, z; \lambda)$ holds true:

$$ {}_SE_n^{(\alpha)}(x, y, z; \lambda) = \sum_{m=0}^{n-2j} \sum_{j=0}^{[\frac{n}{2}]} \frac{E_m^{(\alpha)}(\lambda) S_{n-m-2j}(x, y) z^j n!}{m! j! (n-m-2j)!} \tag{3.22} $$

Remark 3.4.3. By taking $\mu = 1$ and $\nu = 1$ in Theorem (3.4), we immediately deduce the following corollary.

Corollary 3.4.3. The following implicit summation formula for the generalized Apostol type Legendre-Genocchi polynomials ${}_SG_n^{(\alpha)}(x, y, z; \lambda)$ holds true:

$$ {}_SG_n^{(\alpha)}(x, y, z; \lambda) = \sum_{m=0}^{n-2j} \sum_{j=0}^{[\frac{n}{2}]} \frac{G_m^{(\alpha)}(\lambda) S_{n-m-2j}(x, y) z^j n!}{m! j! (n-m-2j)!} \tag{3.23} $$

Theorem 3.5. For any integral $n \geq 1$, $x, y, z \in R$, $\lambda \in C$ and $\alpha \in N$. The following implicit summation formula for the generalized Apostol type Legendre-Based polynomials ${}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu)$ holds true:

$$ {}_SF_n^{(\alpha)}(x, y+1, z; \lambda; \mu, \nu) = \sum_{m=0}^{n} \binom{n}{m} {}_SF_{n-m}^{(\alpha)}(x, y, z; \lambda; \mu, \nu) \tag{3.24} $$

Proof. By the definition of Apostol type Legendre-Based polynomials, we have

$$ \sum_{n=0}^{\infty} {}_SF_n^{(\alpha)}(x, y+1, z; \lambda; \mu, \nu) \frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu) \frac{t^n}{n!} $$

$$ = \left(\frac{2^\mu t^\nu}{\lambda e^t + 1} \right)^\alpha (e^t - 1) e^{yt+zt^2} C_0(-xt^2) $$

$$ = \sum_{n=0}^{\infty} {}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu) \frac{t^n}{n!} \left(\sum_{m=0}^{\infty} \frac{t^m}{m!} - 1 \right) $$

$$ = \sum_{n=0}^{\infty} {}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu) \frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{t^m}{m!} - \sum_{n=0}^{\infty} {}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu) \frac{t^n}{n!} $$

$$ = \sum_{n=0}^{\infty} \sum_{m=0}^{n} {}_SF_{n-m}^{(\alpha)}(x, y, z; \lambda \mu, \nu) \frac{t^n}{m!(n-m)!} - \sum_{n=0}^{\infty} {}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu) \frac{t^n}{n!} $$

Finally equating the coefficient of the like powers of t^n, we get the result (59).

Remark 3.5.1. Replacing $\lambda = -\lambda,\ \mu = 0$ and $\nu = 1$ in Theorem (3.5) and then multiplying $(-1)^\alpha$ on both side of the result, we immediately deduce the following corollary.

Corollary 3.5.1. The following implicit summation formula for the generalized Apostol type Legendre-Bernoulli polynomials ${}_SB_n^{(\alpha)}(x, y, z; \lambda)$ holds true:

$$ {}_SB_n^{(\alpha)}(x, y+1, z; \lambda) = \sum_{m=0}^{n} \binom{n}{m} {}_SB_{n-m}^{(\alpha)}(x, y, z; \lambda) \tag{3.25} $$

Remark 3.5.2. By taking $\mu = 1$ and $\nu = 0$ in Theorem (3.5), we immediately deduce the following corollary.

Corollary 3.5.2. The following implicit summation formula for the generalized Apostol type Legendre-Euler polynomials ${}_SE_n^{(\alpha)}(x, y, z; \lambda)$ holds true:

$$ {}_SE_n^{(\alpha)}(x, y+1, z; \lambda) = \sum_{m=0}^{n} \binom{n}{m} {}_SE_{n-m}^{(\alpha)}(x, y, z; \lambda) \tag{3.26} $$

Remark 3.5.3. By taking $\mu = 1$ and $\nu = 1$ in Theorem (3.5), we immediately deduce the following corollary.

Corollary 3.5.3. The following implicit summation formula for the generalized Apostol type Legendre-Genocchi polynomials ${}_SG_n^{(\alpha)}(x, y, z; \lambda)$ holds true:

$$ {}_SG_n^{(\alpha)}(x, y+1, z; \lambda) = \sum_{m=0}^{n} \binom{n}{m} {}_SG_{n-m}^{(\alpha)}(x, y, z; \lambda) \tag{3.27} $$

Theorem 3.6. For any integral $n \geq 1,\ x, y, z \in R,\ \lambda \in C$ and $\alpha \in N$. The following implicit summation formula for the generalized Apostol type Legendre-Based polynomials ${}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu)$ holds true:

$$ {}_SF_n^{(\alpha)}(x, y, z; \lambda, \mu, \nu) = \sum_{m=0}^{n} F_{n-m}^{(\alpha-1)}(\lambda; \mu, \nu) {}_SF_m(x, y, z; \lambda; \mu, \nu) \tag{3.28} $$

Proof. By the definition of Apostol type Legendre-Based polynomials, we have

$$\sum_{n=0}^{\infty} {}_SF_n^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\frac{t^n}{n!} = \left(\frac{2^\mu t^\nu}{\lambda e^t+1}\right)^{\alpha} e^{yt+zt^2}C_0(-xt^2)$$

$$\sum_{n=0}^{\infty} {}_SF_n^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\frac{t^n}{n!} = \left(\frac{2^\mu t^\nu}{\lambda e^t+1}\right)^{\alpha-1} \left(\frac{2^\mu t^\nu}{\lambda e^t+1}\right) e^{yt+zt^2}C_0(-xt^2)$$

$$\sum_{n=0}^{\infty} {}_SF_n^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\frac{t^n}{n!} =$$

$$\left(\sum_{n=0}^{\infty} F_n^{(\alpha-1)}(\lambda;\mu,\nu)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} {}_SF_m(x,y,z;\lambda;\mu,\nu)\frac{t^m}{m!}\right)$$

Now replacing n by $n-m$ then equating the coefficients of the like powers of t^n, we get the result (63).

Remark 3.6.1. Replacing $\lambda=-\lambda,\ \mu=0$ and $\nu=1$ in Theorem (3.6) and then multiplying $(-1)^\alpha$ on both side of the result, we immediately deduce the following corollary.

Corollary 3.6.1. The following implicit summation formula for the generalized Apostol type Legendre-Bernoulli polynomials ${}_SB_n^{(\alpha)}(x,y,z;\lambda)$ holds true:

$${}_SB_n^{(\alpha)}(x,y,z;\lambda) = \sum_{m=0}^{n} B_{n-m}^{(\alpha-1)}(\lambda)\,{}_SB_m(x,y,z;\lambda) \tag{3.29}$$

Remark 3.6.2. By taking $\mu=1$ and $\nu=0$ in Theorem (3.6), we immediately deduce the following corollary.

Corollary 3.6.2. The following implicit summation formula for the generalized Apostol type Legendre-Euler polynomials ${}_SE_n^{(\alpha)}(x,y,z;\lambda)$ holds true:

$${}_SE_n^{(\alpha)}(x,y,z;\lambda) = \sum_{m=0}^{n} E_{n-m}^{(\alpha-1)}(\lambda)\,{}_SE_m(x,y,z;\lambda) \tag{3.30}$$

Remark 3.6.3. By taking $\mu=1$ and $\nu=1$ in Theorem (3.6), we immediately deduce the following corollary.

Corollary 3.6.3. The following implicit summation formula for the generalized Apostol type Legendre-Genocchi polynomials ${}_SG_n^{(\alpha)}(x,y,z;\lambda)$ holds true:

$${}_SG_n^{(\alpha)}(x,y,z;\lambda) = \sum_{m=0}^{n} G_{n-m}^{(\alpha-1)}(\lambda)\,{}_SG_m(x,y,z;\lambda) \tag{3.31}$$

4 General Symmetry Identities for the Generalized Apostol type Legendre-Based polynomials

In this section, we give general symmetry identities for the generalized Apostol type Legendre-Based polynomials ${}_SF_n^{(\alpha)}(x,y,z;\lambda;\mu,\nu)$ by applying the generating function (2.1). The result extends some known identities of Lu et al. [23], Yang [31], Khan et al. [13]-[15], Pathan [25], Pathan and Khan [26], Yang et al. [32] and Zhang et al. [33]. As it has been mentioned in previous sections, α will be considered as an arbitrary real or a complex parameter.

Theorem 4.1. For any integral $n \geq 1,\ x,y,z \in R,\ \lambda \in C$ and $\alpha \in N$. The following identity for the generalized Apostol type Legendre-Based polynomials ${}_SF_n^{(\alpha)}(x,y,z;\lambda;\mu,\nu)$ holds true:

$$\sum_{m=0}^{n} \binom{n}{m} b^m a^{n-m} {}_SF_{n-m}^{(\alpha)}(x,by,b^2z;\lambda;\mu,\nu) {}_SF_m^{(\alpha)}(x,ay,a^2z;\lambda;\mu,\nu)$$

$$= \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} {}_SF_{n-m}^{(\alpha)}(x,ay,a^2z;\lambda;\mu,\nu) {}_SF_m^{(\alpha)}(x,by,b^2z;\lambda;\mu,\nu) \quad (4.1)$$

Proof. Start with

$$g(t) = \left(\frac{((ab)^\nu 2^{2\mu} t^{2\nu})^2}{(\lambda e^{at}+1)(\lambda e^{bt}+1)} \right)^\alpha e^{(a+b)yt+(a^2+b^2)zt^2}\ (C_0(-xt^2))^2 \quad (4.2)$$

and

$$C_0(abxt) \neq\ C_0(axt)C_0(bxt)$$

Then the expression for $g(t)$ is symmetric in a and b and we can expand $g(t)$ into series in two ways to obtain

$$g(t) = \sum_{n=0}^{\infty} {}_SF_n^{(\alpha)}(x,by,b^2z;\lambda;\mu,\nu) \frac{(at)^n}{n!} \sum_{m=0}^{\infty} {}_SF_m^{(\alpha)}(x,ay,a^2z;\lambda;\mu,\nu) \frac{(bt)^m}{m!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m {}_SF_m^{(\alpha)}(x,by,b^2z;\lambda;\mu,\nu) {}_SF_{n-m}^{(\alpha)}(x,ay,a^2z;\lambda;\mu,\nu) t^n$$

On the similar lines we can show that

$$g(t) = \sum_{n=0}^{\infty} {}_SF_n^{(\alpha)}(x,ay,a^2z;\lambda;\mu,\nu) \frac{(bt)^n}{n!} \sum_{m=0}^{\infty} {}_SF_m^{(\alpha)}(x,by,b^2z;\lambda;\mu,\nu) \frac{(at)^m}{m!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} a^m b^{n-m} {}_S F_{n-m}^{(\alpha)}(x, ay, a^2 z; \lambda; \mu, \nu) {}_S F_m^{(\lambda)}(x, by, b^2 z; \lambda; \mu, \nu) t^n$$

Comparing the coefficient of t^n on the right hand sides of the last two equations we arrive at the desired result.

Remark 4.1.1. Replacing $\lambda = -\lambda$, $\mu = 0$ and $\nu = 1$ in Theorem (4.1) and then multiplying $(-1)^\alpha$ on both side of the result, we immediately deduce the following corollary.

Corollary 4.1.1. The following identity for the generalized Apostol type Legendre-Bernoulli polynomials ${}_S B_n^{(\alpha)}(x, y, z; \lambda)$ holds true:

$$\sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} b^m a^{n-m} {}_S B_{n-m}^{(\alpha)}(x, by, b^2 z; \lambda) {}_S B_m^{(\alpha)}(x, ay, a^2 z; \lambda)$$

$$= \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} a^m b^{n-m} {}_S B_{n-m}^{(\alpha)}(x, ay, a^2 z; \lambda) {}_S B_m^{(\alpha)}(x, by, b^2 z; \lambda) \tag{4.3}$$

Remark 4.1.2. By taking $\mu = 1$ and $\nu = 0$ in Theorem (4.1), we immediately deduce the following corollary.

Corollary 4.1.2. The following identity for the generalized Apostol type Legendre-Euler polynomials ${}_S E_n^{(\alpha)}(x, y, z; \lambda)$ holds true:

$$\sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} b^m a^{n-m} {}_S E_{n-m}^{(\alpha)}(x, by, b^2 z; \lambda) {}_S E_m^{(\alpha)}(x, ay, a^2 z; \lambda)$$

$$= \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} a^m b^{n-m} {}_S E_{n-m}^{(\alpha)}(x, ay, a^2 z; \lambda) {}_S E_m^{(\alpha)}(x, by, b^2 z; \lambda) \tag{4.4}$$

Remark 4.1.3. By taking $\mu = 1$ and $\nu = 1$ in Theorem (4.1), we immediately deduce the following corollary.

Corollary 4.1.3. The following identity for the generalized Apostol type Legendre-Genocchi polynomials ${}_S G_n^{(\alpha)}(x, y, z; \lambda)$ holds true:

$$\sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} b^m a^{n-m} {}_S G_{n-m}^{(\alpha)}(x, by, b^2 z; \lambda) {}_S G_m^{(\alpha)}(x, ay, a^2 z; \lambda)$$

$$= \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} {}_SG_{n-m}^{(\alpha)}(x, ay, a^2z; \lambda) {}_SG_m^{(\alpha)}(x, by, b^2z; \lambda) \tag{4.5}$$

Theorem 4.2. For any integral $n \geq 1$, $x, y, z \in R$, $\lambda \in C$ and $\alpha \in N$. The following identity for the generalized Apostol type Legendre-Based polynomials ${}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu)$ holds true:

$$\sum_{m=0}^{n} \binom{n}{m} b^m a^{n-m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j}$$

$$\times {}_SF_{n-m}^{(\alpha)}\left(x, by + \frac{b}{a}i + j, b^2u; \lambda; \mu, \nu\right) {}_SF_m^{(\alpha)}(x, az, a^2v; \lambda; \mu, \nu)$$

$$= \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-\lambda)^{i+j}$$

$$\times {}_SF_{n-m}^{(\alpha)}\left(x, ay + \frac{a}{b}i + j, a^2u; \lambda; \mu, \nu\right) {}_SF_m^{(\alpha)}(x, bz, b^2v; \lambda; \mu, \nu) \tag{4.6}$$

Proof. Let

$$g(t) = \left(\frac{((ab)^\nu 2^{2\mu} t^{2\nu})^2)^\alpha (C_0(-xt^2))^2 (\lambda(-1)^{a+1} e^{abt} + 1)^2 e^{ab(y+z)t + a^2b^2(u+v)t^2}}{(\lambda e^{at} + 1)^{\alpha+1} (\lambda e^{bt} + 1)^{\alpha+1}} \right)$$

$$g(t) = \left(\frac{2^\mu (at)^\nu C_0(-xt^2)}{\lambda e^{at} + 1} \right)^\alpha e^{abyt + a^2b^2ut^2} \left(\frac{1 - \lambda e^{-abt}}{\lambda e^{bt} + 1} \right)$$

$$\times \left(\frac{2^\mu (bt)^\nu C_0(-xt^2)}{\lambda e^{bt} + 1} \right)^\alpha e^{abzt + a^2b^2vt^2} \left(\frac{1 - \lambda e^{-abt}}{\lambda e^{at} + 1} \right)$$

From where we have

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} b^m a^{n-m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} \right.$$

$$\left. \times {}_SF_{n-m}^{(\alpha)}\left(x, by + \frac{b}{a}i + j, b^2u\lambda; \mu, \nu\right) {}_SF_m^{(\alpha)}(x, az, a^2v; \lambda; \mu, \nu) \right) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-\lambda)^{i+j} \right.$$

$$\times\, {}_SF^{(\alpha)}_{n-m}\left(x, ay+\frac{a}{b}i+j, a^2u;\lambda;\mu,\nu\right) {}_SF^{(\alpha)}_{m}(x, bz, b^2v;\lambda;\mu,\nu)\Big)\frac{t^n}{n!}$$

Our assertion follows from comparing the coefficients of $\frac{t^n}{n}$ on the right hand sides of the last two equations, we arrive at the desired result.

Remark 4.2.1. Replacing $\lambda=-\lambda,\ \mu=0$ and $\nu=1$ in Theorem (4.2) and then multiplying $(-1)^\alpha$ on both side of the result, we immediately deduce the following corollary.

Corollary 4.2.1. The following identity for the generalized Apostol type Legendre-Bernoulli polynomials ${}_SB^{(\alpha)}_n(x,y,z;\lambda)$ holds true:

$$\sum_{m=0}^{n}\binom{n}{m} b^m a^{n-m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j} \times {}_SB^{(\alpha)}_{n-m}\left(x, by+\frac{b}{a}i+j, b^2u;\lambda\right) {}_SB^{(\alpha)}_{m}(x, az, a^2v;\lambda)$$

$$=\sum_{m=0}^{n}\binom{n}{m} a^m b^{n-m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{i+j} \times {}_SB^{(\alpha)}_{n-m}\left(x, ay+\frac{a}{b}i+j, a^2u;\lambda\right) {}_SB^{(\alpha)}_{m}(x, bz, b^2v;\lambda) \quad (4.7)$$

Remark 4.2.2. By taking $\mu=1$ and $\nu=0$ in Theorem (4.2), we immediately deduce the following corollary.

Corollary 4.2.2. The following identity for the generalized Apostol type Legendre-Euler polynomials ${}_SE^{(\alpha)}_n(x,y,z;\lambda)$ holds true:

$$\sum_{m=0}^{n}\binom{n}{m} b^m a^{n-m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j} \times {}_SE^{(\alpha)}_{n-m}\left(x, by+\frac{b}{a}i+j, b^2u;\lambda\right) {}_SE^{(\alpha)}_{m}(x, az, a^2v;\lambda)$$

$$=\sum_{m=0}^{n}\binom{n}{m} a^m b^{n-m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{i+j} \times {}_SE^{(\alpha)}_{n-m}\left(x, ay+\frac{a}{b}i+j, a^2u;\lambda\right) {}_SE^{(\alpha)}_{m}(x, bz, b^2v;\lambda) \quad (4.8)$$

Remark 4.2.3. By taking $\mu = 1$ and $\nu = 1$ in Theorem (4.2), we immediately deduce the following corollary.

Corollary 4.2.3. The following identity for the generalized Apostol type Legendre-Genocchi polynomials ${}_SG_n^{(\alpha)}(x, y, z; \lambda)$ holds true:

$$\sum_{m=0}^{n} \binom{n}{m} b^m a^{n-m} \sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j} \times {}_SG_{n-m}^{(\alpha)}\left(x, by + \frac{b}{a}i + j, b^2u; \lambda\right) {}_SG_m^{(\alpha)}(x, az, a^2v; \lambda)$$

$$= \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} \sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{i+j} \times {}_SG_{n-m}^{(\alpha)}\left(x, ay + \frac{a}{b}i + j, a^2u; \lambda\right) {}_SG_m^{(\alpha)}(x, bz, b^2v; \lambda) \quad (4.9)$$

5 Conclusion and Suggestion

By applying the 2-variable Legendre polynomial $S_n(x, y)$, which are defined by means of a generating function (11), we have introduced and systematically investigated a family of the Legendre-based Apostol-type polynomials ${}_SF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu)$ defined by means of the generating function (30). In the readily-accessible literature on the subject, there exits a more general class of polynomials than the 2-variable Legendre polynomial $S_n(x, y)$. These general 2-variable polynomials $R_n(x, y)$ are popularly known as the 2-variable Legendre polynomial and are defined by means of a generating function (12). Moreover, it is good enough to say that to suitably extend the results asserted in this paper holds true for the generalized Legendre-based Apostol-type polynomials ${}_RF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu)$. The corresponding extension of the result in this paper based on ${}_RF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu)$ are still an open problem derived by means of the generating function (12).

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