Note di Matematica
Note Mat. **37** (2017) no. 2, 11–20.

ISSN 1123-2536, e-ISSN 1590-0932
doi:10.1285/i15900932v37n2p11

On solutions of fractional kinetic equations involving the generalized *k*-Bessel function

Kottakkaran Sooppy Nisar
Department of Mathematics, College of Arts and Science, Prince Sattam bin Abdulaziz University, Wadi Al dawaser, Riyadh region 11991, Saudi Arabia
ksnisar1@gmail.com

Feng Qi
Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China; Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China
qifeng618@gmail.com

Received: 28.2.2017; accepted: 18.4.2017.

Abstract. In the paper, the authors obtain solutions of fractional kinetic equations involving the generalized k-Bessel function given by Mondal. These results are general in nature and are useful to investigate many problems in applied sciences.

Keywords: solution, fractional kinetic equation, Laplace transform, k-Bessel function

MSC 2000 classification: primary 26A33, secondary 33E12, 44A10, 44A20

1 Introduction

The Mittag-Leffler function $E_\alpha(z)$ was defined [11] by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \quad z, \alpha \in \mathbb{C}, \quad |z| < 0, \quad \Re(\alpha) > 0$$

and $E_{\alpha,\beta}(x)$ was defined [22] by

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+\beta)}. \tag{1.1}$$

The k-Bessel function of the first kind was defined [14] by

$$J_{k,v}^{\gamma,\lambda}(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n+v+1)} \frac{1}{(n!)^2} \left(\frac{z}{2}\right)^n,$$

where $k \in \mathbb{R}$, $\lambda, \gamma, v \in \mathbb{C}$, $\Re(\lambda), \Re(v) > 0$,

$$(\gamma)_{n,k} = \begin{cases} \dfrac{\Gamma_k(\gamma + nk)}{\Gamma_k(\gamma)}, & k \in \mathbb{R}, \quad \gamma \in \mathbb{C} \setminus \{0\} \\ \gamma(\gamma + k) \cdots (\gamma + (n-1)k), & n \in \mathbb{N}, \quad \gamma \in \mathbb{C} \end{cases}$$

and

$$\Gamma_k(z) = \int_0^\infty e^{-t^k/k} t^{z-1} \, \mathrm{d}\, t, \quad \Re(z) > 0, \quad k > 0.$$

Recently, the k-Bessel function $J_{k,v}^{\gamma,\lambda}(z)$ was generalized [12] by

$$W_{\nu,c}^k(x) = \sum_{r=0}^\infty \frac{(-c)^r}{\Gamma_k(rk + \nu + k) r!} \left(\frac{x}{2}\right)^{2r + \nu/k}. \tag{1.2}$$

In this paper, we consider the function (1.2) and try to obtain solutions of fractional kinetic equations. For more details about fractional kinetic equations and its solutions, the Bessel functions, the k-Pochhammer symbols, and the Laplace transforms, please refer to [1, 2, 3, 4, 5, 6, 8, 9, 13, 15, 16, 17, 18, 19, 20, 23] and the closely related references therein.

As mentioned in [10], the destruction rate and the production rate is

$$\frac{\mathrm{d}\, N}{\mathrm{d}\, t} = -\mathfrak{d}(N_t) + \mathfrak{p}(N_t),$$

where $N_t(t^*) = N(t - t^*)$ for $t^* > 0$ and

$$\frac{\mathrm{d}\, N_i}{\mathrm{d}\, t} = -c_i N_i(t) \tag{1.3}$$

with $c_i > 0$ and $N_i(0) = N_0$ which means the number of density of species i at time $t = 0$. It is easy to see that the solution of (1.3) is $N_i(t) = N_0 e^{-c_i t}$. Integrating on both sides of (1.3) gives

$$N(t) - N_0 = -c \cdot {}_0D_t^{-1} N(t), \tag{1.4}$$

where ${}_0D_t^{-1}$ is the special case of the Riemann-Liouville integral operator and c is a constant.

The fractional generalization of (1.4) was given by

$$N(t) - N_0 = -c^v {}_0D_t^{-v} N(t),$$

where ${}_0D_t^{-v}$ is defined by

$${}_0D_t^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t - s)^{v-1} f(s) \, \mathrm{d}\, s, \quad \Re(v) > 0.$$

2 Solutions of generalized fractional Kinetic equations

We are now in a position to find solutions of generalized fractional Kinetic equations for the k-Bessel function.

Theorem 2.1. If $d, v > 0$, $l, c, t \in \mathbb{C}$, and $k \in \mathbb{R}^+$, then the equation

$$N(t) - N_0 W^k_{l,c}(t) = -d^v {}_0D_t^{-v} N(t), \tag{2.1}$$

has a solution

$$N(t) = N_0 \sum_{r=0}^{\infty} \frac{(-c)^r \Gamma(2r + l/k + 1)}{\Gamma_k(rk + l + k) r!} \left(\frac{t}{2}\right)^{2r+l/k+1} E_{v,2r+l/k+1}\big(-d^v t^v\big). \tag{2.2}$$

where $E_{v,2r+l/k+1}$ is the generalized Mittag-Leffler function given in (1.1).

Proof. Suppose that $f(t)$ is a real or complex valued function of the (time) variable $t > 0$ and s is a real or complex parameter. The Laplace transform of $f(t)$ is defined by

$$F(p) = L[f(t) : p] = \int_0^{\infty} e^{-pt} f(t)\, \mathrm{d}\, t, \quad \Re(p) > 0.$$

The Laplace transform of the Riemann-Liouville fractional integral operator [7, 21] is

$$L\{D_t^{-v} f(t); p\} = p^{-v} F(p). \tag{2.3}$$

Applying the Laplace transform on both sides of (2.1) and using (1.2) and (2.3) lead to

$$\begin{aligned}
L[N(t); p] &= N_0 L\big[W^k_{l,c}(t); p\big] - d^v L\big[D_t^{-v} N(t); p\big],\\
N(p) &= N_0 \int_0^{\infty} e^{-pt} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_k(rk + l + k) r!} \left(\frac{t}{2}\right)^{2r+l/k} \mathrm{d}\, t - d^v p^{-v} N(p),\\
N(p) + d^v p^{-v} N(p) &= N_0 \sum_{r=0}^{\infty} \frac{(-c)^r 2^{-(2r+l/k)}}{\Gamma_k(rk + l + k) r!} \int_0^{\infty} e^{-pt} t^{2r+l/k}\, \mathrm{d}\, t\\
&= N_0 \sum_{r=0}^{\infty} \frac{(-c)^r (2)^{-(2r+l/k)}}{\Gamma(rk + l + k) r!} \frac{\Gamma(2r + l/k + 1)}{p^{2r+l/k+1}},
\end{aligned}$$

and

$$N(p) = N_0 \sum_{r=0}^{\infty} \frac{(-c)^r (2)^{-(2r+l/k+1)} \Gamma(2r+l/k+1)}{\Gamma(rk+l+k) r!} \times \left\{ p^{-(2r+l/k+1)} \sum_{s=0}^{\infty} (1)_s \frac{[-(p/d)^{-v}]^s}{s!} \right\}. \quad (2.4)$$

Taking the inverse Laplace transform on both sides of (2.4) and using $L^{-1}\{p^{-v}\} = \frac{t^{v-1}}{\Gamma(v)}$ for $\Re(v) > 0$ reveals

$$L^{-1}(N(p)) = N_0 \sum_{r=0}^{\infty} \frac{(-c)^r (2)^{-(2r+l/k)} \Gamma(2r+l/k+1)}{\Gamma(rk+l/k+k) r!} \times L^{-1} \left\{ \sum_{s=0}^{\infty} (-1)^s d^{vs} p^{-(2r+l/k+1+vs)} \right\}$$

and

$$\begin{aligned} N(t) &= N_0 \sum_{r=0}^{\infty} \frac{(-c)^r (2)^{-(2r+l/k)} \Gamma(2r+l/k+1)}{\Gamma(rk+l/k+k) r!} \\ &\quad \times \sum_{s=0}^{\infty} (-1)^s d^{vs} \frac{t^{(2r+l/k+vs)}}{\Gamma(vs+2r+l/k+1)} \\ &= N_0 \sum_{r=0}^{\infty} \frac{(-c)^r \Gamma(2r+l/k+1)}{\Gamma(rk+l/k+k) r!} \left(\frac{t}{2} \right)^{2r+l/k} \\ &\quad \times \sum_{s=0}^{\infty} (-1)^s d^{vs} \frac{t^{vs}}{\Gamma(vs+2r+l/k+1)}. \end{aligned}$$

In view of definition of the generalized Mittag-Leffler function, we obtain the desired result. QED

Corollary 2.1. If $d, v > 0$ and $l, c, t \in \mathbb{C}$, then the equation

$$N(t) - N_0 W_{l,c}^1(t) = -d^v {}_0D_t^{-v} N(t)$$

has a solution

$$N(t) = N_0 \sum_{r=0}^{\infty} \frac{(-c)^r \Gamma(2r+l+1)}{\Gamma(r+l+1) r!} \left(\frac{t}{2} \right)^{2r+l+1} E_{v,2r+l+1}\big(-d^v t^v\big).$$

Proof. This follows from putting $k = 1$ in (2.2). QED

Theorem 2.2. If $d, v > 0$, $c, l, t \in \mathbb{C}$, and $k \in \mathbb{R}^+$, then the equation

$$N(t) - N_0 W^k_{l,c}(d^v t^v) = -d^v {}_0D_t^{-v} N(t),$$

has the solution

$$N(t) = N_0 \sum_{r=0}^{\infty} \frac{(-c)^r \Gamma(2rv + vl/k + 1)}{\Gamma_k(rk + l + k) r!} \left(\frac{d^v}{2}\right)^{2r+l/k} \times t^{2vr+lv/k} E_{v,2rv+lv/k+1}(-d^v t^v),$$

where $E_{v,2rv+lv/k+1}$ is the generalized Mittag-Leffler function defined in (1.1).

Proof. This can be proved by the same method as in the proof of Theorem 2.1. So we omit all details. QED

Corollary 2.2. If $d, v > 0$ and $c, l, t \in \mathbb{C}$, then the equation

$$N(t) - N_0 W^1_{l,c}(d^v t^v) = -d^v {}_0D_t^{-v} N(t)$$

has the solution

$$N(t) = N_0 \sum_{r=0}^{\infty} \frac{(-c)^r \Gamma(2rv + vl + 1)}{\Gamma_k(rk + l + k) r!} \left(\frac{d^v}{2}\right)^{2r+l} t^{2vr+lv} E_{v,2rv+lv+1}(-d^v t^v).$$

Proof. This follows from putting $k = 1$ in Theorem (2.2). QED

Theorem 2.3. If $d, v > 0$, $c, l, t \in \mathbb{C}$, and $\mathfrak{a} \neq d$, then the equation

$$N(t) - N_0 W^k_{l,c}(d^v t^v) = -\mathfrak{a}^v {}_0D_t^{-v} N(t)$$

has the solution

$$N(t) = N_0 \sum_{r=0}^{\infty} \frac{(-c)^r \Gamma(2rv + vl/k + 1)}{\Gamma_k(rk + l + k) r!} \left(\frac{d^v}{2}\right)^{2r+l/k} \times t^{2vr+lv/k} E_{v,2rv+lv/k+1}(-\mathfrak{a}^v t^v),$$

where $E_{v,2rv+lv/k+1}$ is the generalized Mittag-Leffler function defined by (1.1).

Proof. This can be proved by the method similar to the proof of Theorem 2.1. So we omit all details. QED

Corollary 2.3. If $d, v > 0$ and $c, l, t \in \mathbb{C}$, then the equation

$$N(t) - N_0 W^1_{l,c}(d^v t^v) = -\mathfrak{a}^v {}_0D_t^{-v} N(t)$$

has the solution

$$N(t) = N_0 \sum_{r=0}^{\infty} \frac{(-c)^r \Gamma(2rv + vl + 1)}{\Gamma_k(rk + l + k) r!} \left(\frac{d^v}{2}\right)^{2r+l} t^{2vr+lv} E_{v,2rv+lv+1}(-\mathfrak{a}^v t^v).$$

Proof. This follows from setting $k = 1$ in Theorem (2.3). QED

3 Graphical interpretation

In this section, we plot graphs of the solution (2.2). In each graph, we give four special solutions by assigning different values to the parameters.

In Figures 1, we take $k = 1$ and $\upsilon = 0.5, 1, 1.5, 2$.

Similarly, in Figures 2 to 6, we take $k = 2, 3, 4, 5, 10$ and all other parameters are fixed by 1. Hence, the effect of k is shown.

From these figures, we can see that $N_t > 0$ and N_t is an increasing function for $t \in (0, \infty)$. We also can see that $\lim_{t\to0^+} N_t = 0$ and $\lim_{t\to\infty} N_t = \infty$ hold for all chosen parameters.

When plotting these graphs, we choose the first 50 terms of the Mittag-Leffler function and the first 50 terms of our solutions to plot these graphs.

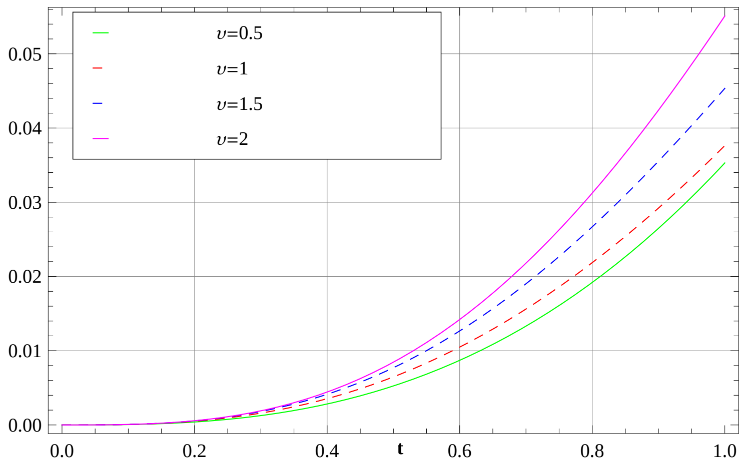

Figure 1. The solution (2.2) for $N(t)$ and $k = 1$

References

[1] Á. Baricz: Generalized Bessel functions of the first kind, Lecture Notes in Mathematics, **1994**, Springer-Verlag, Berlin, 2010.

[2] R. A. Cerutti: *On the k-Bessel functions*, Int. Math. Forum, **7** (2012), no. 37-40, 1851–1857.

[3] V. B. L. Chaurasia, S. C. Pandey: *On the new computable solution of the generalized fractional kinetic equations involving the generalized function for the fractional calculus and related functions*, Astrophys. Space Sci., **317** (2008), no. 3, 213–219.

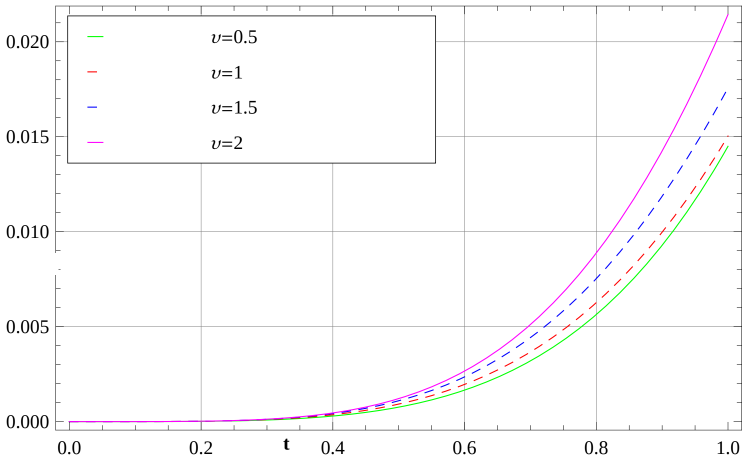

Figure 2. The solution (2.2) for $N(t)$ and $k = 2$

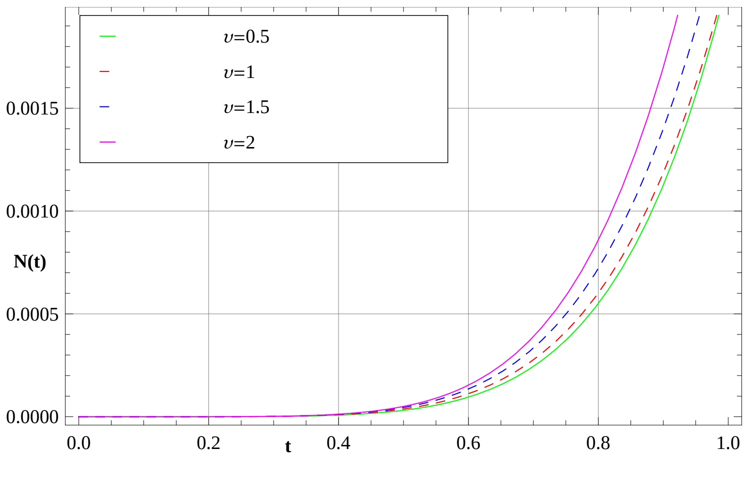

Figure 3. The solution (2.2) for $N(t)$ and $k = 3$

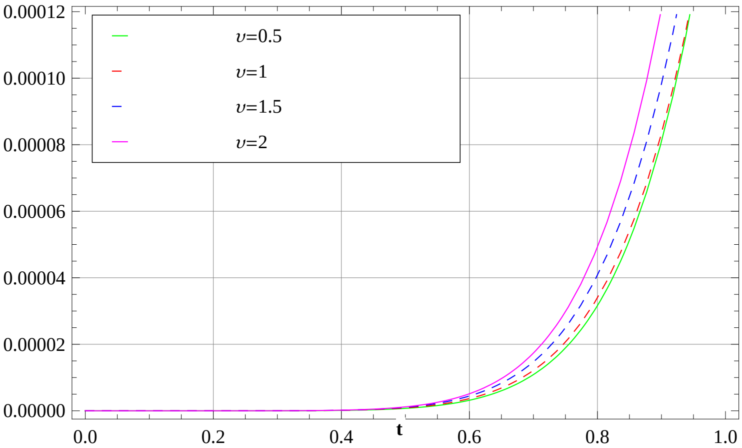

Figure 4. The solution (2.2) for $N(t)$ and $k = 4$

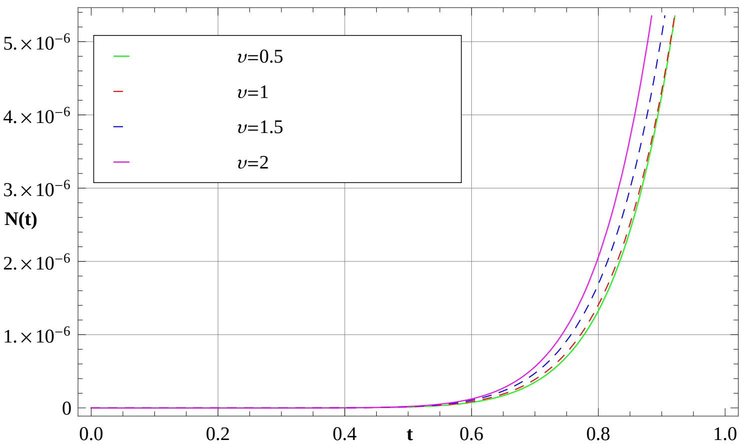

Figure 5. The solution (2.2) for $N(t)$ and $k = 5$

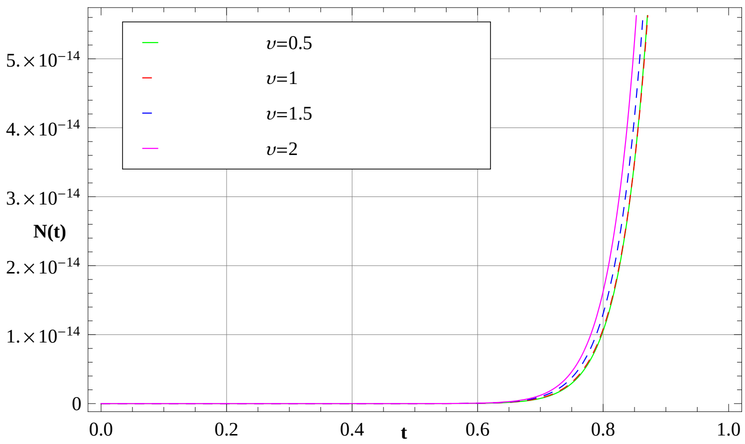

Figure 6. The solution (2.2) for $N(t)$ and $k = 10$

[4] A. Chouhan, S. D. Purohit, S. Saraswat: *An alternative method for solving generalized differential equations of fractional order*, Kragujevac J. Math., **37** (2013), no. 2, 299–306.

[5] A. Chouhan, S. Saraswat: *On solution of generalized kinetic equation of fractional order*, Int. J. Math. Sci. Appl., **2** (2012), no. 2, 813–818.

[6] R. Díaz, E. Pariguan: *On hypergeometric functions and Pochhammer k-symbol*, Divulg. Mat., **15**(2007), no. 2, 179–192.

[7] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi: Tables of Integral Transforms, Vol. I, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954.

[8] A. Gupta, C. L. Parihar: *On solutions of generalized kinetic equations of fractional order*, Bol. Soc. Parana. Mat. (3), **32**(2014), no. 1, 183–191.

[9] V. G. Gupta, Bhavna Sharma, F. B. M. Belgacem: *On the solutions of generalized fractional kinetic equations*, Appl. Math. Sci. (Ruse), **5** (2011), no. 17-20, 899–910.

[10] H. J. Haubold, A. M. Mathai: *The fractional kinetic equation and thermonuclear functions*, Astrophys. Space Sci., **273** (2000), no. 1, 53–63.

[11] G. Mittag-Leffler: *Sur la représentation analytique d'une branche uniforme d'une fonction monogène*, Acta Math., **29** (1905), no. 1, 101–181.

[12] S. R. Mondal: *Representation formulae and monotonicity of the generalized k-Bessel functions*, arXiv:1611.07499, 12 pages, 2016.

[13] K. S. Nisar, S. D. Purohit, S. R. Mondal: *Generalized fractional kinetic equations involving generalized Struve function of the first kind*, J. King Saud Univ. Sci., **28** (2016), no. 2, 167–171.

[14] L. G. Romero, G. A. Dorrego, R. A. Cerutti: *The k-Bessel function of the first kind*, Int. Math. Forum, **7** (2012), no. 37-40, 1859–1864.

[15] A. I. Saichev, G. M. Zaslavsky: *Fractional kinetic equations: solutions and applications*, Chaos, **7** (2997), no. 4, 753–764.

[16] R. K. Saxena, S. L. Kalla: *On the solutions of certain fractional kinetic equations*, Appl. Math. Comput., **199** (2008), no. 2, 504–511.

[17] R. K. Saxena, A. M. Mathai, H. J. Haubold: *On fractional kinetic equations*, Astrophys. Space Sci., **282** (2002), no. 1, 281–287.

[18] R. K. Saxena, A. M. Mathai, H. J. Haubold: *On generalized fractional kinetic equations*, Phys. A, **344** (2004), no. 3-4, 657–664.

[19] R. K. Saxena, A. M. Mathai, H. J. Haubold: *Solution of generalized fractional reaction-diffusion equations*, Astrophys. Space Sci., **305** (2006), no. 305, 305–313.

[20] M. R. Spiegel: Theory and problems of Laplace transforms, Schaum Publishing Co., New York, 1965.

[21] H. M. Srivastava, R. K. Saxena: *Operators of fractional integration and their applications*, Appl. Math. Comput., **118** (2001), no. 1, 1–52.

[22] A. Wiman: *Über den Fundamentalsatz in der Teorie der Funktionen* $E^{a}(x)$, Acta Math., **29** (1905), no. 1, 191–201.

[23] G. M. Zaslavsky: *Fractional kinetic equation for Hamiltonian chaos*, Phys. D, **76** (1994), no. 1-3, 110–122.